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# Quantum Mechanical Potential Step Functions, Barriers, Wells and the Tunneling Effect

**Gilbert A. Ibitola, Olanrewaju Ajanaku**

Department of Physical Sciences, Ondo State University of Science and Technology, Okitipupa, Nigeria

**Email address:**
ibitolaieee@gmail.com (G. A. Ibitola), lanrayus@yahoo.com (O. Ajanaku)

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**Abstract:** In this paper, use is made of the tools of analytical mechanics and the concept of operators to obtain the time-independent and time-dependent Schrodinger wave equations for quantum mechanical systems. Derivations are embarked upon of expressions for reflection and transmission coefficients for a particle of mass m as well as of energy E moving under different potential set-ups across step functions, barriers and well functions. The tunneling effect is then discussed. The transmission probability equation obtained in this research has been observed to be more accurate than the transmission probability expression deduced by some researchers in 2014 for a tunneling barrier. This research work finds applications in nuclear magnetic resonance imaging systems, synchrotrons, gyrators, accelerators, and in electrodynamics.

**Keywords:** Schrodinger Wave Equation (SWE), Potential Step, Potential Barrier, Potential Well, Wave Function, Reflection Coefficient, Transmission Probability and Tunneling Effect

## 1. Introduction

Several texts and past research work on this research area or topic employ diverse and very difficult approaches in the treatment of potential steps, barriers, wells and the tunneling effect [1, 2]. In this paper, use would be made of a simplified model in deriving the time-independent and time-dependent Schrodinger Wave Equations (SWEs) as outlined in Section 2 [3-5]. In Section 3, some illumination will be shed on potential step functions as encountered by moving particles in systems and devices. Further, in Section 4, the potential barrier functions as encountered by moving particles in devices and systems will be treated with derivations and discussions [6, 7]. Section 5 explains the concept of potential well of depth $V_0$ in a scenario in which it is encountered by some particle in motion of mass m and of energy E (where both cases, $E > V_0$ and $E < V_0$, are treated). Section 6 offers to explain the 'tunneling effect' and some of its applications [8-10]. The concluding remarks will be presented in Section 7.

## 2. Schrodinger Wave Equations

The Lagrange's equation for a conservative mechanical system is given by:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial q_i}\right) - \frac{\partial L}{\partial q_i} = \frac{\partial W}{\partial q_i} \tag{1}$$

But $\partial W/\partial q_i = 0$. Therefore, Equation (1) becomes

$$\frac{d}{dt}\left(\frac{\partial L}{\partial q_i}\right) - \frac{\partial L}{\partial q_i} = 0 \tag{2}$$

The Langragian (L) of the mechanical system (quantum or classical) is given by:

$$L = T - U \tag{3}$$

Where T = kinetic energy; U = Potential energy; $q_i$ = generalized coordinate.

$$\frac{\partial L}{\partial q_i} = \frac{\partial (T-U)}{\partial q_i} = \frac{\partial T}{\partial q_i} = mq_i = P_i \tag{4}$$

Hence we have:

$$\sum q_i \frac{\partial L}{\partial q_i} = m(q_i)^2 = 2T \tag{5}$$

The Hamiltonian (H) of a mechanical system is its total energy, and is given by:

$$H = \sum q_i \frac{\partial L}{\partial q_i} - L = 2T - L = 2T - (T - U)$$
$$= 2T - T + U = T + U \quad (6)$$

Let p be the momentum and m the mass of the mechanical system. Then

$$H = \frac{p^2}{2m} + U = E \quad (7)$$

Using operators, equation (7) becomes

$$\hat{H} = \frac{\hat{p}^2}{2m} + \hat{U} = \hat{E} \quad (8)$$

Introducing the concept of wave function $\Psi(x)$ in one dimension, we have:

$$\hat{H}\Psi(x) = \frac{\hat{p}^2\Psi(x)}{2m} + \hat{U}(x)\Psi(x) = \hat{E}\Psi(x) \quad (9)$$

The momentum operator is given by:

$$\hat{p} = i\hbar\frac{\partial}{\partial x} = i\hbar\frac{d}{dx} \quad (10)$$

$$\Rightarrow (\hat{p})^2 = -\hbar^2\frac{\partial^2}{\partial x^2} = -\hbar^2\frac{d^2}{dx^2} \quad (11)$$

By virtue of equation (11), equation (9) becomes:

$$\hat{H}\Psi(x) = -\hbar^2\frac{d^2}{dx^2}\Psi(x) + \hat{U}(x)\Psi(x) = \hat{E}\Psi(x) \quad (12)$$

Equation (12) is the time-independent Schrödinger wave equation (SWE). For time-dependent SWE, the total energy operator $(\hat{E})$ is given by:

$$\hat{E} = -i\hbar\frac{\partial}{\partial t} \quad (13)$$

By virtue of equation (13), equation (12) becomes:

$$H\Psi(x,t) = -\hbar^2\frac{\partial^2}{\partial x^2}\Psi(x,t) + U(x)\Psi(x,t) = -i\hbar\frac{\partial}{\partial t}\Psi(x,t) \quad (14)$$

Equation (14) is called the time-dependent SWE.

SWEs are used for determining the expressions of probabilities and computing values for energy states (En) and wave functions $\Psi n\ for\ quantum\ mechanical\ systems.$

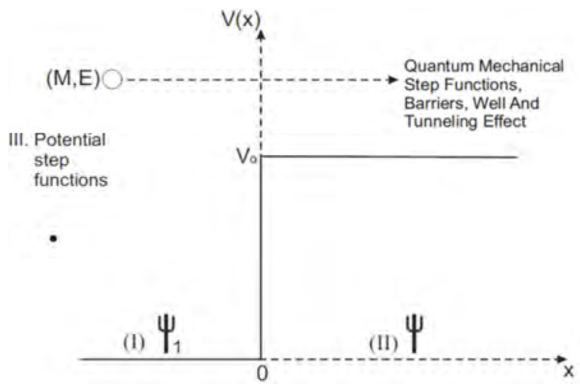


***Figure 1.*** *A potential step with a moving particle of mass m and of energy E (Where E > $V_O$).*

# 3. Potential Step Functions

Let us consider Figure 1 which displays a potential step of height $V_O$ which is encountered by a particle of energy E (Where E > $V_O$). Then the potential step is described by:

$$V(x) = \begin{cases} 0; \ x < 0 \\ V_o; x \geq 0 \end{cases} \quad (15)$$

For region I in figure 1 in the case where E > $V_O$, the SWE is:

$$\frac{d^2\Psi_I(x)}{dx^2} + K_1^2\Psi_I(x) = 0; for\ x < 0 \quad (16)$$

Where $K_1^2 = 2mE/\hbar^2$.

For region (II) in figure 1 in the case where E > $V_O$, the SWE is written as:

$$\frac{d^2\Psi_2(x)}{dx^2} + K_2^2\Psi_2(x) = 0;\ for\ x \geq 0 \quad (17)$$

Where $K_2^2 = 2m(E - V_o)/\hbar^2$.

On solving equations (15) and (16), we get:

$$\Psi_1(x) = Ae^{iK_Ix} + Be^{-iK_1x}; \quad (18)$$

$$\Psi_2(x) = Ce^{iK_2x} + De^{-iK_2x}$$
$$= Ce^{iK_2x}; for\ x \geq 0 \quad (19)$$

Where A, B and C are amplitude constants and D = 0.

The probability that the moving particle will be reflected (R) is:

$$R = J_r/J_i \quad (20)$$

The probability that the moving particle is transmitted (T) is given by:

$$T = J_t/J_i \quad (21)$$

Where:

$$J_i = incident\ current\ density;$$
$$J_r = reflected\ current\ density;$$
$$J_t = transmitted\ current\ density.$$

It has been found that:

$$J_i = \frac{\hbar K_I}{m}|A|^2 \quad (22)$$

$$J_r = \frac{\hbar K_I}{m}|B|^2 \quad (23)$$

$$J_t = \frac{\hbar K_2}{m}|C|^2 \quad (24)$$

Thus, by virtue of Equations 22 - 24, we obtain:

Reflection coefficient,

$$R = J_r/J_i = \frac{|B|^2}{|A|^2} \quad (25)$$

Transmission coefficient,

$$T = J_t/J_i = (K_2/K_1)\frac{|C|^2}{|A|^2} \tag{26}$$

To find the expressions for R and T in terms of $K_1$ and $K_2$, we proceed as follows. Applying the boundary conditions on equations (17) and (18), we get:

$$\Psi_I(x)|_{x=0} = \Psi_2(x)|_{x=0} \tag{27}$$

$$\left.\frac{d\Psi_1(x)}{dx}\right|_{x=0} = \left.\frac{d\Psi_2(x)}{dx}\right|_{x=0} \tag{28}$$

Thus, we get:

$$A+B = C \tag{29}$$

$$K_I A - K_I B = K_2 C. \tag{30}$$

Solving for B, we get:

$$B = \left(\frac{K_1 - K_2}{K_1 + K_2}\right)A$$

$$\Rightarrow \frac{B}{A} = \left[\frac{K_1 - K_2}{K_1 K_2}\right] \tag{31}$$

Solving for C, we obtain:

$$C = \left(\frac{2K_1}{K_1 + K_2}\right)A$$

$$\Rightarrow \frac{C}{A} = \left(\frac{2K_1}{K_1 + K_2}\right) \tag{32}$$

Hence, we get:

$$R = \frac{|B|^2}{|A|^2} = \frac{(K_1 - K_2)^2}{(K_1 + K_2)^2} \tag{33}$$

And

$$T = (K_1/K_2)\frac{|C|^2}{|A|^2}$$

$$= \frac{4K_1K_2}{(K_1 + K_2)^2} \tag{34}$$

At this point, let us find the reflection coefficient, R, and the transmission coefficient, T, for the wave function of a particle of a mass m and of energy E encountering a potential step, $V_O$, represented by:

$$V(x) = \begin{cases} 0; & x < 0 \\ V_o; & x > 0 \end{cases} \tag{35}$$

Where $E < V_o$. See Figure 2.

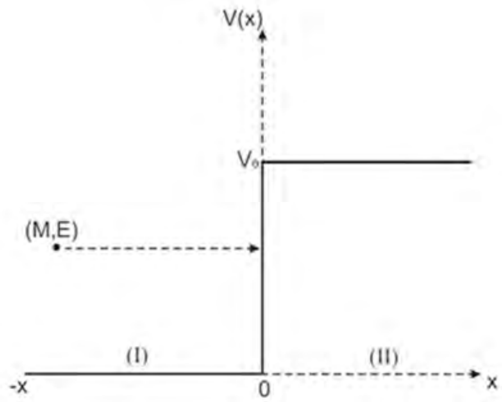


***Figure 2.*** *Potential step of height $V_O$ encountered by a particle of mass m of energy E (where $E < V_O$).*

For region (I), the SWE is:

$$\frac{d^2\Psi_1(x)}{dx^2} + K_1^2\Psi_1(x) = 0; x < 0 \tag{36}$$

On solving equation (36), we get:

$$\Psi_1(x) = \Psi_i(x) + \Psi_r(x) = Ae^{iK_I x} + Be^{-iK_1 x} \tag{37}$$

Where:

$$\Psi_i(x) = Ae^{iK_I x};$$

$$\Psi_r(x) = Be^{-iK_1 x};$$

A,B = amplitude constants; and $K_1^2 = \frac{2mE}{\hbar^2}$.
For region (II), the SWE is written as:

$$\frac{d^2\Psi_2(x)}{dx^2} - K_2^2\Psi_2(x) = 0; x \geq 0 \tag{38}$$

On solving Equation (38), we get:

$$\Psi_2(x) = Ce^{ik_2 x} + De^{-iK_2 x}$$

$$= Ce^{ik_2 x} = \Psi_t(x). \tag{39}$$

Where: D = 0;

C = amplitude constant and $K_2 = \frac{2m(V_o - E)}{\hbar^2}$.

Let

$$J_i = probability\ incident\ current\ density;$$

$$J_r = probability\ reflected\ current\ density;$$

$$J_t = probability\ transmitted\ current\ density.$$

Then

$$R = \frac{J_r}{J_i} = \frac{J_i}{J_i} = 1 \tag{40}$$

$$T = \frac{J_t}{J_i} = \frac{0}{J_i} = 0 \tag{41}$$

This means the reflection is complete.

## 4. Potential Barriers

Let us consider the potential barrier in Figure 3.

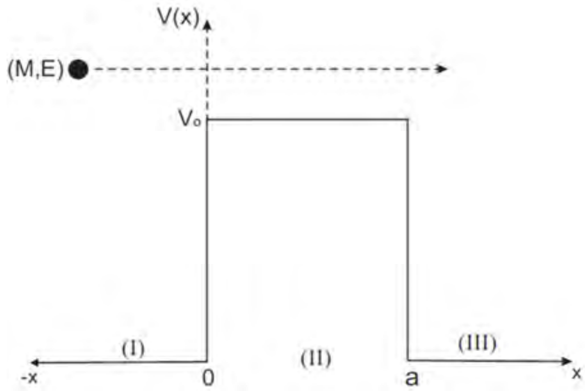


***Figure 3.*** *A potential barrier of width a and of height $V_O$ encountered by a moving object or particle of mass M and of energy E (where E> $V_O$).*

The potential barrier in figure 3 can be represented by:

$$V(x) = \begin{cases} 0;\ x < 0 \\ V_0;\ 0 \le x \le a \\ 0;\ x > a \end{cases} \quad (42)$$

For region (I), the SWE is:

$$\frac{d^2\Psi_I(x)}{dx^2} + K_1^2\Psi_I(x) = 0; for\ x < 0 \quad (43)$$

On solving equation (43), we get:

$$\Psi_1(x) = Ae^{iK_Ix} + Be^{-iK_1x}; x < 0 \quad (44)$$

Where $K_1^2 = {2mE}/{\hbar^2}$;

A,B =amplitude constants.

For region II, the SWE is written as:

$$\frac{d^2\Psi_2(x)}{dx^2} + K_2^2\Psi_2(x) = 0; 0 \le x \le a \quad (45)$$

On solving equation (45), we have:

$$\Psi_2(x) = Ce^{ik_2x} + De^{-iK_2x};\ 0 \le x \le a \quad (46)$$

Where:

$$K_2^2 = \frac{2m(E-V_O)}{\hbar^2};$$

C,D = amplitude constants.
For region (III), the SWE is written as:

$$\frac{d^2\Psi_3(x)}{dx^2} + K_1^2\Psi_3(x) = 0; for\ x > a \quad (47)$$

On solving equation (47), we obtain:

$$\Psi_3(x) = Ee^{iK_1x} + Fe^{-iK_1x};\ x > a$$

$$= Ee^{iK_1x} \quad (48)$$

F= 0, which implies there is no reflection in this region.

To determine A, B, C, D and E, let us use the boundary conditions which work out here to be the following:

$$\Psi_1(0) = \Psi_2(0) \quad (49)$$

$$\frac{d\Psi_1(0)}{dx} = \frac{d\Psi_2(0)}{dx} \quad (50)$$

$$\Psi_2(a) = \Psi_3(a) \quad (51)$$

$$\frac{d\Psi_2(a)}{dx} = \frac{d\Psi_3(a)}{dx} \quad (52)$$

Equations (49-52) work out to give us the following:

$$A + B = C + D \quad (53)$$

$$iK_1(A-B) = iK_2(C-D) \quad (54)$$

$$Ce^{iK_2a} + De^{-iK_2a} = Ee^{iK_1a} \quad (55)$$

$$iK_2Ce^{iK_2a} - iK_2De^{-iK_2a} = iK_1Ee^{iK_1a} \quad (56)$$

$E$ Is found to be:

$$E = 4K_1K_2e^{iK_1a}.A[4K_1K_2cos(K_2a) - 2i(K_1^2 + K_2^2)sin(K_2a)]^{-1} \quad (57)$$

The transmission coefficient T is:

$$T = \frac{|E|^2}{|A|^2}$$

$$= \left[1 + \frac{1}{4}\left(\frac{K_1^2-K_2^2}{K_1K_2}\right)^2 sin^2(K_2a)\right]^{-1} \quad (58)$$

The reflection coefficient R is:

$$R = \frac{|B|^2}{|A|^2}$$

$$= \left[1 + \frac{4\left(\frac{E}{V_O}\right)\left(\frac{E}{V_O}-1\right)}{sin^2\left[a\left(\frac{2mV_O}{\hbar^2}\right)^{\frac{1}{2}}\left(\frac{E}{V_O}-1\right)^{\frac{1}{2}}\right]}\right]^{-1} \quad (59)$$

Now for a potential barrier of height $V_o$, let us consider the case where the energy E of the moving particle of mass m is less than $V_o$ (i.e. E< $V_o$). Refer to figure 4.

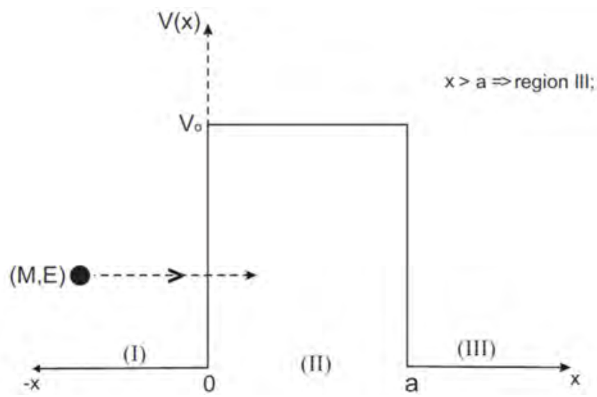


***Figure 4.*** *A potential barrier of height $V_O$ and of width 'a' encountered by a moving particle of mass m and of energy E (where E < $V_O$).*

The potential barrier is described by:

$$V(x) = \begin{cases} 0;\ x < 0 \\ V_0; 0 \le x \le a \\ 0;\ x > a \end{cases} \quad (60)$$

For region (I), the SWE is:

$$\frac{d^2\Psi_I(x)}{dx^2} + K_1^2\Psi_I(x) = 0; for\ x < 0 \quad (61)$$

Upon solving equation (61), we obtain:

$$\Psi_1(x) = Ae^{iK_Ix} + Be^{-iK_1x}; x < 0 \quad (62)$$

Where $K_1^2 = {2mE}/{\hbar^2}$;
For region (II), we have as SWE:

$$\frac{d^2\Psi_2(x)}{dx^2} + K_2^2\Psi_2(x) = 0;\ 0 \le x \le a \quad (63)$$

Where $K_2^2 = \frac{2m(V_0-E)}{\hbar^2}$
The solution for equation (63) is:

$$Ce^{ik_2x} + De^{-iK_2x};\ 0 \le x \le a \quad (64)$$

For region (III), the SWE is:

$$\frac{d^2\Psi_3(x)}{dx^2} + K_1^2\Psi_3(x) = 0;\ x > a \quad (65)$$

On solving equation (65), we get:

$$\Psi_3(x) = Ee^{iK_1x} + Fe^{-iK_1x};\ x > a$$

But F = 0, since there is no reflection in region (III), therefore we have:

$$\Psi_3(x) = Ee^{iK_1x}; x > a \quad (66)$$

The reflection coefficient R is:

$$R = \frac{|B|^2}{|A|^2} \quad (67)$$

The transmission coefficient T is:

$$T = \frac{|E|^2}{|A|^2} \quad (68)$$

To determine A, B and E, let us use the following boundary conditions:

$$\Psi_1(0) = \Psi_2(0) \quad (69)$$

$$\frac{d\Psi_1(0)}{dx} = \frac{d\Psi_2(0)}{dx} \quad (70)$$

$$\Psi_2(a) = \Psi_3(a) \quad (71)$$

$$\frac{d\Psi_2(a)}{dx} = \frac{d\Psi_3(a)}{dx} \quad (72)$$

Solving for A, B and E, we obtain (after substitutions in equations 67 and 68):

$$R = \left(\frac{K_1^2+K_2^2}{K_1K_2}\right)^2 sinh^2(K_2a).\left[4cosh^2(K_2a) + \left(\frac{K_1^2+K_2^2}{K_1K_2}\right)^2 sinh^2(K_2a)\right]^{-1} \quad (73)$$

$$T = 4\left[4cosh^2(K_2a) + \left(\frac{K_1^2-K_2^2}{K_1K_2}\right)^2 sinh^2(K_2a)\right]^{-1} \quad (74)$$

## 5. Potential Wells

Let us consider a moving particle of mass m and of energy E trapped in a potential well of infinite walls as shown in Figure 5.

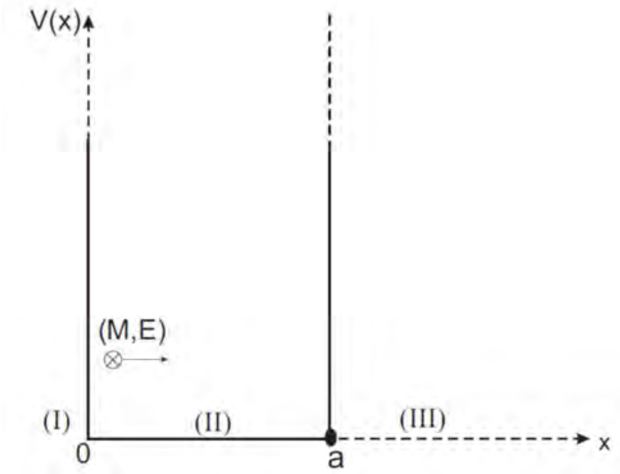


***Figure 5.*** *A potential well of width 'a'.*

The potential well is described by:

$$V(x) = \begin{cases} \infty;\ x < 0 \\ 0\ ; 0 \le x \le a \\ \infty;\ x > a \end{cases} \quad (75)$$

The wave functions in regions (I) and (III) are zero. Hence for region (II), inside the well, we have as the SWE:

$$\frac{\hbar^2}{2m}\frac{d^2\Psi(x)}{dx^2} = E\Psi(x)$$

$$\frac{d^2\Psi(x)}{dx^2} = \frac{-2mE}{\hbar^2}\Psi(x)$$

$$\Rightarrow \frac{d^2\Psi(x)}{dx^2} + K^2\Psi(x) = 0 \quad (76)$$

Where $K^2 = 2mE/\hbar^2$
Solving equation (76), we get:

$$\Psi(x) = Asin(kx) + Bcos(kx) \quad (77)$$

The well is infinite at x = 0 and at x = a. Hence,

$$\Psi(0) = 0$$

$$\Psi(a) = 0$$

$$\Psi(0) = 0 + Bcos\ (0) = 0$$

$$\Rightarrow B = 0 \quad (78)$$

$$\Psi(a) = Asin(Ka) = 0$$

But $A \neq 0$. Therefore:

$$sin(Ka) = 0$$

$$Ka = n\pi;\ n = 1,2,3,4, \dots \dots$$

$$\Longrightarrow K = (n\pi/a)$$

$$\text{But } K^2 = \frac{2mE}{\hbar^2} = n^2\pi^2/a^2$$

$$\therefore E = E_n = \frac{K^2\hbar^2}{2m}$$

$$= \frac{n^2\pi^2\hbar^2}{2ma^2} \quad (79)$$

(Where $n = 1,2,3,4, \dots \dots$). Equation (79) shows that the energy states in a potential well (and in all quantum systems) are quantized. Thus, the wave function is quantized also in a potential well. It is given by:

$$\Psi_n(x) = \left(\frac{2}{a}\right)^{\frac{1}{2}} sin\left(\frac{n\pi x}{a}\right); n = 1, 2, 3, 4, \dots. \quad (80)$$

## 6. The Tunneling Effect

Let us investigate the motion of a particle of mass m and of energy E in the presence of a square potential barrier of the form:

$$V(x) = V_0(a - |x|) \quad (81)$$

As shown in Figure 6.

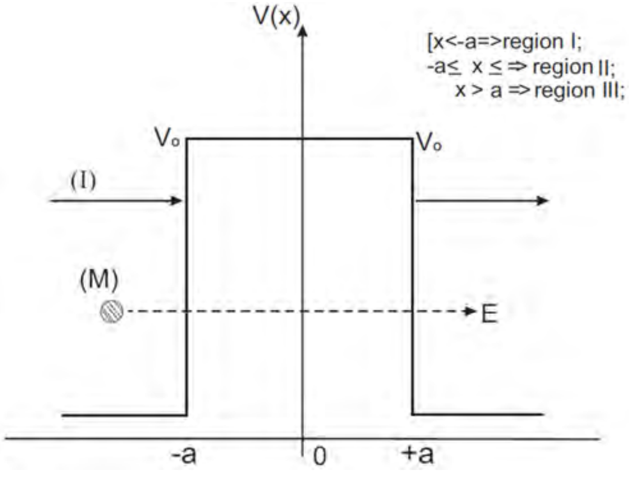


***Figure 6.*** *A potential barrier.*

A classical particle will be reflected from the potential barrier if its energy E is less than the barrier height $V_0$. According to quantum theory, there is a finite probability of the particle penetrating the potential barrier and being transmitted even if E < $V_0$. This purely quantum mechanical phenomenon is known as "tunneling effect". Important examples of tunneling effect are alpha decay of nuclei, the cold emission or the tunneling of cooper pairs between superconductors separated by a thin insulating layer.

If two superconductors are connected by a thin layer of insulating material with no electrostatic potential present, there is a quantum mechanical penetration, through the layer of electrons existing as cooper pairs. This arrangement (of superconductor-thin insulator-superconductor) is called a Josephon junction, and the tunneling is known as the dc Josephon effect.

If a steady potential difference V is applied, an oscillatory current passes through the layer of angular frequency $w = 2\pi f = 2\pi(eV/h)$, where e is the effective charge of the cooper pairs. This is known as the AC Josephson effect.

Let us solve the SWE for the three regions indicated around/in the potential barrier of Figure 6. The general form of the solution is:

$$\Psi(x) = \begin{cases} Ie^{iK_1x} + Re^{-iK_1x}; x < -a \\ Ce^{-K_2x} + De^{K_2x};\ -a \leq x \leq a \\ Te^{iK_1x} + Ge^{-iK_1x};\ x > a \end{cases} \quad (82)$$

Where:

$$K_1 = {(2mE)^{\frac{1}{2}}}/{\hbar};\ K_2 = \frac{[2m(V_0 - E)]^{\frac{1}{2}}}{\hbar}$$

In order to relate the amplitude constants I, R, C, D and G, we use the continuity requirements for the wave function $\Psi$ and its derivatives.

Thus, at x = -a:

$$Ie^{-iK_1a} + Re^{iK_1a} = Ce^{K_2a} + De^{-K_2a}; \quad (83)$$

$$iK(Ie^{-iK_1a} + Re^{iK_1a}) = -K_2(Ce^{K_2a} - De^{-K_2a}) \quad (84)$$

The above can be put in the following matrix form

$$\begin{bmatrix} e^{-iK_1a} & e^{iK_1a} \\ e^{-iK_1a} & -e^{iK_1a} \end{bmatrix}\begin{bmatrix} I \\ R \end{bmatrix} = \begin{bmatrix} e^{K_2a} & e^{-K_2a} \\ \frac{iK_2e^{K_2a}}{K_1} & \frac{-iK_2e^{-K_2a}}{K_1} \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix} \quad (85)$$

$$\begin{bmatrix} I \\ R \end{bmatrix} = \frac{1}{2}\begin{bmatrix} e^{iK_1a} & e^{iK_1a} \\ e^{-iK_1a} & -e^{iK_1a} \end{bmatrix}\begin{bmatrix} e^{K_2a} & e^{-K_2a} \\ \frac{iK_2e^{K_2a}}{K_1} & -\frac{iK_2e^{-K_2a}}{K_1} \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix}$$

$$= m(a)\begin{bmatrix} C \\ D \end{bmatrix} \quad (86)$$

Where:

$$M(a) = \frac{1}{2}\begin{bmatrix} \left(\left(1 + \frac{iK_2}{K_1}\right)e^{K_2a + iK_1a}\right) & \left(\left(1 - \frac{iK_2}{K_1}\right)e^{-K_2a + iK_1a}\right) \\ \left(\left(1 - \frac{iK_2}{K_1}\right)e^{K_2a - iK_1a}\right) & \left(\left(1 + \frac{iK_2}{K_1}\right)e^{-K_2a - iK_1a}\right) \end{bmatrix} \quad (87)$$

At x=a:

$$\begin{bmatrix} T \\ G \end{bmatrix} = M(-a)\begin{bmatrix} C \\ D \end{bmatrix} \quad (88)$$

Thus, from Equations (86) and (88), we have:

$$\begin{bmatrix} I \\ R \end{bmatrix} = M(a)M^{-1}(-a)\begin{bmatrix} T \\ G \end{bmatrix} \quad (89)$$

Where:

$$M^{-1}(-a) = \frac{1}{2}\begin{bmatrix} \left(\left(1-\frac{iK_1}{K_2}\right)e^{K_2a+iK_1a}\right) & \left(\left(1+\frac{iK_1}{K_2}\right)e^{K_2a-iK_1a}\right) \\ \left(\left(1+\frac{iK_1}{K_2}\right)e^{-K_2a+iK_1a}\right) & \left(\left(1-\frac{iK_1}{K_2}\right)e^{-K_2a-iK_1a}\right) \end{bmatrix} \quad (90)$$

Hence we have:

$$\begin{bmatrix} I \\ R \end{bmatrix} = \begin{bmatrix} \left(cosh2K_2a \quad + \frac{i\propto}{2}sinh2K_1a\right)e^{2iK_1a} & \left(\frac{i\beta}{2}sinh2K_2a\right) \\ \left(-\frac{i\beta}{2}sinh2K_2a\right) & \left(cosh2K_2a \ - \frac{i\propto}{2}sinh2K_1a\right)e^{-2iK_1a} \end{bmatrix} . \begin{bmatrix} T \\ G \end{bmatrix} \quad (91)$$

Where:

$$\propto = \frac{K_2}{K_1} - \frac{K_1}{K_2} \ ; \ \beta = \frac{K_1}{K_2} + \frac{K_2}{K_1} \quad (92)$$

Let us now confine ourselves to the situation where:

A particle of mass m and of energy E is incident from the left-hand side of the potential barrier.

For this case, G=O since there is no reflection in region III, and Eq. (91) becomes

$$I = T.\left(cosh2K_2a + \frac{i\propto}{2}sinh2K_2a\right)e^{2iK_1a} \quad (93)$$

$$R = T.\left(\frac{-i\beta}{2}\right)sinh2K_2a \quad (94)$$

The transmission coefficient, Tr(E), is defined as the ratio of the transmitted wave amplitude to the incident wave amplitude:

$$Tr(E) = \frac{T}{I} = \frac{e^{-2iK_1a}}{cosh2K_2a + \frac{i\propto}{2}sinh2K_2a} \quad (95)$$

The modulus squared of the above represents the probability (Pt) that a particle incident on the potential barrier will penetrate it. Thus,

$$P_t = |Tr(E)|^2 = \frac{1}{1+\left(1+\left(1+\frac{\propto^2}{4}\right)sinh^2 2K_2a\right)} \neq 0. \quad (96)$$

Thus, there is a finite probability $P_t$ ($\neq 0$) that a particle will be transmitted quantum mechanically even if its energy E is less than the potential barrier height, Vo.

If the potential barrier is very high and very wide, then

$$Ka \gg 1$$

And

$$sinh2K_2a \simeq \frac{1}{2}e^{2K_2a} \gg 1$$

Hence, we have:

$$P_t \cong \left(1+\frac{\propto^2}{4}\right)^{-1}.\left(4e^{-4K_2a}\right) \quad (97)$$

or

$$P_t = \frac{16E(V_o-E)}{{V_o}^2}.exp\left\{\frac{-4\propto(2m(V_o-E))^{\frac{1}{2}}}{\hbar}\right\} \quad (98)$$

According to the research finding of some researchers [Ref. 19] in 2014, the transmission probability (T(E)) through a tunneling barrier is:

$$T(E) = \begin{cases} \dfrac{4k_rk_b^2k_p}{k_r+k_p)^2k_b^2 + [k_r^2k_p^2 + k_b^2(k_b^2 - k_r^2 + k_p^2)]sin^2(k_bL)} \ for\ E > V_b \\ \dfrac{4k_rk_b^2k_p}{k_r+k_p)^2k_b^2 + [k_r^2k_p^2 + k_b^2(k_b^2 - k_r^2 + k_p^2)]sinh^2(k_bL)} \ for\ V_o < E < V_b \end{cases} \quad (99)$$

Where: ($V_b$ - $V_o$) = the barrier height; $K_r = [2mE]^{1/2}$; $K_p = [2m(E-V_b)]^{1/2}$; $K_b = [2m(V_b - E)]^{1/2}$; and

L = the barrier width.

It has been found that equation (99) is less accurate than equation (96) in the description of the tunneling effect. The tunneling effect is utilized in the design of the devices such as tunnel diodes.

## 7. Conclusion

A simplified approach has been employed in deriving the time- independent and time-dependent Schrodinger wave Equations (SWEs). The popular concepts of quantum mechanical potential steps, barriers and wells have been treated clearly. The quantum phenomenon of tunneling effect has been explained clearly and its applications have been mentioned. The transmission probability equation obtained in this research is observed to be more accurate than the transmission probability expression deduced by some researchers in 2014.

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