International Journal of Wireless Communications and Mobile Computing
2020; 8(1): 9-17
http://www.sciencepublishinggroup.com/j/wcmc
doi: 10.11648/j.wcmc.20200801.12
ISSN: 2330-1007 (Print); ISSN: 2330-1015 (Online)

New Orthogonal Binary Sequences Using Quotient Rings Z/*n*Z Where *n* Is a Multiple of Some Prime Numbers

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To cite this article:
Ahmad Hamza Al Cheikha. New Orthogonal Binary Sequences Using Quotient Rings Z/*n*Z Where n Is a Multiple of Some Prime Numbers. *International Journal of Wireless Communications and Mobile Computing*. Vol. 8, No. 1, 2020, pp. 9-17. doi: 10.11648/j.wcmc.20200801.12

Received: July 17, 2020; **Accepted**: September 27, 2020; **Published**: October 14, 2020

Abstract: Orthogonal Sequences (as M-Sequences, Walsh Sequences,…) are used widely at the forward links of communication channels to mix the information on connecting to and at the backward links of these channels to sift through this information is transmitted to reach the receivers this information in a correct form, especially in the pilot channels, the Sync channels, and the Traffic channel. This research is useful to generate new sets of orthogonal sequences (with the bigger lengths and the bigger minimum distance that assists to increase secrecy of these information and increase the possibility of correcting mistakes resulting in the channels of communication) from quotient rings Z/*n*Z, where Z is the integers and *n* is not of the form $p^{m,}$ where *p* is prime, replacing each event number by zero and each odd number by one, also, the increase in the natural number does not necessarily lead to an increase in the size of the biggest orthogonal set in the corresponding quotient ring. The length of any sequence in a biggest orthogonal set in the quotient ring Z/*n*Z is *n* and the minimum distance is between (*n*-3)/2 and (*n*-1)/2 and the sequences can be used as keywords or passwords for secret messages.

Keywords: Walsh Sequences, M-sequences, Additive Group, Coefficient of Correlation, Orthogonal Sequences, Quotient Ring

1. Introduction

Shannon's classic articles, 1948-1949, were followed by many research papers on the question of finding successful ways to encode a successful encoding of the media to allow it to be transmitted correctly through jammed channels. [1]

The main obstacle to encoding and decoding is the complexity of decoding and decoding. For this reason, efforts have been made to design cryptographic and decoding methods in an easy way. The works of Hocquenghem in 1959, Reed Solomon 1960, Chaudhuri and Bose in 1960, BCH codes or Bose–Chaudhuri–Hocquenghem codes and others as Goppa, and Petrson 1961 were a new starting point for solving this issue. [2-8]

In all stages of encoding and decoding the orthogonal sequences play the main role in these processes n all stages of encoding and decoding, the orthogonal sequences play the main role in these processes, including: the sequences with maximum period M-Sequences, the Walsh sequences, the Reed-Solomon sequences, and the other. [9-12]

In 2018 Al Cheikha A. H. publish an article "Generating New Binary Sequences Using Quotient Rings Z/p^mZ "and current article is extending to this article. [13]

Orthogonal Sequences are used widely at the forward links of communication channels to sift through this information is transmitted to reach the receivers this information in a correct form, especially in the pilot channels, the Sync channels, and the Traffic channel. [14-17]

2. Research Method and Material

Definition 1. The complement of the binary vector $X=(x_1,x_2,...,x_n), x_i \in F_2\{0,1\}$ is the vector $\overline{X}=(\overline{x_1},\overline{x_2},...,\overline{x_n})$, where:

$$\overline{x_i}=\begin{cases}1 & if \quad x_i=0\\ 0 & if \quad x_i=1\end{cases}. [1, 3]$$

Definition 2. (Euler function φ). $\varphi(n)$ is the number of the natural numbers that are relatively prime with *n*.

Definition 3. Suppose $x=(x_0,x_1,...,x_{n-1})$ and

$y=(y_0,y_1,...,y_{n-1})$ are binary vectors of length n on $GF(2)=\{0,1\}$. The coefficient of correlations function of x and y, denoted by $R_{x,y}$ is:

$$R_{x,y}=\sum_{i=0}^{n-1}(-1)^{x_i+y_i} \tag{1}$$

Where x_i+y_i is computed *mod* 2. It is equal to the number of agreements components minus the number of disagreements corresponding to components or if $x_i, y_i \in \{1,-1\}$ (usually, replacing in binary vectors x and y each "1" by "-1" and each "0" by "1") then

$$R_{x,y}=\sum_{i=0}^{n-1}x_i y_i \text{ , [1, 2, 14-17]} \tag{2}$$

Definition 4. Suppose $x=(x_0,x_1,\ldots,x_{n-1})$ and $y=(y_0,y_1,...,y_{n-1})$ are binary vectors of length n on $GF(2)=\{0,1\}$, or components belong to $\{1,-1\}$, is said strictly orthogonal or briefly orthogonal if $R_{x,y}=0$, (Usually, said orthogonal if $R_{x,y}\in\{-1,0.1\}$). [6-8, 12-16]

Definition 5. Suppose G is a set of binary vectors of length n:

$$G=\{X;X=(x_0,x_1,...,x_{n-1}),x_i\in F_2=\{0,1\},i=0,1,...,n-1\}$$

Let's $1^*=-1$ and $0^*=1$, The set G is said to be orthogonal if the following two conditions are Satisfied:

$$1.\ \forall X\in G,\ \sum_{i=0}^{n-1}x_i^*\in\{-1,0,1\},\ or\ \left|R_{x,0}\right|\in\{-1,0,1\}. \tag{3}$$

Or; the difference between the number of "0.s" and the number of "1.s" is at most one.

$$2.\ \forall X,Y\in G(X\neq Y),\sum_{i=0}^{n-1}x_i^*y_i^*\in\{-1,0,1\}\ or\left|R_{x,y}\right|\in\{-1,0,1\}. \tag{4}$$

That is, the absolute value of "the number of agreements minus the number of disagreements" is at most equal to one. [3, 7], [9-14]

Theorem 1.

$\varphi(p^s)=p^s\left(1-\frac{1}{p}\right)$, where p is prime.

$\varphi(m.n)=\varphi(m).\varphi(n)$, if $gcd\ (m, n)=1$.

$\varphi(m)=m\prod_{p|m}\left(1-\frac{1}{p}\right)$, p is prime or if $n=\prod_{i=1}^{k}p_i^{m_i}$ then $\varphi(n)=\prod_{i=1}^{k}p_i^{m_i-1}(p_i-1)$.

$\sum_{d|m}\varphi(d)=m$, where d is all divisors of m including 1 and m. [11, 12, 16]

Result. if n larger than 1 to $\varphi(n)$ is even except $\varphi(2)=1$.

3. Results and Discussion

In this study we restrict our self $n\neq p^m$, p is prime.

3.1. n Is Odd

The best method for getting the binary representation of the multiplication table of the quotient ring $Z/(nZ)$, where Z is the integers and n is natural number larger than 1, is replacing each event number by "0" and replacing each odd number by "1', by this way each row with the index i relatively prime with n contains $(n+1)/2$ of "0.s' and $(n-1)/2$ of "1.s" and the row ri is the conjugate of $r(n-i)$ that is the entries in $ri+r(n-i)$ are equal to zero by *mod n*.

We searching between these rows about a comfortable subset of rows which with the null row form additional subgroups achieve the number of "0.s" and the number of "1.s" or orthogonal conditions in the vector space 2^n, where the addition is performed by *mod* 2.

3.1.1. n = (3) (5) = 15

For the quotient ring $Z/15Z$ the following table 1 showing the multiple in the ring $Z/15Z$:

Table 1. *Multiplication Table of* $Z/15Z$.

	*	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
*r*0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
*r*1	1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
*r*2	2	0	2	4	6	8	10	12	14	1	3	5	7	9	11	13
*r*3	3	0	3	6	9	12	0	3	6	9	12	0	3	6	9	12
*r*4	4	0	4	8	12	1	5	9	13	2	6	10	14	3	7	11
*r*5	5	0	5	10	0	5	10	0	5	10	0	5	10	0	5	10
*r*6	6	0	6	12	3	9	0	6	12	3	9	0	6	12	3	9

	*	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
*r*7	7	0	7	14	6	13	5	12	4	11	3	10	2	9	1	8
*r*8	8	0	8	1	9	2	10	3	11	4	12	5	13	6	14	7
*r*9	9	0	9	3	12	6	0	9	3	12	6	0	9	3	12	6
*r*10	10	0	10	5	0	10	5	0	10	5	0	10	5	0	10	5
*r*11	11	0	11	7	3	14	10	6	2	13	9	5	1	12	8	4
*r*12	12	0	12	9	6	3	0	12	9	6	3	0	12	9	6	3
*r*13	13	0	13	11	9	7	5	3	1	14	12	10	8	6	4	2
*r*14	14	0	14	13	12	11	10	9	8	7	6	5	4	3	2	1

Where "*" is the multiplication on $Z/15Z$ and we can see that 1, 2, 4, 7, 8, 11, 13, 14 are relatively prime with 15 and $r1+r14 = r2+ r13 = r4+r11= [00\ldots0]_{15}$.

Table 2 showing the binary representation of table 1, when in table 1 each even number replaced by "0" and each odd number replaced by "1" and the binary row RI is a binary representation of the Row *ri*:

Table 2. *Binary Representation of* $Z/15Z$.

R0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
R1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0
R2	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1
R3	0	1	0	1	0	0	1	0	1	0	0	1	0	1	0
R4	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1
R5	0	1	0	0	1	0	0	1	0	0	1	0	0	1	0
R6	0	0	0	1	1	0	0	0	1	1	0	0	0	1	1
R7	0	1	0	0	1	1	0	0	1	1	0	0	1	1	0
R8	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1
R9	0	1	1	0	0	0	1	1	0	0	0	1	1	0	0
R10	0	0	1	0	0	1	0	0	1	0	0	1	0	0	1
R11	0	1	1	1	0	0	0	0	1	1	1	1	0	0	0
R12	0	0	1	0	1	0	0	1	0	1	0	0	1	0	1
R13	0	1	1	1	1	1	1	1	0	0	0	0	0	0	0
R14	0	0	1	0	1	0	1	0	1	0	1	0	1	0	1

From table 2:

1) All entries in the R0 are "0" and does not meet the conditions of orthogonal.
2) Each row of the R3, R5, R6, R9, R10, R12 contains 9 or 10 of "0.s" and 6 or 5 of "1.s" and does not meet the first conditions of orthogonal.
3) Each row of the R1, R2, R4, R7, R8, R11, R13, 14 contains (15+1)/2 of "0.s" and (15-1)/2 of "1.s" and the first condition of orthogonal is verified.
4) The basic rows R1, R2, R4, R7 are the conjugates of R14, R13, R11, R8 respectively and R1+R14 = R2+R13 = R4+R11 = R7+R8= $[011\ldots1]_{15}$.

The following table 3 showing the addition of some of the rows in the set where R*i* + R*j* denoted by R*i*+*j*. able 3 showing the addition between the rows{R1, R2, R3, R4}.

Table 3. *Addition between the rows {R1, R2, R4}.*

R1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0
R2	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1
R4	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1
R1+2	0	1	0	1	0	1	0	1	1	0	1	0	1	0	1
R1+4	0	1	0	1	1	0	1	0	0	1	0	1	1	0	1
R2+4	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0
R1+2+4	0	1	0	1	1	0	1	0	1	0	1	0	0	1	0
R1+7	0	0	0	1	1	0	1	0	1	0	1	1	1	0	0
R2+7	0	1	0	0	1	1	0	0	0	0	1	1	0	0	1

From table 3 we can see R2+7 contains 9 of "0.s" and 6 of "1.s" and does not satisfy the condition of orthogonal, also *Span*{R1, R2, R4}without R0 is {R1, R2, R4, R1+2, R1+4, R2+4} is a maximum closed orthogonal set contained in $F_{2^{15}}$, where {R1+2, R1+4, R2+4}is not including in binary representation of Z/15Z, and the number of these maximum closed orthogonal sets is at most $\binom{\varphi(3).\varphi(5)}{3} = \binom{2(4)}{3} = \binom{8}{3} = 56$ sets with the dimension 3, size or capacity is 7, and minimum distance 6 of each a set.

3.1.2. n = (3) (7) = 21

For the quotient ring $Z/21Z$ the multiplication table of $Z/21Z$ is the following table 4:

Table 4. *Multiplication Table of* $Z/21Z$.

	*	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
*r*o	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
*r*1	1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
*r*2	2	0	2	4	6	8	10	12	14	16	18	20	1	3	5	7	9	11	13	15	17	19
*r*3	3	0	3	6	9	12	15	18	0	3	6	9	12	15	18	0	3	6	9	12	15	18
*r*4	4	0	4	8	12	16	20	3	7	11	15	19	2	6	10	14	18	1	5	9	13	17
*r*5	5	0	5	10	15	20	4	9	14	19	3	8	13	18	2	7	12	17	1	6	11	16
*r*6	6	0	6	12	18	3	9	15	0	6	12	18	3	9	15	0	6	12	18	3	9	15
*r*7	7	0	7	14	0	7	14	0	7	14	0	7	14	0	7	14	0	7	14	0	7	14
*r*8	8	0	8	16	3	11	19	6	14	1	9	17	4	12	20	7	15	2	10	18	5	13
*r*9	9	0	9	18	6	15	3	12	0	9	18	6	15	3	12	0	9	18	6	15	3	12
*r*10	10	0	10	20	9	19	8	18	7	17	6	16	5	15	4	14	3	13	2	12	1	11
*r*11	11	0	11	1	12	2	13	3	14	4	15	5	16	6	17	7	18	8	19	9	20	10
*r*12	12	0	12	3	15	6	18	9	0	12	3	15	6	18	9	0	12	3	15	6	18	9
*r*13	13	0	13	5	18	10	2	15	7	20	12	4	17	9	1	14	6	19	11	3	16	8
*r*14	14	0	14	7	0	14	7	0	14	7	0	14	7	0	14	7	0	14	7	0	14	7
*r*15	15	0	15	9	3	18	12	6	0	15	9	3	18	12	6	0	15	9	3	18	12	6
*r*16	16	0	16	11	6	1	17	12	7	2	18	13	8	3	19	14	9	4	20	15	10	5
*r*17	17	0	17	13	9	5	1	18	14	10	6	2	19	15	11	7	3	20	16	12	8	4
*r*18	18	0	18	15	12	9	6	3	0	18	15	12	9	6	3	0	18	15	12	9	6	3
*r*19	19	0	19	17	15	13	11	9	7	5	3	1	20	18	16	14	12	10	8	6	4	2
*r*20	20	0	20	19	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1

We can see that 1, 2, 4, 5, 8, 10, 11, 13, 16, 17, 19, 20 are relatively prime with 21 and $r1+r20 = r2+r19= r4+r17= r5+r16 = r8+r13 = r10+r11= [00\ldots0]_{21}$.

Table 5 showing the binary representation of table 4, when in the table 4 each even number replaced by "0" and each odd number replaced by "1". The binary representation of table 4 is the table 5:

Table 5. *Binary Representation of Multiplication table of* $Z/21Z$.

Ro	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
R1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
R2	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1
R3	0	1	0	1	0	1	0	0	1	0	1	0	1	0	0	1	0	1	0	1	0	1
R4	0	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0	1	1	1	1	1	1
R5	0	1	0	1	0	0	1	0	1	1	0	1	0	0	1	0	1	1	0	1	0	1
R6	0	0	0	0	1	1	1	0	0	0	0	1	1	1	0	0	0	0	1	1	1	1
R7	0	1	0	0	1	0	0	1	0	0	1	0	0	1	0	0	1	0	0	1	0	1
R8	0	0	0	1	1	1	0	0	1	1	1	0	0	0	1	1	0	0	0	1	1	1
R9	0	1	0	0	1	1	0	0	1	0	0	1	1	0	0	1	0	0	1	1	0	1
R10	0	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	1
R11	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1
R12	0	0	1	1	0	0	1	0	0	1	1	0	0	1	0	0	1	1	0	0	1	1
R13	0	1	1	0	0	0	1	1	0	0	0	1	1	1	0	0	1	1	1	0	0	1
R14	0	0	1	0	0	1	0	0	1	0	0	1	0	0	1	0	0	1	0	0	1	1
R15	0	1	1	1	0	0	0	0	1	1	1	0	0	0	0	1	1	1	0	0	0	1
R16	0	0	1	0	1	1	0	1	0	0	1	0	1	1	0	1	0	0	1	0	1	1
R17	0	1	1	1	1	1	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0	1
R18	0	0	1	0	1	0	1	0	0	1	0	1	0	1	0	0	1	0	1	0	1	1
R19	0	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	2
R20	0	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	1

From table 5:

1) All entries in the R0 are "0" and does not meet the conditions of orthogonal.
2) Each row of the R3, R6, R7, R9, R12, R14, R15, R18 contains 12 or 14 of "0.s" and 9 or 7 of "1.s" and does not meet the first conditions of orthogonal except R9 contains 11 of "0.s" and 10 of "1.s". R2+R9 contains 12 of "0.s" and 9 of "1.s"
3) Each row of the R1, R2, R5, R8, R10, R11, R13, R14, R16, R17, R19, R20 contains (21+1)/2 of "0.s" and (21-1)/2 of "1.s" and the first condition of orthogonal is verified.

4) The basic rows R1, R2, R4, R5, R8, R10 are the conjugates of R20, R19, R17, R16, R13, R11 respectively and R1+R20 = R2+R19 = R4+R17 = R5+R16 = R8+R13 = R10+R11= $[011\ldots1]_{21}$.
5) R*i* and its conjugate R(*n-i*) can't be in one *Span*, each of R1 + R4 and R2+R8 contain 9 of '0.s" and 12 of "1.s", R2+R9 contains 12 of "0.s" and 9 of "1.s", and each of R1+R5 and R2+R10 contain 13 of "0.s" and 8 of "1.s".

Table 6 showing Sum R1 with R4 and R5 and table 7 showing *the s*um R2 with R8, R9, and R10

Table 6. *Sum R1 with R4 and R5.*

R1+4	0	1	0	1	0	1	1	0	1	0	1	1	0	1	0	1	1	0	1	0	1	
R1+5	0	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	

Table 7. *Sum R2 with R8, R9, and R10.*

R2+R8	0	0	0	1	1	1	0	0	1	1	1	1	1	1	0	0	1	1	1	0	0	
R2+R9	0	1	0	0	1	1	0	0	1	0	0	0	0	1	1	0	1	1	0	0	1	
R2+R10	0	0	0	1	1	0	0	1	1	0	0	0	0	1	1	0	0	1	1	0	0	

Thus; R1, R4, R5 can't be in one *Span* and the same for R2, R8, R9, R10.
6) The following table showing the addition between some of the rows in the set where R*i* + R*j* denoted by R*i+j*.
Table 8 showing the sum of the row R1 with the row R2:

Table 8. *Sum R1 with R2.*

R1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	
R2	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	
R1+2	0	1	0	1	0	1	0	1	0	1	0	0	1	0	1	0	1	0	1	0	1	

Span{R1, R2}without R0 is {R1, R2, R1+2} is a maximum closed orthogonal set contained in $F_{2^{21}}$, where {R1+2, R1+4, R2+4}is not including in binary representation of Z/21Z, and the number of these maximum closed orthogonal sets is at most $\binom{\varphi(3).\varphi(7)}{2}=\binom{2(6)}{2}=\binom{12}{2}=66$ sets with the dimension 2 and the size 3 of each a set, and minimum distance 10, while the expected dimensions and sizes for each of set are at least 4 and 17 respectively.

3.1.3. n = 5 (7) = 35

The following table 9 showing the multiple in the quotient ring $Z/35Z$ with restriction over the basic useful numbers which are relatively prime with 35 (are half of the numbers which are relatively prime with 35).

Table 9. *Multiplication table of* $Z/35Z$.

	*	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34
r1	1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34
r2	2	0	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	13	5	7	9	11	13	15	17	19	21	23	25	27	29	31	33	
r3	3	0	3	6	9	12	15	18	21	24	27	30	33	1	4	7	10	13	16	19	22	25	28	31	34	2	5	8	11	14	17	20	23	26	29	32
r4	4	0	4	8	12	16	20	24	28	32	1	5	9	13	17	21	25	29	33	2	6	10	14	18	22	26	30	34	3	7	11	15	19	23	27	31
r6	6	0	6	12	18	24	30	17	13	19	25	31	2	8	14	20	26	32	3	9	15	21	27	33	4	10	16	22	28	34	5	11	17	23	29	
r8	8	0	8	16	24	32	5	13	21	29	2	10	18	26	34	7	15	23	31	4	12	20	28	1	9	17	25	33	6	14	22	30	3	11	19	27
r9	9	0	9	18	27	1	10	19	28	2	11	20	29	3	12	21	30	4	13	22	31	5	14	23	32	6	15	24	33	7	16	25	34	8	17	26
r11	11	0	11	22	33	9	20	31	7	18	29	5	16	27	3	14	25	1	12	23	34	10	21	32	8	19	30	6	17	28	4	15	26	2	13	24
r12	12	0	12	24	1	13	25	2	14	26	3	15	27	4	16	28	5	17	29	6	18	30	7	19	31	8	20	32	9	21	33	10	22	34	11	23
r13	13	0	13	26	4	17	30	8	21	34	12	25	3	16	29	7	20	33	11	24	2	15	28	6	19	32	10	23	1	14	27	5	18	31	9	22
r16	16	0	16	32	13	29	10	26	7	23	4	20	1	17	33	14	30	11	27	8	24	5	21	2	18	34	15	31	12	28	9	25	6	22	3	19
r17	17	0	17	34	16	33	15	32	14	31	13	30	12	29	11	28	10	27	9	26	8	25	7	24	6	23	5	22	4	21	3	20	2	19	1	18

The binary representation of table 9 showing in the following table 10:

Table 10. *Binary representation of table 9.*

R1	0	1 0 1 0 1	0 1 0 1 0	1 0 1 0 1	0 1 0 1 0	1 0 1 0 1	0 1 0 1 0	1 0 1 0														
R2	0	0 0 0 0 0	0 0 0 0 0	0 0 0 0 0	0 0 1 1 1	1 1 1 1 1	1 1 1 1 1	1 1 1 1														
R3	0	1 0 1 0 1	0 1 0 1 0	1 1 0 1 0	1 0 1 0 1	0 1 0 0 1	0 1 0 1 0	1 0 1 0														
R4	0	0 0 0 0 0	0 0 0 1 1	1 1 1 1 1	1 1 0 0 0	0 0 0 0 0	0 1 1 1 1	1 1 1 1														
R6	0	0 0 0 0 0	1 1 1 1 1	1 0 0 0 0	0 0 1 1 1	1 1 1 0 0	0 0 0 0 1	1 1 1 1														
R8	0	0 0 0 0 1	1 1 1 0 0	0 0 0 1 1	1 1 0 0 0	0 1 1 1 1	1 0 0 0 0	1 1 1 1														
R9	0	1 0 1 1 0	1 0 0 1 0	1 1 0 1 0	0 1 0 1 1	0 1 0 0 1	0 1 1 0 1	0 0 1 0														
R11	0	1 0 1 1 0	1 1 0 1 1	0 1 1 0 1	1 0 1 0 0	1 0 0 1 0	0 1 0 0 1	0 0 1 0														
R12	0	0 0 1 1 1	0 0 0 1 1	1 0 0 0 1	1 1 0 0 0	1 1 1 0 0	0 1 1 1 0	0 0 1 1														

R13	0	10010	01001	10110	11001	00100	11011	0110
R16	0	00110	01100	11100	11001	10001	10011	0011
R17	0	10011	00110	01100	11001	10011	00110	0110

From table 10:

1) All entries in the R0 are "0" and does not meet the conditions of orthogonal.
2) Each row of the R5, R7, R10, R14, R15, R20, R21, R25, R28, R30, contains 20 or 21 of "0.s" and 15 or 14 of "1.s" and does not meet the first conditions of orthogonal.
3) Each row of the R1, R2, R3, R4, R6, R8, R9, R11, R12, R13, R16, R17 or their conjugates (by mode 35) contains (35+1)/2 of "0.s" and (35-1)/2 of "1.s" and the first condition of orthogonal is verified.
4) R*i* and its conjugate R(*n-i*) can't be in one *Span*, each of R1 + 3, R1+4, R1+9, R1+12, R1+13, R1+17, R2+6, R2+8, and R2+11 does not meet the first conditions of orthogonal and can't be in one *Span*.

Table 11 showing the sum of R1 and R2 with the some other rows.

Table 11. *Sum R1 and R2 with some other rows.*

R1+3	0	00000	00000	01111	11111	11100	00000	0000
R1+4	0	10101	01001	01010	10010	10100	00101	0101
R1+9	0	00011	11000	01111	00001	11100	00011	1000
R1+12	0	10010	01001	00100	10010	01001	00100	1001
R1+13	0	00111	00011	00011	10011	10001	10001	1100
R1+17	0	00010	01100	11001	10011	00110	01100	1100
R2+6	0	00000	11110	10000	00000	00011	11110	0000
R2+8	0	00001	11100	00011	11111	10000	01111	0000
R2+11	0	10110	11011	01101	10011	01101	10110	1101

5) The following table showing the addition between some of the rows in the set where R*i* + R*j* denoted by R*i+j*. Table 12 showing the of *Span* R1, R2 and R16.

Table 12. *Span R1, R2 and R16.*

R1	0	10101	01010	10101	01010	10101	01010	1010
R2	0	00000	00000	00000	00111	11111	11111	1111
R16	0	00110	01100	11100	11001	10001	10011	0011
R1+2	0	10101	01010	10101	01101	01010	10101	0101
R1+16	0	10111	00110	01001	10011	00100	11001	1001
R2+16	0	00110	01100	11100	11110	01110	01100	1100
R1+2+16	0	10011	00110	01001	10100	11011	00110	0110

Span {R1, R2, R16} without R0 is {R1, R2, R16, R1+2, R1+16, R2+16, R1+2+16} is a maximum closed orthogonal set contained in $F_{2^{35}}$, where {R1+2, R1+16, R2+16, R1+2+16}is not including in binary representation of Z/35Z, and the number of these maximum closed orthogonal sets is at most $\binom{\varphi(5).\varphi(7)}{3} = \binom{4(6)}{3}\binom{24}{3} = 2024$ sets with the dimension 3 and the size 7 of each a set, and minimum distance 17, while the expected dimension and size for each set are at least 5 and 31 respectively.

3.1.4. n = 3² (5) = 45

The following table 8 showing the multiple in the ring $Z/45Z$ with restriction over the basic useful numbers which are relatively prime with 45 (are half of the numbers which are relatively prime with 45.

Table 13. *Multiplication table in* $Z/45Z$.

	*	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44
*r*1	1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44
*r*2	2	0	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40	42	44	1	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31	33	35	37	39	41	43
*r*4	4	0	4	8	12	16	20	24	28	32	36	40	44	3	7	11	15	19	23	27	31	35	39	43	2	6	10	14	18	22	26	30	34	38	42	1	5	9	13	17	21	25	29	33	37	41
*r*7	7	0	7	14	21	28	35	42	4	11	18	25	32	39	1	8	15	22	29	36	43	5	12	19	26	33	40	2	9	16	23	30	37	44	6	13	20	27	34	41	3	10	17	24	31	38
*r*8	8	0	8	16	24	32	40	3	11	19	27	35	43	6	14	22	30	38	1	9	17	25	33	41	4	12	20	28	36	44	7	15	23	31	39	2	10	18	26	34	42	5	13	21	29	37
*r*11	11	0	11	22	33	44	10	21	32	43	9	20	31	42	8	19	30	41	7	18	29	40	6	17	28	39	5	16	27	38	4	15	26	37	3	14	25	36	2	13	24	35	1	12	23	34
*r*13	13	0	13	26	39	7	20	33	1	14	27	40	8	21	34	2	15	28	41	9	22	35	3	16	29	42	10	23	36	4	17	30	43	11	24	37	5	18	31	44	12	25	38	6	19	32
*r*14	14	0	14	28	42	11	25	39	8	22	36	5	19	33	2	16	30	44	13	27	41	10	24	38	7	21	35	4	18	32	1	15	29	43	12	26	40	9	23	37	6	20	34	3	17	31
*r*16	16	0	16	32	3	19	35	6	22	38	9	25	41	12	28	44	15	31	2	18	34	5	21	37	8	24	40	11	27	43	14	30	1	17	33	4	20	36	7	23	39	10	26	42	13	29
*r*17	17	0	17	34	6	23	40	12	29	1	18	35	7	24	41	13	30	2	19	36	8	25	42	14	31	3	20	37	9	26	43	15	32	4	21	38	10	27	44	16	33	5	22	39	11	28

*r*19	19	0	19	38	12	31	5	24	43	17	36	10	29	3	22	41	15	34	8	27	1	20	39	13	32	6	25	44	18	37	11	30	4	23	42	16	35	9	28	2	21	40	14	33	7	26
*r*22	22	0	22	44	21	43	20	42	19	41	18	40	17	39	16	38	15	37	14	36	13	35	12	34	11	33	10	32	9	31	8	30	7	29	6	28	5	27	4	26	3	25	2	24	1	23

The binary representation of table 8 showing in the following table 14

Table 14. *Binary representation of table 13.*

R1	0	1 0 1 0 1	0 1 0 1 0	1 0 1 0 1	0 1 0 1 0	1 0 1 0 1	0 1 0 1 0	1 0 1 0 1	0 1 0 1 0	1 0 1 0
R2	0	0 0 0 0 0	0 0 0 0 0	0 0 0 0 0	0 0 0 0 0	0 0 1 1 1	1 1 1 1 1	1 1 1 1 1	1 1 1 1 1	1 1 1 1
R4	0	0 0 0 0 0	0 0 0 0 0	0 1 1 1 1	1 1 1 1 1	1 1 0 0 0	0 0 0 0 0	0 0 0 1 1	1 1 1 1 1	1 1 1 1
R7	0	1 0 1 0 1	0 0 1 0 1	0 1 1 0 1	0 1 0 1 1	0 1 0 1 0	0 1 0 1 0	1 0 0 1 0	1 0 1 1 0	1 0 1 0
R8	0	0 0 0 0 0	1 1 1 1 1	1 0 0 0 0	0 1 1 1 1	1 1 0 0 0	0 0 0 1 1	1 1 1 0 0	0 0 0 0 1	1 1 1 1
R11	0	1 0 1 0 0	1 0 1 1 0	1 0 0 1 0	1 1 0 1 0	0 1 0 1 1	0 1 0 0 1	0 1 1 0 1	0 0 1 0 1	1 0 1 0
R13	0	1 0 1 1 0	1 1 0 1 0	0 1 0 0 1	0 1 1 0 1	1 0 1 0 0	1 0 0 1 0	1 1 0 1 1	0 1 0 0 1	0 0 1 0
R14	0	0 0 0 1 1	1 0 0 0 1	1 1 0 0 0	0 1 1 1 0	0 0 1 1 1	0 0 0 1 1	1 1 0 0 0	1 1 1 0 0	0 1 1 1
R16	0	0 0 1 1 1	0 0 0 1 1	1 0 0 0 1	1 1 0 0 1	1 1 0 0 0	1 1 1 0 0	1 1 1 0 0	0 1 1 1 0	0 0 1 1
R17	0	1 0 0 1 0	0 1 1 0 1	1 0 1 1 0	0 1 0 0 1	0 0 1 1 0	1 1 0 1 1	0 0 1 0 0	1 0 0 1 1	0 1 1 0
R19	0	1 0 0 1 1	0 1 1 0 0	1 1 0 1 1	0 0 1 1 0	1 1 0 0 1	0 0 1 1 0	0 1 0 0 1	1 0 0 1 0	0 1 1 0
R22	0	0 0 1 1 0	0 1 1 0 0	1 1 0 0 1	1 0 0 1 1	0 0 1 1 0	0 1 1 0 0	1 1 0 0 1	1 0 0 1 1	0 0 1 1

From table 13 and binary representation of other rows in Z/45Z:

1) All entries in the R0 are "0" and does not meet the conditions of orthogonal.
2) Each row of the R3, R5, R6, R9, R10, R12, R15, R18, R20, R21 and their conjugates(by *mode* 45) does not meet the first conditions of orthogonal.
3) Each row of the R1, R2, R4, R7, R8, R11, R13, R14, R16, R17, R19, R22 or their conjugates contains (45+1)/2 of "0.s" and (45-1)/2 of "1.s" and the first condition of orthogonal is verified.
4) R*i* and its conjugate R(*n*-*i*) can't be in one *Span* and where R*i* + R*j* denoted by R*i*+*j*,

each of R1 + 7, R1+11, R1+13, R1+4, R1+16, R1+19, R2+8, R2+17 and R2+22 does not meet the first conditions of orthogonal and can't be in one *Span*. Table 15 showing the sum of R1 and R2 with some other rows.

Table 15. *Sum R1 and R2 with some other rows.*

R1+7	0	00000	01111	11000	00001	11111	00000	00111	11100	0000
R1+11	0	00001	11100	00111	10000	11110	00011	11000	01111	0000
R1+13	0	00011	10000	11100	00111	00001	11000	01110	00011	1000
R1+14	0	10110	11011	01101	00100	10010	01001	01101	10110	1101
R1+16	0	10010	01001	00100	10011	01101	10110	01001	00100	1001
R1+19	0	00110	00110	01110	01100	01100	01100	11100	11000	1100
R2+8	0	00000	11111	100 00	01111	1111 1	11100	00011	11110	0 000
R2+17	0	10010	01101	10110	01001	00001	00100	11011	01100	1001
R2+22	0	00110	01100	11001	10011	00001	10011	00110	01100	1100

The following table 16 showing the Span of R1, R2 and R4;

Table 16. *Span R1, R2 and R4.*

R1	0	1 0 1 0 1	0 1 0 1 0	1 0 1 0 1	0 1 0 1 0	1 0 1 0 1	0 1 0 1 0	1 0 1 0 1	0 1 0 1 0	1 0 1 0
R2	0	0 0 0 0 0	0 0 0 0 0	0 0 0 0 0	0 0 0 0 0	0 0 1 1 1	1 1 1 1 1	1 1 1 1 1	1 1 1 1 1	1 1 1 1
R4	0	0 0 0 0 0	0 0 0 0 0	0 1 1 1 1	1 1 1 1 1	1 1 0 0 0	0 0 0 0 0	0 0 0 1 1	1 1 1 1 1	1 1 1 1
R1+2	0	1 0 1 0 1	0 1 0 1 0	1 0 1 0 1	0 1 0 1 0	1 0 0 1 0	1 0 1 0 1	0 1 0 1 0	1 0 1 0 1	0 1 0 1
R1+4	0	1 0 1 0 1	0 1 0 1 0	1 1 0 1 0	0 1 0 1 0	0 1 1 0 1	0 1 0 1 0	1 0 1 1 0	1 0 1 0 1	0 1 0 1
R2+4	0	0 0 0 0 0	0 0 0 0 0	0 1 1 1 1	1 1 1 1 1	1 1 1 1 1	1 1 1 1 1	1 1 1 0 0	0 0 0 0 0	0 0 0 0
R1+2+4	0	1 0 1 0 1	0 1 0 1 0	1 1 0 1 0	1 0 1 0 1	0 1 0 1 0	1 0 1 0 1	0 1 0 0 1	0 1 0 1 0	1 0 1 0

Span{R1, R2, R4}without R0 is {R1, R2, R4, R1+2, R1+4, R2+4, R1+2+4} is a maximum closed orthogonal set contained in $F_{2^{45}}$, where {R1+2, R1+4, R2+4, R1+2+4}is not including in binary representation of Z/45Z, and the number of these maximum closed orthogonal sets is at most $\binom{\varphi(9).\varphi(5)}{3} = \binom{6(4)}{3}\binom{24}{3} = 2024$ sets with the dimension 3 and the size 7 of each a set, and minimum distance 22, while the expected dimensions and sizes for each of set are at least 5 and 31 respectively.

3.2. n Is Even

1) If the index *i* is even then the entries in the row *ri* in the table Z/*n*Z contains only even numbers and its corresponding binary representation R*i* is zero row and

does not satisfy the first condition of orthogonal.

2) If the index *i* is odd then the entries in the row *ri* in the table Z/*n*Z contains one even entry and after one odd entry periodically and its corresponding binary representation R*i* the entries are one o and after 1 periodically and satisfies the first condition of orthogonal.

3) If the indexes *i*, *j* is odd numbers the distribution of odd and even numbers the same in *ri* and *rj* and the distribution of "0.s" and "1.s" in R*i* and R*j* also the same and R*i*+*j* is the zero row. Thus if *n* is even to Z/*n*Z don't have orthogonal sets and the following tables of representation of Z/10Z illustrated the ideas.

The following table 17 showing the multiplication on quotient ring Z/10Z:

Table 17. *multiplication on quotient ring Z/10Z.*

	*	0	1	2	3	4	5	6	7	8	9
r0	0	0	0	0	0	0	0	0	0	0	0
r1	1	0	1	2	3	4	5	6	7	8	9
r2	2	0	2	4	6	8	0	2	4	6	8
r3	3	0	3	6	9	2	5	8	1	4	7
r4	4	0	4	8	2	6	0	4	8	2	6
r5	5	0	5	0	5	0	5	0	5	0	5
r6	6	0	6	2	8	4	0	6	2	8	4
r7	7	0	7	4	1	8	5	2	9	6	3
r8	8	0	8	6	4	2	0	8	6	4	2
r9	9	0	9	8	7	6	5	4	3	2	1

Table 18 showing the binary representation of multiplication on quotient ring Z/10Z

Table 18. *Binary Representation of Z/10Z.*

r0	0	0	0	0	0	0	0	0	0	0
r1	0	1	0	1	0	1	0	1	0	1
r2	0	0	0	0	0	0	0	0	0	0
r3	0	1	0	1	0	1	0	1	0	1
r4	0	0	0	0	0	0	0	0	0	0
r5	0	1	0	1	0	1	0	1	0	1
r6	0	0	0	0	0	0	0	0	0	0
r7	0	1	0	1	0	1	0	1	0	1
r8	0	0	0	0	0	0	0	0	0	0
r9	0	1	0	1	0	1	0	1	0	1

4. Conclusions

When studying the quotient rings Z/15Z, Z/21Z, Z/35Z, Z/45Z and Z/10Z and their binary representation we found the following results:

4.1. For n Is Odd, and $n \neq p^m$ Where p Is Prime

(1) In binary representation of Z / nZ, the length of each row is *n*, started by zero, each row with index relatively prime with *n* has $(n+1)/2$ of "0.s", $(n-1)/2$ of "1.s", satisfy the first condition of orthogonal, the number of these rows is $\varphi(n)$, the first half of them is basic and the second half is their conjugates where the indexes computed *mod n*.

(2) If *i* is prime then R*i*+ R(*n*-*i*) = $[0\ 1\ 1\ 1 \ldots 1]_n$ that is $Ri = \overline{R(n-i)}$ except the first entry is zero in both of them

(3) In Z/15Z; the number of the biggest binary orthogonal closed sets (in the space 2^{15}*)* which we can get them from Z/15Z is at most $\binom{8}{3} = \binom{\varphi(10)}{3} = \binom{\varphi(3)\varphi(5)}{3} = 56$ sets with; dimension 3, length 15, and size or capacity 7 and minimum distance 6, of each set.

(4) In Z/21Z; the number of the biggest binary orthogonal closed sets (in the space 2^{21}*)* which we can get them from Z/21Z is at most $\binom{12}{2} = \binom{\varphi(12)}{2} = \binom{\varphi(3)\varphi(7)}{2} = 66$ sets with; dimension 2, length 21, and size or capacity 3, and minimum distance 10, of each set while the expected dimensions and sizes for each of set are at least 4 and 17 respectively.

(5) In Z/35Z; the number of the biggest binary orthogonal closed sets (in the space 2^{35}*)* which we can get them from Z/35Z is at most $\binom{24}{3} = \binom{\varphi(35)}{3} = \binom{\varphi(5)\varphi(7)}{3} = 2024$ sets with; dimension 3, length 35, and size or capacity 7, and minimum distance 17, of each set, while the expected dimensions and sizes for each of set are at least 5 and 31 respectively.

(6) In Z/45Z; the number of the biggest binary orthogonal closed sets (in the space 2^{45}*)* which we can get them from Z/45Z is at most $\binom{24}{3} = \binom{\varphi(45)}{3} = \binom{\varphi(3^3)\varphi(5)}{3} = 2024$ sets with; dimension 3, length 35, and size or capacity 7, and minimum distance 22, of each set, while the expected dimensions and sizes for each of set are at least 5 and 31 respectively.

4.2. For n Is Even

(*a*) The number of the biggest binary orthogonal closed sets (in the space 2^n) which we can get them from Z/nZ is only n/2 sets with; dimension 1, length *n*, and size or capacity 1 of each set, and this case is very trivial.

(*b*) From above in the quotient ring Z/nZ, and *n* is odd, the dimension of orthogonal set is don't increase or increase very slaw with increasing *n* and consequently their capacity but we can get orthogonal sets with biggest lengths and biggest minimum distances (*n*-1)/2.

(*c*) The increase in the natural number does not necessarily lead to an increase in the size of the biggest orthogonal set in the corresponding quotient ring (see 3.1.1 and 3.1.2).

(*d*) In Z/nZ the length any sequence in orthogonal set is n and the minimum distance is between (*n*-3)/2 and (*n*-1)/2.

Limitation: This method of compose sequences is useful for only binary sequences and the addition on the sequences computed by "*mod* 2 " also used Microsoft Word 2010 and the Microsoft equation 3.0 for written the math equations.

The method for reading a page which has a block will be according to the following direction as in figure 1.

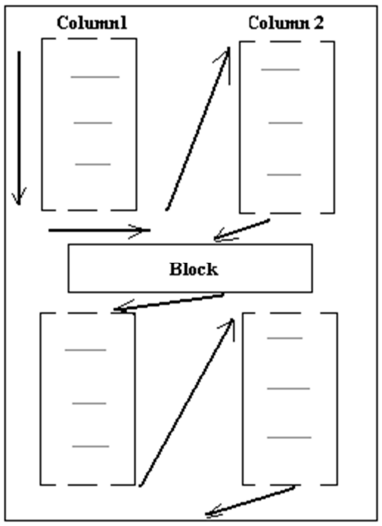

Figure 1. *How reading pages with two columns and horizontal block.*

Acknowledgements

The author express their gratitude to Prof. Abdulla Y Al Hawaj, President of Ahlia University for all the support.

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