

Contribution to the diagnostics of university students' knowledge and competencies in linear algebra

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Abstract: We study knowledge and competencies of Brazilian university students after an one year course in linear algebra, concerning various aspects: the global landscape of linear algebra as seen by students, how they do cope with modelization problems through linear algebra, what are their knowledge and competencies about the duality of representation equations/parametrization of subspaces of \mathbb{R}^n , their ability in calculations for solving linear equations and their understanding of the symbolic algebra used in linear algebra. The results are coherent with such previous studies, which underlined that the learning of linear algebra by students is generally poor after a one year course.

Keywords: Mathematics Education, Linear Algebra, Teaching and Learning at University Level, Diagnostic of Knowledge

1. Introduction

From more than twenty years ago, a lot of studies in didactics have been devoted to the issue of the teaching and the learning of linear algebra at the university level.

In France, the pioneer work of A. Robert and J. Robinet (1989) showed that the learning of linear algebra by students was bad for a majority of them. Their study concerned three universities, and its conclusions were severe: "Less than one quarter of students can use the notions of image and kernel of a linear map, less than one half of them is able to solve a 4x4 simple linear system [...] A third of students is not able to compute the matrix of a linear map, if the space is not \mathbb{R}^2 or \mathbb{R}^3 [...] For a majority of students, linear algebra is only a list of very abstract notions that they are unable to represent and to deal with; moreover, they are submerged under a flow of new words, new symbols, new definitions and new theorems."

The works by J. L. Dorier (1990, 1991) about the teaching and the learning of linear algebra during an one year course at university underlined the "obstacle of formalism": notions of linear algebra are of type "formalizing, unifying and generalizing" (FUG), and this specificity is not taken into account in the classical teaching, which goes in an opposite direction to the historical development of linear algebra. This mathematical domain

appears actually as an abstract formalism which slowly unified several previous fields:

- the linearity, with linear equations, algebraic curves, determinants' theory, quadratic forms, linear transformations, matrices;
- the calculations with coordinates, transition to the vector calculus and the space \mathbb{R}^n ;
- the vector calculus, stemmed from geometry and in the goal to formalize it;
- the final form of linear algebra, first in \mathbb{R}^n , and then in axiomatic form, which shall be accepted by mathematicians only when the infinite dimension in analysis shall be inevitable.

R. Ousman (1998) gave also a contribution to the evaluation of performances of French students in linear algebra, and studied their concept of "inclusive dependence", that is the idea due to Euler that a linear equation depends from others if its solutions contain the ones of the others, without reference to linear combinations. See Dorier (2006) for the historical development of this question from Euler until Frobenius.

In the book coordinated by J. L. Dorier (1997), French authors developed this understanding of linear algebra as FUG theory. They proposed to use what they called "the

meta lever" (meta discourse of teachers, meta activities for students, that is epistemological reflexions and activities about the role and the nature of linear algebra). They analyzed other student productions, in the context of an experimental teaching of linear algebra (Rogalski 1991).

In the same book, G. Harel, J. Hillel and A. Sierpinska presented some aspects of their researches on linear algebra, in particular about different forms of "thinking" in this domain, in particular the difficulty for students to pass from "practice" points of view to theoretical ones. See (Hillel, 2000), (Harel, 2000), (Sierpinska, 2000), (Harel, 1987), (Harel, 1989), (Hillel and Sierspinska, 1994) and (Sierpinska, Nnadozie, & Oktac, 2002).

M. Alves Dias and M. Artigue (1995) announced a research program about "flexibility" among several settings, registers of representation or viewpoints in linear algebra. In her thesis, M. Alves Dias (1998) studied the change of viewpoints between two types of representations of subspace: by equations or by a parametrization. In particular, she showed that classical textbooks of linear algebra contain few developments on this subject. She studied also productions of French and Brazilian students about the duality equations/parameters, and showed that, in the two countries, performances of students were poor.

In the present paper, we study productions of Brazilian students at the end of their first year course in linear algebra. Our goal is to give a survey of some aspects of knowledge and performances of students. Because of some constraints in the possibility for students of taking our test, we keep only seven themes: finding a matrix for a linear map, awareness of the duality equations/parameters for representing subspaces, ability in modeling problems through linear algebra, understanding the use of symbolic algebra, possibility of initiative in solving problems in linear algebra, competencies in calculations with linear equations; more globally, we study the general landscape that the students have about linear algebra.

So our paper gives contributions to the researches about diagnostics of students' learning of linear algebra. In conclusion of this study, we will compare its results with the diagnostics of the previous searchers with students in France in 1987 and in Brazil in 1998.

2. Description of the Student Population via Their Recruitment and Their Course

2.1. General Information

The population under study was formed by third year students of a private university in São Paulo, taking their first course in linear algebra. They were from 20 to 35 old. In general, this kind of universities recruits students who want to become mathematics school teachers, and frequently can only study in evening classes. During the day, all of them teach mathematics or work in another

activity. The mathematical course is less ambitious than in public universities, where the students are interested by high level mathematics. Some characteristics of this particular three years course are: emphasis in mathematical analysis, few proofs and reasonings, little formalism, little abstraction, little abstract algebra. A full description of this course is given in Annex 1.

2.2. The Third Year Course in Linear Algebra

We give a description of the discipline Linear Algebra, which is seen in the 3rd year and has six chapters.

The first chapter treated linear equations. The first objective was to familiarize students with the modeling of solutions to problems via linear systems. The second objective was to discuss the resolution methods for linear systems, with an emphasis in Gauss elimination method. The third one was to confront the students with situations in which the set of solutions is not finite. The main function of this part was to be the operational support for the discipline: many problems in linear algebra come down to solving linear systems.

The second chapter of the course studied the structure of vector spaces. Beyond the classical spaces \mathbb{R}^n , the matrix spaces with real entries and the polynomial spaces with real coefficients were presented. After the introduction of the notion of subspace, the following themes were studied: intersection and sum of subspaces, linear combination of vectors, generators of subspaces, linear independence, basis, dimension, and the formula about the dimension of the sum of two subspaces.

The third chapter was concerned by linear maps between vector spaces. The emphasis was put on the maps from \mathbb{R}^n to \mathbb{R}^m . The notions of kernel and image of a linear map and its relations with the notion of injective and surjective functions were explored. The linear structure of the space of linear maps was taught, and the composition of maps and the invertible operators were studied.

In the fourth chapter the notion of the matrix associated to a linear map related to a couple of basis was introduced. The isomorphism of algebras between the matrices and the linear operators was given (with the correspondence of inverses) and the notion of similar matrices was presented.

The fifth chapter of the course studied the diagonalization problem of linear operators and matrices. The eigenvalues and the eigenvectors associated to an operator and a matrix were studied.

Finally, the sixth chapter treated the notion of inner product in vector spaces. The notions of orthogonal subspaces, orthonormal basis and orthogonal matrices were introduced.

The construction of this Linear Algebra program goes in the direction of this citation: "One can distinguish roughly two main traditions in linear algebra teaching (the teaching of linear algebra): one focuses on the study of formal vector spaces while the other proposes a more analytical approach based on the study of \mathbb{R}^n and matrix calculus. Between these two orientations, there exists a continuum of teaching

designs, in which each pole is more or less dominant.” (Dorier, J.L. Teaching Linear Algebra at University, ICM 2002 · Vol. III · 1–3, p. 875.)

At first glance, this one year program is a very ambitious course (it corresponds to a two years course in France), and the practice has shown that there is a time problem for the teaching-learning process. Also, partly as a consequence of the preceding comment, students focus mainly on the procedures to insure their success to the examination instead of a real understanding of the subject. Probably these two points had a big influence on the results to the test, and are to be taken into account for understanding the results. In fact, the test was taken when only the four first chapters of the Linear Algebra course were being taught, and only these chapters were concerned by the topics of the test.

3. The Test, Conditions of Taking, Analysis a Priori

3.1. The Test and its Conditions of Taking

The group of students was composed of two classes (on four in this university). There were a total of 56 students in these groups, and 36 took the test, but not all the test. In fact, we wanted to study different versions of some questions, and so we had in this case less students' papers to study. Moreover, it was impossible to study all the points that we wanted to see in a classroom test which would last 100 minutes only. So some questions were given for a work at home, in particular when students need abstract reflexion and/or more time to solve it. But in this case, we had only 14 students which took the home questions of the test. Moreover, some students' papers were anonymous, and some not, so it is possible to find relations between classroom questions and home questions for seven of them only. At the end, our sample was on certain points limited, and the conclusions must to be taken with caution.

The test was taking just after the 4 first chapters of the course, and no item was concerned by chapters 5 and 6.

Four questions were asked in classroom test: (1) a modelization of an economic situation by linear equations (with 2 versions); (2) the abstract algebra symbolism about associativity and the neutral element; (3) the duality equations/parametrization for a subspace; (4) the dimension of $E+F$ for some subspaces E and F , with three versions, following the nature of the given hint.

The four questions for the home-test were: an open question about what students consider important results in linear algebra; two questions about magic or special matrices for testing ability to model situations in linear algebra; one question on the determination of the matrix of the orthogonal projection on a line in the geometric space (see Annex 2 for the complete test).

3.2. Analysis a Priori of the Goals of Questions of the Test

3.2.1. Classroom-Test

3.2.1.1. Question 1

This question asked for the modeling and the resolution of a concrete problem whose data were presented in a table. The objectives were to identify the following students' competencies:

- the capacity of modeling of a concrete problem ;
- the ability in reading data presented in a tabular form ; two forms of tables were presented to students, one being the transpose of the other, in order to verify if it is a variable of success or error ;
- the competency in solving a 3 by 3 system of linear equations, by the Gauss elimination method or by the Cramer's rule ;
- the capacity for interpreting the results obtained.

3.2.1.2. Question 2

The objectives of the question were to investigate how much the students knew about the notion of associativity of an operation, if they were able to extend the associativity from real numbers to the space \mathbb{R}^4 and if they could recognize the neutral element for the addition of vectors in \mathbb{R}^4 . We anticipated that some students might confuse associativity with commutativity, due to the low development of abstract algebra in the three years course.

3.2.1.3. Question 3

A first objective was to see if students were able to give an answer to question a) about the duality equations/parametrization for the definition of a subspace (see Annex 2). This theorem was not explicitly given in the course, but the practice of Gauss method gives it implicitly – and in all particular numerical cases studied in exercises. But for this it is necessary to understand that the practice of Gauss gives a theoretical general result, and we wanted to evaluate the capacity of students to make this step.

A second objective was to see if students could solve directly question b) with the use of the answer to question a), without using again Gauss method as a routine (in fact the subspace which was proposed was very simple and could be directly parameterized).

3.2.1.4. Question 4

The goal was to see if the students could determine the dimension of the sum $E+F$ of two subspaces of \mathbb{R}^4 given by their equations, with three versions, depending of the hint given in the text:

- i) in version V1 no hint was given, and we wanted to see if only the algorithmic procedure using the formula $\dim(E+F) = \dim(E) + \dim(F) - \dim(E \cap F)$ would appear, or if other methods were available to students ;
- ii) in version V2, we gave a hint which should direct students to an easy and intuitive but unusual method: add to the hyperplane F a vector of E not in F , which nevertheless needed a use of inclusion on subspaces and inequalities on dimensions, not totally evident ;

iii) in version V3, we directed the students to the classical method using the formula for $\dim(E+F)$.

These versions were given respectively to 15, 9 and 12 students.

3.2.2. Home-Test

3.2.2.1. Question 1

This question was: "Write two results that in your opinion are the most important in this Linear Algebra Course until now". The term "until now" is motivated by the fact that the test was before the two last parts of the course. We expected to find in the answers some items of a list of some basic results, notions or methods, as: notions of dimension, of basis, the theorem of basis completion, the isomorphism with \mathbb{R}^n in finite dimension, the general linear equation $T(x) = y$, the Gauss' method, the formula for the dimension of $E+F$, the structure of $L(\mathbb{R}^n, \mathbb{R}^m)$, etc. And we hoped to see, from these answers, what landscape students have of linear algebra.

3.2.2.2. Questions 2 and 3

The first goal of these similar questions was to see if a type of modelization by linear algebra was available to students. Precisely, the modeling consisted in representing the studied set of matrices by a subspace of \mathbb{R}^{16} or \mathbb{R}^9 defined by 10 or 3 equations. The second goal was to see if this modelization, if realized by students, could help them to solve the problem by using an inequality on the dimension (as in classroom question 3) to compare with the teacher's affirmation.

3.2.2.3. Question 4

The objectives were to see how the students could recognize a linear operator given in a geometric formulation (different from the classical algebraic ones), if they could relate the given definition to the other formal notions of linear operators: matrix, formulas, basis of kernel, and if they could see some geometric properties related to the linear operator which is defined.

4. The Global Analysis of the Results

4.1. Results to Some Questions

We will here focus on some interesting facts. The lector will find in the tables in Annex 3 the details of the results of each question and a more global analysis by themes in § IV.B.

The first result is the medium success to the modelization in the "industrial" problem by linear equations: 44% of good answer, and a little more for a good interpretation of the calculations (even false): 53%.

The second is the poor score of success to the question on symbolic algebra: less of 6% of students could explain what associativity is.

Then, only 11% of students gave the good answer $n-k$ to the dimension of solutions of k independent homogeneous equations, while 69% gave no answer.

Another characteristic students' behavior in front of the problem without hint is visible in classroom question 4: there is only one complete correct solution, and no student tried another method than the formula for $\dim(E+F)$.

For the modelization of an object (a set of matrices) by a subspace of \mathbb{R}^n defined by m equations, the success was not very high: 36%.

In fine, in spite of the fact that students had much time for solving this question at home, 50% of them gave no answer to the problem of determining the matrix of the orthogonal projection on a line in \mathbb{R}^3 .

A more synthetic analysis is given in the following section.

4.2. Analysis of Results by Themes

4.2.1. Failure of Knowledge about the Duality "Number of Equations/Dimension" for a Subspace of \mathbb{R}^n : The Total Unsuccessfulness to the Classroom Question 3, Many False Answers in Home Questions 2 and 3

The results to classroom question 3 were dramatically bad. Only 4 students among 36 gave the good answer $n-k$ for the dimension of a subspace of \mathbb{R}^n defined by k independent homogeneous linear equations (see the tables in Annex 3), and 25/36 gave no answer. Among the 4 students who gave the good answer, only one tried (without success) to use this theoretical result for the proposed subspace, whose equations were given by a triangular system.

In home questions 2 and 3, 5/14 students tried to give a modelization by a subspace of \mathbb{R}^{16} defined by 10 equations [resp. \mathbb{R}^9 defined by 3 equations], but no one of them succeeded to give the conclusion about the possible dimension of the space of solutions.

This diagnostic is worrying, because the duality "number of equations/dimension" was essential. It highlights, with some easy procedures in the case of linear algebra, the general duality "number of degrees of freedom/ number of constraints". This duality is important in analytical geometry (in \mathbb{R}^3 , 1 equation \leftrightarrow 2 parameters – a surface, 2 equations \leftrightarrow 1 parameter – a curve, 3 equations \leftrightarrow 0 parameter – a point), in differential geometry, in mechanics, in technology, etc.

In many textbooks and courses this duality is absent or only implicit (see Alves Dias (1998), Alves Dias and Rogalski (2011)). In theory, it follows (in a direction) from the Gauss' method applied to a system of linear equations, which gives a basis of the space of solutions, and the basis gives then the dimension.

But the students couldn't find by themselves the direct road "independent equations \rightarrow dimension" from the procedure "Gauss' method \rightarrow basis \rightarrow dimension", because Gauss' method is for them a procedure whose goal is numerical resolution and not the proof of a theoretical theorem; it has even rarely the goal of description of a set, because the term "parametrization" of a subspace is not often present in textbooks or courses in linear algebra. And to find a basis in this situation requires explicit calculations, which are not useful for the theoretical result. The

possibility to use Gauss' method without explicit calculations is for students a cause of trouble.

So, the results to this question exemplify the fact that if a theoretical goal for Gauss' method is not explicitly given to students (accompanied with "meta" discourse about it), they cannot find and use a theorem which is the abstract version of their procedures. This point was in the conclusion of (Alves Dias (1998)).

Of course, the characteristic of the three years course, cited in § II.A, can explain in part the difficulty of students to "pass from practice points of view to theoretical ones". This passage is a great challenge in linear algebra, for having access to the abstract point of view of its notions. It is probably the case for all knowledge which is FUG. For this type of difficulty in linear algebra, see Sierpiska (2000).

4.2.2. The Possibility to Take Initiatives in Solving Problems

Two occurrences in the test allow seeing this possibility via student productions.

4.2.2.1. What do Students do when they Have No Hint for a Problem? They Try to Use a Formula: Classroom Question 4 Version V1

Classroom question 4 asked to calculate the dimension of $E+F$ for two subspaces E , defined by two independent equations, and F , defined by one equation. This question had three versions, and the first of them (V1) gave no indications. The objective was to see which procedures students were able to use for solving such a question without hint. Three were possible:

- to determine a basis B of E , a basis B' of F , and calculate the rank of $B \cup B'$;
- to calculate the dimension of $E \cap F$ and to use the classical formula $\dim(E+F) = \dim(E) + \dim(F) - \dim(E \cap F)$;
- to add to F a vector of E not belonging to F .

In fact, we did not think that the third procedure could have been found, and this anticipation was of course verified. But none of the 8 student who answered tried to use the first method, and 7 did not answer the question. So only 1/15 students succeed to find the good response (because there was many errors of calculations in the Gauss' method for finding the dimensions). But the important result, we think, is that no student was able to plane an answer without the formula which gives directly the dimension.

4.2.2.2. The Incapacity to Use an Unusual Method, Even when it is Proposed by the Text of the Problem, and is More Intuitive than other Methods: Classroom Question 4 Version V2

The third possible procedure in the version V1 of classroom question 4 was the procedure proposed to the students in version V2 of this question. It was the simplest of the possible solutions, even without calculation if one guesses the easy solution $(3,1,1,1) \in E \setminus F$. But this simplicity was only for intuition: a hyperplane and a vector

not in it generate the whole space. But the proof requires using inclusions of spaces and inequalities about dimensions linked to these inclusions. If inclusions and inequalities are easy to find, the problem is: how to think to them for solving the question by the proposed method? Probably this difficulty is linked to the lack, in the course, of a structured set of methods of resolution of types of problems, in particular, here, about the question: how to compare subspaces of a space?

The result of the analysis of the students' responses is without ambiguity: among the 9 concerned students, there were 8 without answer, and the only student who responds cannot give a vector in E and not in F .

4.2.3. Mean Level Competencies of Students in Calculations: Classroom Questions 1, 4 (Versions V1 and V3), Home Question 3, etc.

In many questions of the test, it was necessary to solve a simple system of linear equations. In the best case, only nearly 2/3 of students could make such a calculation without error, and sometimes only half of them succeed to solve the system. It is a poor competency, because many problems in linear algebra use systems of linear equations.

4.2.4. Not Good Results in Task of Modelization of Problems in Linear Algebra: Classroom Question 1, Home Questions 2 and 3

In the test, there were three problems of this sort.

The first problem was in classroom question 1. In fact, the problem was only to make a correct modelization by linear equations. In version 1+3, 16/23 were correct. In version 2, 5/9 were correct. Due to the smallness of the populations, the difference is not significative (make change one student of category in version 2, from error to correct, and in the reverse direction in version 1+3: 6/9 is similar to 15/23). So the presentation seems to have no effect. So we present in Annex 3 only the global results to this question (versions 1+2+3), and we see a rate of success of 16/36, that is only 44%.

In the home questions 2 and 3, the problem was to find a modelization of the types of studied matrices as subspaces of R^{16} [resp. R^9] defined by 10 [resp. 3] equations. In each case, 5/14 students succeed to make this modelization. It is not a bad score, because there was neither a particular teaching in the course about this use of linear algebra, nor a "meta" discourse about the fact that some concepts of linear algebra come from "concrete problems" about linear equations (or about problems of functional analysis, which come from applied mathematics or from physics).

We think that problems of modelization in linear algebra are very useful for the motivation of some basic questions in linear algebra: the linear combinations and their property of "transitivity", the concept of rank and of linear independence, the choice of a basis, the equivalence of injectivity and surjectivity in some cases etc. And some sophisticated modelizations may be good tools for the understanding by students that a theoretical detour is often a good way to solve concrete problems. For students'

motivation, this may be very useful, if some “meta” discourse is made about this point, and if some “meta” situations to make them think about how these theoretical detours are used (see Rogalski, 2011).

4.2.5. The Landscape of Linear Algebra which Appears from Students, through their Responses to Home Question 1, is Very Scattered

The question was: “Write two results that in your opinion are the most important in this Linear Algebra Course until now”.

What is for us a structuration of linear algebra, which can come from students’ course, would be organized around, for example, the following results:

- (1) The Gauss procedure gives a basis and the dimension of the space of solutions of a system of linear homogeneous equations.
- (2) Vectors u_1, \dots, u_k which are linearly independent and generators form a basis.
- (3) With a basis, every n -dimensional linear space is isomorphic to \mathbb{R}^n .
- (4) The theorem of basis completion.
- (5) The rank theorem for a linear map $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$: $\dim(\text{Ker}(T)) + \dim(\text{Im}(T)) = n$.
- (6) The equivalence between injectivity and surjectivity in some cases.
- (7) The formula for the dimension of $E+F$: $\dim(E+F) = \dim(E) + \dim(F) - \dim(E \cap F)$.

The result (7) is less important, but we thought that it can be given by students because they like “beautiful and close formulas”.

So we anticipated that most of students’ answers would be in this list.

It is not the case! The 12 (among 14) students who answered this question do not give exactly “results”, but evoke concepts or methods in linear algebra. Moreover, they are very scattered in 12 categories (see the Annex 3), and none of them is a theorem. Of course, as we anticipated, 50% of answering students cite Gauss’ method, but they do not say anything about its usefulness; in the context of the answers to all the test, it is possible to think that the only role of this method is solving linear equations (see the comments on classroom question 3). The other notions cited more than one time are: matrices (3 among 12), linear maps (3 among 12), algebraic and logical reasoning used in linear algebra (2 among 12), and dimension (2 among 12).

The answers about reasoning probably reflects the gap that students feel between linear algebra, and analytic geometry and calculus that are taught in the university two first years, where there are essentially calculations, with not many algebraic reasoning and a poor use of logical reasoning.

It seems that students are focused in their answers on concepts or methods frequently used or on some aspects which are difficult for them, but it appears no links between these concepts or methods. For example, if some answers cite dimension, basis or generators, none of them give links between these concepts. And the answers which cite

“variety of tools” or “variety of registers” say nothing about relationships between them.

One can think that this split aspect of the landscape of linear algebra for students is probably an obstacle for solving problems which are not direct applications of one theorem or direct uses of one algorithm, and probably only the presence of intermediate questions making the links that students cannot make by themselves enables success to more difficult problems.

It is also interesting and disturbing to see that there is no answer concerning the applications of linear algebra, in other mathematics or in other domains in sciences or economy, although this question is sometimes cited in third place in the answer of students.

4.2.6. The Lack of Understanding of the Symbolic Algebra Used in Linear Algebra

The results were not very good: only 2/36 students could answer correctly the 3 parts of the question. The surprise was that 7 students (more than our prevision) confused “associativity” with “commutativity”, and, even in this situation, some of them proved part b) only for particular cases. It was supposed that in this stage of the course the students had clarified the difference between an “associative law” and a “commutative law”, and it is not the case. One supposition for this confusion between properties, other than that they in fact didn’t know the notion of associativity, was that for commutativity we use only two elements to formulate and prove the property, while for the associativity, we need three vectors, and in the definition of the addition in the text of the exercise we presented only two general vectors. They could not imagine a third vector!

The answers reflect the student difficulties with symbolic notations and abstract notions, and suggest that the “obstacles of formalism” in this case are bigger than in the situations studied by Dorier (1990).

Moreover it is strange to note that the 2 students who gave a good answer to this question on associativity have very bad papers: wrong answer or no answer to almost all questions. It seems that there are two distinct worlds for students: the one of calculations, and a formal-symbolic one, with no relationship between them.

4.2.7. Competencies in the Determination of the Matrix of a Linear Map of Geometric Origin

Only half the students answered completely this question, and most of them preferred to use a numerical and symbolic approach of the definition of the operator to get the solution instead of the definition and properties of the inner product. The correct answers came from the fact that the students first have found the formula for the calculation of the function $P_u(x,y,z)$ and then they construct the matrix. This behavior seems to show that they only knew how to build a matrix associated to a linear map when the map is given in an algebraic presentation, in the same way the construction was presented in the course; and also, it seems to show a lack of maturity and experience in working with this kind of situation.

4.3. Analysis of Complete Papers from Some Students

In this part, we analyze the complete answers of the seven identified students who have taken the two part of the test, in classroom and at home, in the goal to have, with some individual landscapes of the competencies in linear algebra, a first idea of the coherence of the results with the meaning of linear algebra for students.

4.3.1. Student 1-18 (V3)

First, the competencies in calculation are very bad: there is no answer to questions with calculations, except to classroom questions 1 and 3, where the calculations are wrong.

Algebraic symbolism is not understood by this student: associativity of addition has the meaning of “to one vector, we associate another vector”. Only the vector 0 is correct.

There is a no answer to any question which uses modelization.

There is also no answer to classroom question 3 about the relation between the number of independent equations and the dimension.

In spite of the fact that the question was to be solved at home, with time, there is no answer to the determination of the matrix of the orthogonal projection.

In fine, this student understands the “most important results in this linear algebra course” as “the most important results on my own capacities”: “to be prepared for teaching the matter with its precise content” and “to have success in using content when it is necessary”!

In summary, it is a bad paper in all the themes that we have studied.

4.3.2. Student 6-21 (V3)

The only good calculation is in classroom question 3.

For the associativity, this student does not understand that it concerns 3 vectors, he writes: “ $u+(u+v)$ ”... and does not give an answer to the value of this expression; and he also does not the question about the null vector. See also below his answer below to classroom question 3.

The modelization in classroom question 1 is wrong; the two other questions with modelization are not answered.

There is a surprising meaningless answer to classroom question 3 about the dimension of a subspace of \mathbb{R}^n defined by k linear homogeneous independent equations: “ $\mathbb{R}^n \rightarrow \mathbb{R} - n = \text{number of dimensions}$ ”!

As for the precedent student, the question about the matrix of the orthogonal projection is not answered. The student only writes: “it is a linear transformation because it permits to make a linear transformation of U into V ”.

For important results in linear algebra course he cited Gauss method for systems, which allows to find “the values of the unknowns of a problem”, and a formulation without meaning about linear transformation of “a given dimension of \mathbb{R}^3 for \mathbb{R}^2 or \mathbb{R}^2 for \mathbb{R}^3 ”, with a sentence which means that linear algebra is a good exercise for the mind and a pastime!

So we have the same global comments as about the previous student.

4.3.3. Student 9-27 (V3)

Most of the calculations are good, except for classroom question 3 (it was very easy!), and no answer to classroom question 4.

She confuses associativity and commutativity, and gives no answer to the question about vector 0.

The three asked modelizations have good answers (but she is not able to use it at home in questions 2 and 3 about matrices).

She gives no answer to the question about the duality: number of equations/dimension.

There is a good answer to questions a), b), c) about the matrix (no answer for d) nor e).

She gives two interesting answers for important results in linear algebra: “a critical revision of algebra, with accent about logic-algebraic reasoning”, and “a diversification of types of used registers of representation”. But what is exactly her judgment on these aspects?

In summary, this student has good practice competencies, good capacities in modelization, but symbolic algebra is difficult for her, and the duality equations/dimension is not a theorem for her.

4.3.4. Student 10-34 (V2)

This student makes good calculations, except in classroom question 4: he gives good parametrization of spaces E and F but by the same names for the parameters z and t , and this classical error makes the final result wrong.

He confuses associativity and commutativity, gives a good response to the vector 0, but with a proof in a particular case.

The three modelizations are good.

He gives the good answer ($n-k$) to part a) of classroom question 3, but without justification, and does not use it in question b).

The question about the matrix of the projection is entirely solved.

For the important results in linear algebra, his first answer evokes the understanding of subspaces, basis, dimensions, basis of $\text{Ker}(T)$, basis and dimension of $\text{Im}(T)$; probably, these points are cited as difficult. The second cited notion is linear transformation, with matrices.

This is an example of a student which has good competencies and in part good understanding on linear algebra.

4.3.5. Student 12-10 (V1)

This student makes good calculations with Gauss' method (except one error in the classroom question 1).

She gives no answer to the question about associativity.

If the modelization in classroom question 1 is good, she does not answer the two questions about magic and special matrices.

She gives no answer to the question about the dimension of a subspace defined by k independent equations.

In spite of the fact that she had time at home, she gives no answer to the question about the matrix of the projection onto a line.

As important results in linear algebra, she cites only Gauss' method.

One can think that for this student linear algebra consists only of procedures.

4.3.6. Student 13-13 (V1)

This student has medium results in calculation.

He confuses associativity with commutativity.

He makes good modelization to the three questions about this theme.

He gives the good answer $n-k$ to classroom question 3a), without justification, but he does not use it to solve part b) of the question.

The matrix of the projection in home question 4 is good, he find it by the images of the basis vectors, in spite he has the algebraic formula; he proves linearity from the formula and not from the geometric definition; all the answers are correct.

The results in linear algebra which are presented as important to learn: "associate matrix and vectors", "think to spaces of finite dimension", groups and fields.

This student has good competencies, and probably has the potentiality to understand the role of symbolic algebra in linear algebra.

4.3.7. Student 14-8 (V1)

This student has good capacities in calculations.

About associativity, she asserts and proves that $\alpha(u+v) = \alpha(v+u)$.

She makes good modelization in first classroom question and about magic or special matrices, and in this case she uses the relation $\dim(\text{magic matrices}) = 6$ for her reasoning, but without prove.

In contrast, she does not give the dimension of a subspace defined by k independent equations.

She solves completely the question about the matrix of the projection on a line.

At last, she cites dimension and subspaces as most important in linear algebra. But she adds: "use an algebraic language in science for expressing a relation between quantities and modeling a situation-problem".

In summary it is a good paper in procedure and with good understanding of the stake of linear algebra.

The balance for these seven students is not negative: 5 of them have good practice competencies, 3 of them are probably able to understand theoretical aspects of linear algebra. But when one sees the global results of Annex 3, one can think that this sample has a bias, probably because these are students which have put their name of their two papers (in classroom and at home), which is perhaps due to their self-confidence.

5. Conclusions from the Students' Productions

First, we recall that we said in the presentation about the conditions of the taking of the test: some versions have

been taken by few students, in the classroom activity, and only 14 students took the home test. So the conclusions are to be considered with caution. But if we look the landscape of some of the seven themes what we have tried to evaluate, some possible conclusions appear.

First, the mean students' results in calculations seem a little bit worrying. But, if we compare with the results in Robert & Robinet (1989), which have been obtained in 1987, Brazilian students' results seem a little better. Perhaps this difference results only from the fact that at this date the teaching of linear algebra in France was "à la Bourbaki": abstract algebra first, linear equations were only a practical subject which appeared as a not very interesting application. This is not the case in Brazilian teaching.

Secondly, the very poor results on the question about associativity of addition in \mathbb{R}^4 show that students have no access to the symbolic and algebraic aspect of linear algebra, which would probably be an obstacle to conceptualization: this is the "obstacle of formalism" detected by Dorier. But this comparison is not right, because Dorier's study concerns students in first year, and they have seen little formalism at secondary school. In our study, students had a two years course in university. So one can think that also little formalism is developed in this course. Thus it can be a good objective to improve the course in this direction.

The scattered landscape of linear algebra which appears by students is also in coherence with what said Robert and Robinet (1989): "[students] are submerged under a flow of new words, new symbols, new definitions and new theorems." In the teaching of a domain which is "formalizing, unifying and generalizing" (FUG), it is probably necessary to underline the links between some different aspects and the changes of point of view in the mathematical organization of the knowledge and the methods. So a global structure of linear algebra is to be given to students.

Finally, in the important field of modelization of situations by linear algebra, the results of our study are encouraging. So it would be necessary to make more research to confirm this point.

In summary, our global results are coherent with previous results in France in 1987, in France and Brazil in 1998. This means that the problem of didactics to introduce substantial modifications in teaching linear algebra in order to improve learning remains an open problem, despite the existence of concrete propositions and experiments (see, for example, Dorier (1997), Harel (1989, 2000), Rogalski (1991, 2011)).

Annex 1. The Global Course of Students during the Three Years

The course is a Mathematics teacher formation night course in a private university in São Paulo.

The Disciplines

1st. Year:

- Introduction to calculus: functions, composite functions, inverse functions, elementary real functions, limits, derivatives.
- Introduction to algebra: notable products, polynomials, complex numbers.
- Fundaments of mathematics: equations, inequations of the 1st and 2nd degrees.
- Fundaments of education.
- Methodology and practice of fundamental teaching (7-15 old).

2nd. Year:

- Differential and integral calculus: applications of derivatives, integrals.
- Plane geometry.
- Analytical geometry: vectors in dimension 3, lines and planes.
- Statistics.
- Structure and functioning of the medium (16-18 old) and fundamental Teaching

3rd. Year:

- Mathematics analysis: numerical series, function series, ordinary differential equations.
- Advanced calculus: functions of two variables, partial derivatives, gradient.
- Linear algebra : linear equation systems, Gauss elimination, real vector spaces, subspaces, basis and dimension, linear transformations, kernel and image of linear transformations, inner products (see details in § II.B).
- Didactics.
- Methodology and practice of medium teaching
- Sign language.

Annex 2. The Test and its Three Versions

Classroom Questions, Version 1

Question 1

A metallurgy industry has 3 machines A, B and C which produce nails, screws and screw nuts simultaneously. The following table shows the production of each machine in one hour.

	Machine A	Machine B	Machine C
Nails	2.000	3.000	1.000
Screws	1.000	2.000	1.000
Screw nuts	500	1.000	1.000

The industry has received an order for 15.000 nails, 9.000 screws and 5.000 screw nuts.

How long each machine must be used for producing exactly the quantities of this asking?

Question 2

In \mathbb{R}^4 the addition of two vectors $u = (u_1, u_2, u_3, u_4)$ and $v = (v_1, v_2, v_3, v_4)$ is usually defined by

$$u + v = (u_1 + v_1, u_2 + v_2, u_3 + v_3, u_4 + v_4)$$

- Explain what you understand as the associative property for this operation in \mathbb{R}^4 .
- Assuming that the addition of real numbers is associative, prove that the addition of vectors in \mathbb{R}^4 is also associative.
- Which is the neutral element of the addition of vectors in \mathbb{R}^4 ? Why?

Question 3

- A vector subspace W of \mathbb{R}^n is defined by k linear equations with $k < n$ and which are homogeneous and independent. Which is the dimension of W ? Justify.
- Determine the dimension of subspace W of \mathbb{R}^5 defined by

$$W = \{ (x, y, z, t, w) \in \mathbb{R}^5 \mid x + y + z + t + w = 0; y - z - 2t - 3w = 0 \text{ and } z + 2t - w = 0 \}$$

Question 4

Let E and F be subspaces of \mathbb{R}^4 defined by

$$E = \{ (x, y, z, t) \in \mathbb{R}^4 \mid x - 3z = 0 \text{ and } y - t = 0 \} \text{ and } F = \{ (x, y, z, t) \in \mathbb{R}^4 \mid x + y - z = 0 \}.$$

Determine the dimension of the subspace $E+F$.

Classroom questions, version 2

Question 1

A metallurgy industry has 3 machines A, B and C which produce nails, screws and screw nuts simultaneously. The following table shows the production of each machine in one hour.

	Nails	Screws	Screw nuts
Machine A	2.000	1.000	500
Machine B	3.000	2.000	1.000
Machine C	1.000	1.000	1.000

The industry has received an order for 15.000 nails, 9.000 screws and 5.000 screw nuts.

How long each machine must be used for producing exactly the quantities of this asking?

Question 2

The same version.

Question 3

- A vector subspace W of \mathbb{R}^n is defined by k linear equations with $k < n$ and which are homogeneous and independent. Which is the dimension of W ? Justify.
- Determine the dimension of subspace W of \mathbb{R}^5 defined by

$$W = \{ (x, y, z, t, w) \in \mathbb{R}^5 \mid x + y + z + t + w = 0; z - t + 2w = 0 \text{ and } t - 3w = 0 \}$$

Question 4

Let E and F be subspaces of \mathbb{R}^4 defined by

$$E = \{ (x,y,z,t) \in \mathbb{R}^4 \mid x - 3z = 0 \text{ and } y - t = 0 \} \text{ and}$$

$$F = \{ (x,y,z,t) \in \mathbb{R}^4 \mid x + y - z = 0 \}.$$

- Find a vector of E that doesn't belong to F.
- Determine the dimension of the subspace $E+F$.

Classroom questions, version 3	
Question 1	The same version as version 1.
Question 2	The same version.
Question 3	The same as version 2.
Question 4	Let E and F be subspaces of \mathbb{R}^4 defined by $E = \{ (x,y,z,t) \in \mathbb{R}^4 \mid x - 3z = 0 \text{ and } y - t = 0 \}$ and $F = \{ (x,y,z,t) \in \mathbb{R}^4 \mid x + y - z = 0 \}$. <ol style="list-style-type: none"> Determine the dimension of the subspace $E \cap F$. Determine the dimension of the subspace $E+F$.

Home Questions**Question 1**

Write two results that in your opinion are the most important in this Linear Algebra Course until now.

Question 2

A 4x4 matrix is called a *magic matrix* if it verifies the following conditions:

- The sum of the elements in each row is zero.
 - The sum of the elements in each column is zero.
 - The sum of the elements in each diagonal is zero.
- Give an example of a magic matrix different from the null matrix.
 - If the teacher affirms that any magic matrix is a linear

combination of at least 5 particular magic matrixes, do you think that this is possible or impossible? Explain your answer.

Question 3

A 3x3 matrix is called a special matrix if the sum of the elements in each row is zero.

- Give an example of a special matrix different from the null matrix.
- If the teacher affirms that any special matrix is a linear combination of at least 5 particular magic matrixes, do you think that this is possible or impossible? Explain your answer.

Question 4

Let u be a unitary vector in \mathbb{R}^3 .

We call orthogonal projection in the direction of the vector u the function $P_u : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by

$$P_u(v) = (v \cdot u)u$$

where $v \cdot u$ indicates the inner product of the vectors v and u , that is, if $v = (v_1, v_2, v_3)$ and $u = (u_1, u_2, u_3)$ then $v \cdot u = v_1 \cdot u_1 + v_2 \cdot u_2 + v_3 \cdot u_3$.

- Prove that P_u is a linear operator in \mathbb{R}^3 .

Let $u = (1/3, 2/3, 2/3) \in \mathbb{R}^3$ and let B be the canonical basis of \mathbb{R}^3 .

- Determine the matrix of P_u in relation to the basis B.
- Determine a formula for the calculation of $P_u(x,y,z)$.
- Determine a basis of $\text{Ker}(P_u)$.
- Prove that any vector in $\text{Ker}(P_u)$ is orthogonal to u .

Annex 3. Detailed Analysis of the Results by Questions

Classroom Question 1 Version 1+2+3 (36 papers)

Modelization		Solution of the system		Interpretation		Total success
Correct	24	Correct	16	Correct	19	16
		Wrong or No Answer	8			
Wrong or No Answer	12	Correct		Wrong or No Answer	17	
		Wrong or No Answer	12			

Classroom Question 2 (36 papers)

a)		b)		c)		Total success
Correct	2	Correct	3	Correct	8	2
Other error	14	Other error	8	Error	9	
No answer	13	No answer	14	No answer	18	
		Part. case associativity		Part. case	1	
Confusion with commutativity	7	Part. case commutativity	4			
		Correct comm.	7			

Response a)		b)		Total success
n-k	4	NCalGsCor	13	
		NCGss. Wrg.	14	
k	1	App. a) proof indep.	With Gau.	
Other	6		Without c.	
No answer	25	App. a) Wrong	1	
		No answer	8	

Classroom Question 4 Version 1 (without hint) (15 papers)

Proced. Gauss $B \cup B'$ + rank		basis E	Corr.						Total success.	
			Error							
		basis F	Corr.							
			Error							
		rank	Cor	Err		Nr				
Proced. formule	8	dim(E)	Corr.		7				1	
			Error		1					
		dim(F)	Corr.		7					
			Error		1					
		NR after EF	1							
		dimens. $E \cap F$	Gauss basis		Corr.		1			
					Err.		1			
			Indep. and 4–3		Corr.					
					Err.					
				Final step	Cr	1				
Gl. No resp.	7	Er	4							

a)				b)									
Guess	Parametr.		NA	Again Vers.1	dim(F) = 3			Inclusions			Inequal. dim.		
	Cor.	Wrg.	1		Cor.	Wrg.	NA	Cor.	Wrg.	NA	Cor.	Wrg.	NA
					1								
							Wrong final resp.			1			
Global NA		8											

a)				b)							
Correct		Error	NA	dim(E)			Appl. formula			NA	
Gauss without calc.	Calc.	7		Cor.	Wrg.	NA	Cor.	Wrg.	NA	4	
				2	1	1	1	1	1		
Gl. NA	5										

All the procedures are Gauss basis dimension

Home Question 1 (14 papers)

Student paper num.	NR	Cited results											
		Gauss Solv.	Mat.	Lin. Tranf.	Alg/ Log. Reas.	Dim	Kern linT	Generator	Basis	Vec. Not.	Var. tools	Var. Reg.	Notion of subs
1	x												
2		x	x										
3				x									
4							x	x			x		
5	x												
6		x		x									
7		x								x			
8		x				x							
9					x							x	
10				x					x				
11		x	x										
12		x											
13			x		x								
14						x							x
Freq.		6/12	3/12	3/12	2/12	2/12							

Home Question 2 (14 papers)

a)			b)			
Correct example	Wrong example	No answer	Wrong with arg. or without arg.	Corr. without arg.	No answer	Try mod $\mathbb{R}^{16} + \text{éq.}$
14			6	1	7	5

Home Question 3 (14 papers)

a)			b)			
Correct example	Wrong example	No answer	Err. without arg. or with wrong arg.	Corr. Without arg.	No answer	Try mod $\mathbb{R}^9 + \text{éq.}$
13	1		9		5	5

Home Question 4 (14 papers)

Question	Procedure	Posit. resp. with proc.	Wrong answer	No answer to the question
a)	Numerical with $\sum u_p v_p$		2	1
	Symbolic with $(v \times u)u$			
	Answer more or less "intuitive"	4		
b)	Use of c) and $[(v_1 + 2v_2 + 2v_3)/3]$ 1/3, etc		1	2
	Calculation of the $P_u(e_q)$	4		
c)	(no choice)	3	3	1
d)	Symbolic		1	3
	Numerical with 3 equations	3		
e)	Symbolic		1	3
	Calculation with a basis or with a parametrization	3		
Global No Answer	7			

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