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Estimating the Parameters of (POGE-G) Distribution and Its Application to Egyptian Mortality Rates

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Abstract: In this paper, we consider power odd generalized exponential-Gompertz (POGE-G) distribution which is capable of life tables to calculate death rates (failure). Based on simulated data from the PPOGE-G distribution, we consider the problem of estimation of parameters under classical approaches and Bayesian approaches. In this regard, we obtain maximum likelihood (ML) estimates, maximum product of spacing (MPS) and Bayes estimates under squared error loss function. We also compute 95% asymptotic confidence interval and highest posterior density interval estimates. The Monte Carlo simulation will be conduct to study and compare the performance of the various proposed estimators (simulation study indicates that the performance of MPS estimates is better MLE estimates and the performance of Bayes estimates is also better). Finally, application of a real data from the projections of the future population for the total of the Egyptian Arabic Republic for the period 2017-2052, depending on the book which introduced from the central agency for public mobilization and statistics in Feb (2019) from this application it could be said that this distributions can be applied to mortality rate data set. The present paper can also be extended to design of progressive censoring sampling plan and other censoring schemes can also be considered.

Keywords: Power Odd Generalized Exponential-Gompertez Distribution, Maximum Likelihood Estimation, Maximum Product Spacing, Bayesian Estimation, Metropolis-Hasting Algorithm, Mortality Rates in Egypt

1. Introduction

There are several new families of probability distributions which are proposed by several authors. Such families have great flexibility and generalize many well-known distributions. So several classes have been proposed, in the statistical literature, by adding one or more parameters to generate new distributions. Among this literature exponential Lomax El-Bassiouny *et al.* [10], exponentiated Weibull-Lomax Hassan and Abd-Allah [13], the odd lomax generator Cordeiro *et al.* [8] The generalized odd inverted exponential-G family Chesneau *et al.* [5], The odd log-logistic Lindley-G Alizadeh *et al.* [2] and The Odd Dagum Family of Distributions Afify *et al.* [1]. The exponentiated power lindley distribution by Ashour *et al.* [3] the beta exponentiated power lindley distribution by Pararai *et al.* [14].

Ghitany *et al.* [12] introduced an extension of Lindley distribution by using the power transformation $x = t^{\gamma}$. Hence, it is of interest to know what would be the distribution of similar power transformation of odd generalized exponential Gompertz distribution by using the transformation and the odd generalized exponential Gompertz distribution see El-Damcese *et al.* [11]

The cumulative distribution function (cdf) of the PPOGE-G distribution is given by

$$F(t;\lambda,\alpha,\beta,c,\gamma) = \left\{1 - e^{-\alpha\left[e^{\frac{\lambda}{c}\left(e^{ct^{\gamma}}-1\right)}-1\right]}\right\}^{\beta}, \mathrm{t} > 0, \alpha, \beta, \lambda, \mathrm{c}, \gamma > 0 \tag{1}$$

where β is the shape parameter and $(\lambda,\alpha,\gamma,c)$ are the scale parameters.

The probability density function (pdf) of the POGE-G distribution is given by

$$f(t) = \alpha\beta\lambda\gamma t^{\gamma-1}\, e^{ct^{\gamma}}\, e^{\frac{\lambda}{c}\left(e^{ct^{\gamma}}-1\right)}\, e^{-\alpha\left[e^{\frac{\lambda}{c}\left(e^{ct^{\gamma}}-1\right)}-1\right]} \left\{1 - e^{-\alpha\left[e^{\frac{\lambda}{c}\left(e^{ct^{\gamma}}-1\right)}-1\right]}\right\}^{\beta-1} \tag{2}$$

Figure 1 illustrated the behavior of the pdf of POGE-G distribution at different values of $\beta, \lambda, \alpha, \gamma, c$.
The hazard rate function of POGE-G distribution can be obtained from

$$h(t) = \frac{\alpha\beta\lambda\gamma t^{\gamma-1}\, e^{ct^{\gamma}}\, e^{\frac{\lambda}{c}\left(e^{ct^{\gamma}}-1\right)}\, e^{-\alpha\left[e^{\frac{\lambda}{c}\left(e^{ct^{\gamma}}-1\right)}-1\right]} \left\{1-e^{-\alpha\left[e^{\frac{\lambda}{c}\left(e^{ct^{\gamma}}-1\right)}-1\right]}\right\}^{\beta-1}}{1-\left\{1-e^{-\alpha\left[e^{\frac{\lambda}{c}\left(e^{ct^{\gamma}}-1\right)}-1\right]}\right\}^{\beta}} \tag{3}$$

and its shape is illustrated in Figure 2 some various values of $\beta, \lambda, \alpha, \gamma, c$.

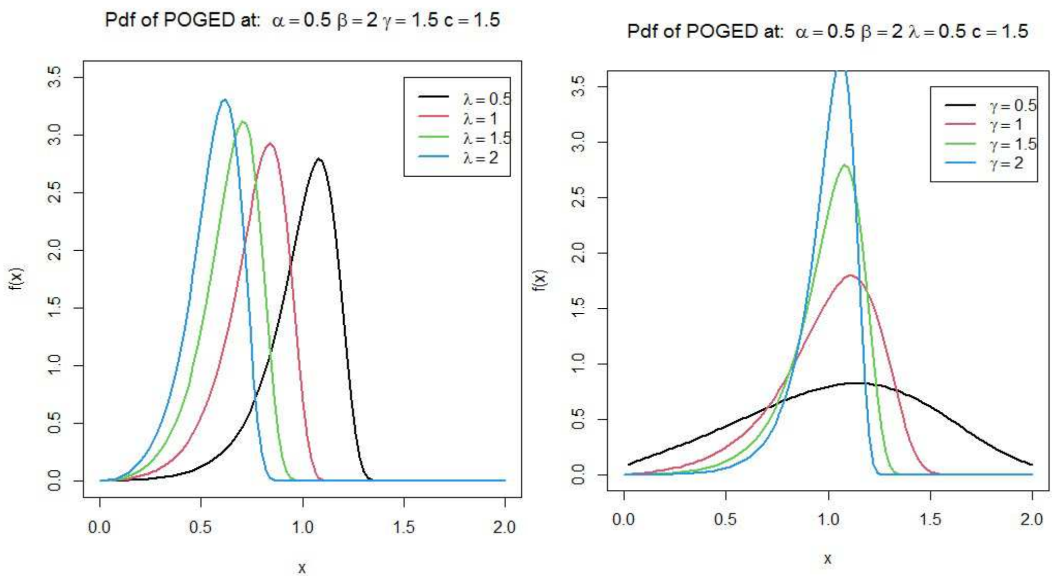

Figure 1. *Density function of POGE-G distribution at different parameters.*

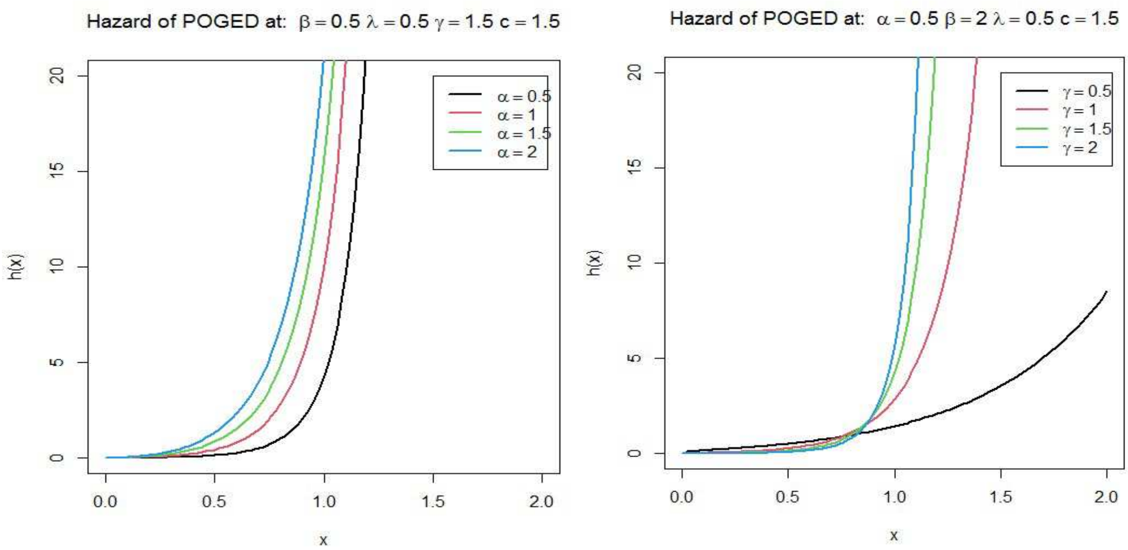

Figure 2. *Hazard rate function of POGE-G distribution at different parameters.*

For parameter estimation of the unknown parameters of the POGE-G distribution $(\lambda,\alpha,\beta,\gamma,c)$ there are three methods: Maximum likelihood estimation, maximum product spacing and Bayesian estimation.

2. Maximum Likelihood Estimation

Suppose that a random sample of n units whose lifetime follow POGE-G distribution with cdf given in Eq. (1) and its pdf given in Eq. (2) and the likelihood function is defined as

$$L(\underline{\omega})=\prod_{i=1}^{n} f\left(t_{(i)};\ \underline{\omega}\right)$$

where $\underline{\omega}=(\lambda,\alpha,\beta,\gamma,c)$. Thus, the likelihood function for the POGE-G distribution can be written as

$$l(\underline{\omega},t)\propto(\lambda\alpha\,\beta\delta)^{n}\prod_{i=1}^{n} t^{\delta-1}\,e^{ct^{\delta}}e^{\frac{\lambda}{c}(e^{ct^{\delta}}-1)}e^{-\alpha\left[e^{\frac{\lambda}{c}(e^{ct^{\delta}}-1)}-1\right]}\left\{1-e^{-\alpha\left[e^{\frac{\lambda}{c}(e^{ct^{\delta}}-1)}-1\right]}\right\}^{\beta-1} \tag{4}$$

By taking logarithm of $L(\lambda,\alpha,\beta,\gamma)$ to obtain log-likelihood $\mathcal{L}$ as

$$\mathcal{L}=nln(\alpha)+nln(\lambda)+nln(\beta)+nln(\delta)+(\delta-1)\sum_{i=1}^{n}\ln(t)+c\sum_{i=1}^{n}t^{\delta}+\frac{\lambda}{c}\sum_{i=1}^{n}\left[e^{ct^{\delta}}-1\right]-\alpha\sum_{i=1}^{n}\left[e^{\frac{\lambda}{c}\left(e^{ct^{\delta}}-1\right)}-1\right]+$$
$$(\beta-1)\sum_{i=1}^{n}\left\{1-e^{-\alpha\left[e^{\frac{\lambda}{c}\left(e^{ct^{\delta}}-1\right)}-1\right]}\right\} \tag{5}$$

by differentiating the associated log-likelihood $\mathcal{L}$ with respect to $\lambda,\alpha,\beta,\gamma,c$ and equating them to zero, we get:

$$\frac{\partial\mathcal{L}}{\partial\lambda}=\frac{n}{\lambda}+\frac{1}{\theta}\sum_{i=1}^{n}\left(e^{\theta t^{\gamma}}-1\right)-\frac{\alpha}{\theta}\sum_{i=1}^{n}e^{\frac{\lambda}{\theta}\left(e^{\theta t^{\gamma}}-1\right)}\left(e^{\theta t^{\gamma}}-1\right)+\frac{(\beta-1)\alpha}{\theta}\sum_{i=1}^{n}e^{-\alpha\left[e^{\frac{\lambda}{\theta}\left(e^{\theta t^{\gamma}}-1\right)}-1\right]}\left[e^{\frac{\lambda}{\theta}\left(e^{\theta t^{\gamma}}-1\right)}-1\right]=0 \tag{6}$$

$$\frac{\partial\mathcal{L}}{\partial\alpha}=\frac{n}{\alpha}-\sum_{i=1}^{n}\left[e^{\frac{\lambda}{\theta}\left(e^{\theta t^{\gamma}}-1\right)}-1\right]+(\beta-1)\sum_{i=1}^{n}e^{-\alpha\left[e^{\frac{\lambda}{\theta}\left(e^{\theta t^{\gamma}}-1\right)}-1\right]}\left[e^{\frac{\lambda}{\theta}\left(e^{\theta t^{\gamma}}-1\right)}-1\right]=0 \tag{7}$$

$$\frac{\partial\mathcal{L}}{\partial\beta}=\frac{n}{\beta}+\sum_{i=1}^{n}\left\{1-e^{-\alpha\left[e^{\frac{\lambda}{\theta}\left(e^{\theta t^{\gamma}}-1\right)}-1\right]}\right\}=0 \tag{8}$$

$$\frac{\partial\mathcal{L}}{\partial\gamma}=\frac{n}{\gamma}+\sum_{i=1}^{n}\ln(t)+\theta\sum_{i=1}^{n}t^{\gamma}\ln(t)+\lambda\sum_{i=1}^{n}e^{\theta t^{\gamma}}\ln(t)-\alpha\sum_{i=1}^{n}\lambda e^{\frac{\lambda}{\theta}\left(e^{\theta t^{\gamma}}-1\right)}e^{\theta t^{\gamma}}t^{\gamma}\ln(t)+\lambda\alpha$$
$$(\beta-1)\sum_{i=1}^{n}e^{\frac{\lambda}{\theta}\left(e^{\theta t^{\gamma}}-1\right)}e^{\theta t^{\gamma}}t^{\gamma}\,[\ln(t)]e^{-\alpha\left[e^{\frac{\lambda}{\theta}\left(e^{\theta t^{\gamma}}-1\right)}-1\right]}=0 \tag{9}$$

$$\frac{\partial\mathcal{L}}{\partial c}=\sum_{i=1}^{n}t^{\gamma}+\left[\frac{\lambda}{\theta}\sum_{i=1}^{n}e^{\theta t^{\gamma}}t^{\gamma}+\sum_{i=1}^{n}\left(-\frac{\lambda}{\theta}\right)\left(e^{\theta t^{\gamma}}-1\right)\right]-\alpha\sum_{i=1}^{n}e^{\frac{\lambda}{\theta}\left(e^{\theta t^{\gamma}}-1\right)}\left[\frac{\lambda}{\theta}e^{\theta t^{\gamma}}-\frac{\lambda}{\theta}\left(e^{\theta t^{\gamma}}-1\right)\right]+$$
$$(\beta-1)\sum_{i=1}^{n}e^{\frac{\lambda}{\theta}\left(e^{\theta t^{\gamma}}-1\right)}\,e^{-\alpha\left[e^{\frac{\lambda}{\theta}\left(e^{\theta t^{\gamma}}-1\right)}-1\right]}\left[\frac{\lambda}{\theta}t^{\gamma}e^{\theta t^{\gamma}}-\frac{\lambda}{\theta}\left(e^{\theta t^{\gamma}}-1\right)\right]=0 \tag{10}$$

where $\hat{\lambda},\hat{\alpha},\hat{\beta},\hat{\gamma}$ and $\hat{c}$ are the MLEs of $\lambda,\alpha,\beta,\gamma$ and c respectively. Now, the asymptotic variance-covariance matrix of the MLEs of $\lambda,\alpha,\beta,\gamma$ and c can be obtained by inverting the observed information matrix $I(\omega)$ under standard regularity conditions, the multivariate normal $N_5(0;\ I(\widehat{\omega})^{-1})$ distribution can be used to construct approximate confidence intervals for the parameters. Here, $I(\widehat{\omega})$ is the total observed information matrix evaluated at $\underline{\widehat{\omega}}$.

$$I(\widehat{\omega})=-\begin{bmatrix}\left(\frac{\partial^2\mathcal{L}}{\partial\lambda^2}\right) & \left(\frac{\partial^2\mathcal{L}}{\partial\lambda\partial\alpha}\right) & \left(\frac{\partial^2\mathcal{L}}{\partial\lambda\partial\beta}\right) & \left(\frac{\partial^2\mathcal{L}}{\partial\lambda\partial\gamma}\right)\\ \left(\frac{\partial^2\mathcal{L}}{\partial\alpha\partial\lambda}\right) & \left(\frac{\partial^2\mathcal{L}}{\partial\alpha^2}\right) & \left(\frac{\partial^2\mathcal{L}}{\partial\alpha\partial\beta}\right) & \left(\frac{\partial^2\mathcal{L}}{\partial\alpha\partial\gamma}\right)\\ \left(\frac{\partial^2\mathcal{L}}{\partial\beta\partial\lambda}\right) & \left(\frac{\partial^2\mathcal{L}}{\partial\beta\partial\alpha}\right) & \left(\frac{\partial^2\mathcal{L}}{\partial\beta^2}\right) & \left(\frac{\partial^2\mathcal{L}}{\partial\beta\partial\alpha}\right)\\ \left(\frac{\partial^2\mathcal{L}}{\partial\gamma\partial\lambda}\right) & \left(\frac{\partial^2\mathcal{L}}{\partial\gamma\partial\alpha}\right) & \left(\frac{\partial^2\mathcal{L}}{\partial\gamma\partial\beta}\right) & \left(\frac{\partial^2\mathcal{L}}{\partial\gamma^2}\right)\end{bmatrix}_{|\lambda=\hat{\lambda},\alpha=\hat{\alpha},\beta=\hat{\beta},\gamma=\hat{\gamma},c=\hat{c}} \tag{11}$$

According to particular regularity conditions, the two-sided $100(1-p)\%$, $0<p<1$, asymptotic confidence intervals for the parameters $\lambda, \alpha, \beta, \gamma$ and c can be obtained.

3. Bayesian Estimation

For Bayesian parameter estimation we will considered squared error loss function. We propose to use independent gamma priors for both λ and α having pdfs

$$\pi_1(\lambda) \propto \lambda^{a_1-1}\, exp(-b_1\lambda)\ \ \lambda>0, a_1, b_1>0 \tag{12}$$

$$\pi_2(\alpha) \propto \alpha^{a_2-1}\, exp(-b_2\alpha)\ \ \alpha>0, a_2, b_2>0 \tag{13}$$

$$\pi_3(c) \propto \alpha^{a_5-1}\, exp(-b_5 c)\ \ c>0, a_5, b_5>0 \tag{14}$$

And for parameters β and γ using an exponential priors with pdfs

$$\pi_4(\beta) \propto \beta\, exp(-c_1\beta)\ \ \beta>0, c_1>0 \tag{15}$$

$$\pi_5(\gamma) \propto \gamma\, exp(-d\gamma)\ \ \gamma>0, d>0 \tag{16}$$

All the hyper-parameters $a_1, b_1, a_2, b_2, a_5, b_5, c_1, d$ are chosen to be known and non-negative. Different priors like exponential and gamma has been used to obtain Bayesian Estimate of λ, α, β, γ and c using MCMC methods it has been found that gamma prior gives quite good estimates compare to other priors. The joint prior for $\lambda, \alpha, c, \beta$and γ is given by

$$\pi(\lambda,\alpha,\beta,\gamma) = \pi_1(\lambda)\,\pi_2(\alpha)\pi_3(c)\,\pi_4(\beta)\pi_5(\delta)$$

$$\pi(\lambda,\alpha,c,\beta,\gamma) \propto\ \beta\gamma\,\lambda^{a_1-1}\,\alpha^{a_2-1}\,c^{a_5-1}\,exp(-b_1\lambda-b_2\alpha-b_5c-c_1\beta-d\gamma) \tag{17}$$

The corresponding posterior density given the observed data $t=(t_1,t_2,\dots,t_n)$ can be written as

$$(\lambda,\alpha,c,\beta,\gamma|\boldsymbol{t}) = \frac{\pi(\lambda,\alpha,c,\beta,\gamma)L(\lambda,\alpha,c,\beta,\gamma)}{\int_0^\infty\int_0^\infty\int_0^\infty \pi(\lambda,\alpha,c,\beta,\gamma)L(\lambda,\alpha,c,\beta,\gamma)\,d\lambda\,d\alpha\,dcd\beta d\gamma}$$

The posterior density function can be written as

$$\pi(\lambda,\alpha,c,\beta,\gamma|\boldsymbol{t}) = K^{-1}(\beta^n\,\gamma^n\lambda^{n+a_1-1}\,\alpha^{n+a_2-1}c^{n+a_5-1})\exp(-b_1\lambda-b_2\alpha-b_5c-c_1\beta-d\gamma)\sum_{i=1}^n t_i{}^\gamma \exp\left(\sum_{i=1}^n\left[\ ct_i{}^\gamma + \frac{\lambda}{c}(\exp(ct_i{}^\gamma)-1)-\alpha\left(\exp\left(\frac{\lambda}{c}\,(\exp(t_i{}^\gamma)-1)\right)-1\right)\right]\right)\sum_{i=1}^n\left(1-\exp\left[-\alpha\left(\exp\left(\frac{\lambda}{c}\,(\exp(t_i{}^\gamma)-1)\right)-1\right)\right]\right)^{\beta-1} \tag{18}$$

where

$$K=\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty(\beta^n\,\gamma^n\lambda^{n+a_1-1}\,\alpha^{n+a_2-1})\,exp(-b_1\lambda-b_2\alpha-b_5c-c_1\beta-d\gamma)\sum_{i=1}^n t_i{}^\gamma\, exp\left(\sum_{i=1}^n\left[\ ct_i{}^\gamma + \frac{\lambda}{c}(exp(ct_i{}^\gamma)-1)-\alpha\left(exp\left(\frac{\lambda}{c}\,(exp(t_i{}^\gamma)-1)\right)-1\right)\right]\right)\sum_{i=1}^n\left(1- exp\left[-\alpha\left(exp\left(\frac{\lambda}{c}\,(exp(t_i{}^\gamma)-1)\right)-1\right)\right]\right)^{\beta-1} d\lambda\,d\alpha dc\,d\beta d\gamma$$

Thus, the posterior density can be rewritten as

$$\pi(\lambda,\alpha,c,\beta,\gamma|\boldsymbol{t}) \propto (\beta^n\,\gamma^n\lambda^{n+a_1-1}\,\alpha^{n+a_2-1}c^{n+a_5-1})\,exp(-b_1\lambda-b_2\alpha-c_1\beta-d\gamma)\sum_{i=1}^n t_i{}^\gamma$$

$$exp\left(\sum_{i=1}^n\left[\ ct_i{}^\gamma + \frac{\lambda}{c}(exp(ct_i{}^\gamma)-1)-\alpha\left(exp\left(\frac{\lambda}{c}\,(exp(ct_i{}^\gamma)-1)\right)-1\right)\right]\right)$$

$$\sum_{i=1}^n\left(1-exp\left[-\alpha\left(exp\left(\frac{\lambda}{c}\,(exp(ct_i{}^\gamma)-1)\right)-1\right)\right]\right)^{\beta-1}$$

The Bayes Estimator of any loss function, say $g(\lambda,\alpha,\beta)$ under the squared error, is given by

$$\tilde{g}(\lambda,\alpha,c,\beta,\gamma) = \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty g(\lambda,\alpha,c,\beta,\gamma)\pi(\lambda,\alpha,c,\beta,\gamma|t)\,d\lambda\,d\gamma d\alpha\,d\beta\,dc \tag{19}$$

Unfortunately, Eq. (19) cannot be computed for general $g(\lambda,\alpha,c,\beta,\gamma)$. Thus, the Bayes estimates of $\lambda,\alpha,c,\beta,\gamma$ can be obtained numerically by using Markov Chain Monte Carlo (MCMC).

3.1. Markov Chain Monte Carlo (MCMC)

Markov Chain Monte Carlo (MCMC) is a computer-driven sampling method. It allows one to characterize a distribution without knowing all of the distribution mathematical properties by random sampling values out of the distribution Ravenzwaaij et al. [15].

MCMC is particularly useful in Bayesian inference because of the focus on posterior distributions which are often difficult to work with via analytic examination. In these cases, MCMC allows the user to approximate aspects of posterior distributions that cannot be directly calculated (e.g., random samples from the posterior, posterior means, etc.). To draw samples from a distribution using MCMC:

1. Starting with an initial guess: just one value that might be plausibly drawn from the distribution.
2. Producing a chain of new samples from this initial guess. Each new sample is produced by two simple steps:
   a) Proposal: a proposal for the new sample is created by adding a small random perturbation to the most recent sample.
   b) Acceptance: the new proposal is either accepted as the new sample, or rejected (in which case the old sample retained).

Proposal Distribution: A distribution for randomly generating new candidate samples, to be accepted or rejected.

There are many ways of adding random noise to create proposals, and also different approaches to the process of accepting and rejecting, such as: Gibbs-sampling and Metropolis-Hastings algorithm.

3.2. Metropolis-Hastings Algorithm

Metropolis-Hastings (MH) algorithm is a useful method for generating random samples from the posterior distribution using a proposal density. To implement the MH algorithm we have to define a proposal distribution $q(\zeta'|\zeta)$ and an initial values $\zeta^{(0)}$ of the unknown parameters. For the proposal distribution, we consider a bivariate normal distribution, that is $q(\zeta'|\zeta)\equiv N_2(\zeta,S_\zeta)$, where $\zeta=(\lambda,\alpha,\beta,\gamma,c)$ and S_ζ represent the variance-covariance matrix, we may get negative observations which are not acceptable. For the initial values, we guess an appropriate values to $\lambda,\alpha,\beta,\gamma$ and c. Therefore, we propose the following steps of MH algorithm to draw sample from the posterior density $\pi(\lambda,\alpha,\beta,\gamma,c|x)$ (Dey and Pradhan [9]

Step 1. Set initial value of ζ as $\zeta=\zeta^{(0)}$.

Step 2. For $i=1,2,\ldots,M$ repeat the following steps:

1. Set $\zeta=\zeta^{(i-1)}$.
2. Generate a new candidate parameter value δ from $N_2(\ln\zeta, S_\zeta)$.
3. Set $\zeta=\exp(\delta)$.
4. Calculate

$$\rho=\min\left(1,\frac{\pi(\zeta'|x)}{\pi(\zeta\,|x)}\right)$$

Update $\zeta^{(i)}=\zeta'$ with probabilityρ; otherwise set $\zeta^{(i)}=\zeta$.

The initial value for ζ is considered to be the MLE $\hat{\zeta}=(\hat{\lambda},\hat{\alpha},\hat{\beta},\hat{\gamma},\hat{c})$ of $\zeta=(\lambda,\alpha,\beta,\gamma,c)$. While, the selection of S_ζ is considered to be the asymptotic variance-covariance matrix $I^{-1}(\lambda,\alpha,\beta,\gamma,c)$, where $I(.)$ is the Fisher information matrix. Notice that, the selection of S_ζ is an important issue in the MH algorithm where the acceptance rate is depends on upon this.

Finally, from the random samples of size M drawn from the posterior density, some of the initial samples (burn-in) can be discarded, and remaining samples can be further utilized to compute Bayes estimates. More precisely the Eq. (5) can be evaluated as

$$\tilde{g}_{MH}(\lambda,\alpha,\beta,\gamma,c)=\frac{1}{M-l_B}\sum_{i=1}^{M}g(\lambda_i,\alpha_i,\beta_i,\gamma_i,c_i)\quad(20)$$

where l_B represent the number of burn-in samples.

3.3. Highest Posterior Density

We suggest utilizing the technique of Chen and Shao [7] to calculate highest posterior density (HPD) interval estimates for the unknown parameters of the GIE distribution. The technique of Chen and Shao has been broadly utilized for constructing HPD intervals for the unknown parameters of the distribution of interest. In the present study, we will employ the samples drawn using the proposed MH algorithm to construct the interval estimates. More accurately, let us assume that $\Pi(\theta|x)$ denotes the posterior distribution function of θ. Let us further suppose that $\theta^{(p)}$ be the pth quantile of γ, that is, $\theta^{(p)}=\inf\{\theta:\Pi(\theta|x)\geq p\}$, where $0<p<1$. Notice that for a given θ^*, a simulation consistent estimator of $\Pi(\theta^*|x)$ can be estimated as

$$\Pi(\theta^*|\boldsymbol{x})=\frac{1}{M-l_B}\sum_{i=l_B}^{M}I_{\theta\leq\theta^*}$$

Here $I_{\gamma\leq\gamma^*}$ is the indicator function. Then the corresponding estimate is obtained as

$$\widehat{\Pi}(\theta^*|\boldsymbol{x})=\begin{cases}0 \text{ if } \theta^*\leq\theta_{(l_B)}\\ \displaystyle\sum_{j=l_B}^{i}w_j \text{ if } \theta_{(i)}\leq\theta^*\leq\theta_{(i+1)}\\ 1 \text{ if } \theta^*>\theta_{(M)}\end{cases}$$

where $w_j=\frac{1}{M-l_B}$ and $\theta_{(j)}$ are the ordered values of θ_j. Now, for $i=l_B,\ldots,M,\theta^{(p)}$ can be approximated by

$$\hat{\theta}^{(p)} = \begin{cases} \theta_{(l_B)} \text{ if } p = 0 \\ \theta_{(i)} \text{ if } \sum_{j=l_B}^{i-1} w_i < p < \sum_{j=l_B}^{i} w_i \end{cases}$$

Now to obtain a $100(1-p)\%$ HPD credible interval for θ, let

$$R_j = \left(\hat{\theta}^{\left(\frac{j}{M}\right)}, \hat{\theta}^{\left(\frac{j+(1-p)M}{M}\right)}\right), j = l_B, \ldots, [p_M],$$

here $[a]$ denotes the largest integer less than or equal to a.

Then choose R_{j^*} among all the R_j's such that it has the smallest width.

4. Maximum Product of Spacing

Suppose that an ordered random sample $t_1, \ldots, t_n$ drawn from POGE-G distribution with parameters $\underline{\epsilon} = (\lambda, \gamma, \beta, \alpha, c)$ and the cdf of the spacing is constructed as:

$$D_i(\alpha,\beta,\gamma,\lambda,c) = F(t_{i:n} \mid \alpha,\beta,\gamma,\lambda,c) - F(t_{i-1:n} \mid \alpha,\beta,\gamma,\lambda,c), \tag{21}$$

where

$$F(t_{0:n} \mid \alpha,\beta,\gamma,\lambda,c) = 0 \; and \; F(t_{n+1:n} \mid \alpha,\beta,\gamma,\lambda,c) = 1.$$

Clearly

$$\sum_{i=1}^{n+1} D_i(\alpha,\beta,\gamma,\lambda,c) = 1$$

$$G(\alpha,\beta,\gamma,\lambda,c) = \left[\prod_{i=1}^{n+1} D_i(\alpha,\beta,\gamma,\lambda,c)\right]^{\frac{1}{n+1}}$$

or, equivalently, by maximizing the function,

$$H(\alpha,\beta,\gamma,\lambda,c) = \frac{1}{n+1}\sum_{i=1}^{n+1} log D_i(\alpha,\beta,\gamma,\lambda,c) \tag{22}$$

The spacings are defined as follows:

$$D_1 = F(t_1) = \left[1 - e^{-\alpha\left(e^{\frac{\lambda}{c}\left(e^{ct^{\gamma}}{}_{1}{-1}\right)}-1\right)}\right]^{\beta}, D_{(n+1)} = F(t_n) = \left[1 - e^{-\alpha\left(e^{\frac{\lambda}{c}\left(e^{ct^{\gamma}}{}_{n}{-1}\right)}-1\right)}\right]^{\beta}$$

And the general term of spacing's is given by,

$$D_i = F(t_i) - F(t_{i-1}) = \left[1 - e^{-\alpha\left(e^{\frac{\lambda}{c}\left(e^{ct^{\gamma}}{}_{i}{-1}\right)}-1\right)}\right]^{\beta} - \left[1 - e^{-\alpha\left(e^{\frac{\lambda}{c}\left(e^{ct^{\gamma}}{}_{i}{-1}\right)}-1\right)}\right]^{\beta} \tag{23}$$

Let $\psi_i = e^{\left(\frac{\lambda}{c}\left(e^{ct^{\gamma}}{}_{i}{-1}\right)\right)}$

Thus, D_i can be rewritten as follows

$$D_i = F(x_i) - F(x_{i-1}) = \left[1 - e^{-\alpha(\psi_i - 1)}\right]^{\beta} - \left[1 - e^{-\alpha(\psi_i - 1)}\right]^{\beta}$$

Hence the corresponding H is given by

$$H = \frac{1}{n+1}\{lnD_1 + \sum_{i=2}^{n} lnD_i + lnD_{n+1}\} = \frac{1}{n+1}\left\{ln\left[1 - e^{-\alpha(\psi_i - 1)}\right]^{\beta} + \sum_{i=2}^{n} ln\left[\left[1 - e^{-\alpha(\psi_i - 1)}\right]^{\beta} - \left[1 - e^{-\alpha(\psi_i - 1)}\right]^{\beta}\right]\right\} + \frac{1}{n+1}\left\{ln\left[1 - e^{-\alpha(\psi_n - 1)}\right]^{\beta}\right\} \tag{24}$$

Following Cheng and Amin [6], the maximum product of spacings estimators $\hat{\alpha}MPS, \hat{\beta}MPS, \hat{\lambda}MPS, \hat{\gamma}MPS \; and \; \hat{c}MPS$ of the parameters α, β, λ, γ and c are obtained by maximizing the geometric mean of the spacings with respect to α, β, λ, γ and c. By solving the nonlinear equations,

$$\frac{\partial}{\partial\alpha}H(\alpha,\beta,\gamma,\lambda,c)=\frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1}{Di(\alpha,\beta,\gamma,\lambda,c)}\left[\Delta_1\left(x_{i,n}\middle|\alpha,\beta,\gamma,\lambda,c\right)-\Delta_1\left(x_{i-1,n}\middle|\alpha,\beta,\gamma,\lambda,c\right)\right]=0 \quad (25)$$

$$\frac{\partial}{\partial\beta}H(\alpha,\beta,\gamma,\lambda,c)=\frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1}{Di(\alpha,\beta,\gamma,\lambda,c)}\left[\Delta_2\left(x_{i,n}\middle|\alpha,\beta,\gamma,\lambda,c\right)-\Delta_2\left(x_{i-1,n}\middle|\alpha,\beta,\gamma,\lambda,c\right)\right]=0 \quad (26)$$

$$\frac{\partial}{\partial\lambda}H(\alpha,\beta,\gamma,\lambda,c)=\frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1}{Di(\alpha,\beta,\gamma,\lambda,c)}\left[\Delta_3\left(x_{i,n}\middle|\alpha,\beta,\gamma,\lambda,c\right)-\Delta_3\left(x_{i-1,n}\middle|\alpha,\beta,\gamma,\lambda,c\right)\right]=0 \quad (27)$$

$$\frac{\partial}{\partial c}H(\alpha,\beta,\gamma,\lambda,c)=\frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1}{Di(\alpha,\beta,\gamma,\lambda,c)}\left[\Delta_4\left(x_{i,n}\middle|\alpha,\beta,\gamma,\lambda,c\right)-\Delta_4\left(x_{i-1,n}\middle|\alpha,\beta,\gamma,\lambda,c\right)\right]=0 \quad (28)$$

$$\frac{\partial}{\partial\gamma}H(\alpha,\beta,\gamma,\lambda,c)=\frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1}{Di(\alpha,\beta,\gamma,\lambda,c)}\left[\Delta_5\left(x_{i,n}\middle|\alpha,\beta,\gamma,\lambda,c\right)-\Delta_5\left(x_{i-1,n}\middle|\alpha,\beta,\gamma,\lambda,c\right)\right]=0 \quad (29)$$

where

$$\Delta_1=\beta e^{\frac{\lambda}{c}\left(e^{ct^\gamma}-1\right)}e^{-\alpha\left[e^{\frac{\lambda}{c}\left(e^{ct^\gamma}-1\right)}-1\right]}\left\{1-e^{-\alpha\left[e^{\frac{\lambda}{c}\left(e^{ct^\gamma}-1\right)}-1\right]}\right\}^{\beta-1}$$

$$\Delta_2=\left\{1-e^{-\alpha\left[e^{\frac{\lambda}{c}\left(e^{ct^\gamma}-1\right)}-1\right]}\right\}^{\beta} ln\left\{1-e^{-\alpha\left[e^{\frac{\lambda}{c}\left(e^{ct^\gamma}-1\right)}-1\right]}\right\}$$

$$\Delta_3=\frac{\beta\alpha}{c}\left(e^{ct^\gamma}-1\right)e^{\frac{\lambda}{c}\left(e^{ct^\gamma}-1\right)}e^{-\alpha\left[e^{\frac{\lambda}{c}\left(e^{ct^\gamma}-1\right)}-1\right]}\left\{1-e^{-\alpha\left[e^{\frac{\lambda}{c}\left(e^{ct^\gamma}-1\right)}-1\right]}\right\}^{\beta-1}$$

$$\Delta_4=\beta e^{-\alpha\left[e^{\frac{\lambda}{c}\left(e^{ct^\gamma}-1\right)}-1\right]}\left\{1-e^{-\alpha\left[e^{\frac{\lambda}{c}\left(e^{ct^\gamma}-1\right)}-1\right]}\right\}^{\beta-1}+\alpha e^{\frac{\lambda}{c}\left(e^{ct^\gamma}-1\right)}\left[\frac{\lambda}{c}t^\gamma e^{ct^\gamma}-\frac{\lambda}{c^2}\left(e^{ct^\gamma}-1\right)\right]$$

$$\Delta_5=\beta\alpha\lambda t^\gamma e^{ct^\gamma}e^{\frac{\lambda}{c}\left(e^{ct^\gamma}-1\right)}e^{-\alpha\left[e^{\frac{\lambda}{c}\left(e^{ct^\gamma}-1\right)}-1\right]}\left\{1-e^{-\alpha\left[e^{\frac{\lambda}{c}\left(e^{ct^\gamma}-1\right)}-1\right]}\right\}^{\beta-1}[\ln(t)]$$

For asymptotic confidence intervals using MPS method, it's necessary to obtain the elements of the Fisher information matrix $I(\widehat{\omega})$ by taking the second derivatives of the function H with respect to $\widehat{\omega}$. The approximate $(1-p)\,100\%$ confidence intervals for the parameters α, λ, γ, β and c are given. see Anatolyev and Kosenok [4].

5. Simulation Study and Data Analysis

The aim of this section is to set a comparison the performance of the different methods of estimation discussed in the previous sections. We analyze a real data set for illustrative purpose; also, a simulation study is employed to check the behavior of the proposed methods as well as to assess the statistical performances of the estimators. We used *R*-statistical programming language for calculation. Further, we utilize *bbmle* and *HDInterval* packages to compute MLEs, MPS and HPD interval in *R*-language.

5.1. Simulation Study

We employ a Monte Carlo simulation study to compare the performance of proposed methods of estimation. We simulate 1000 generating data from POGE-G distribution with initial values:

1) Case I: $\lambda=0.5,\alpha=0.5,\beta=2,c=1.5,\gamma=1.5$
2) Case II: $\lambda=0.5,\alpha=2,\beta=2,c=1.5,\gamma=1.5$

Based on the generated data, firstly, we calculate maximum likelihood estimates and associated 95% asymptotic confidence interval estimates. Note that the initial guess values are considered to be same as the true parameter values while obtaining maximum likelihood estimates.

Based on the generated data, firstly, we calculate maximum product of spacing estimates and associated 95% asymptotic confidence interval estimates. Note that the initial guess values are considered to be same as the true parameter values while obtaining maximum product spacing estimates.

For Bayesian estimation, we calculate Bayes estimates using the MH algorithm under the informative prior. These priors are then plugged-in to calculate the desired estimates. While utilizing MH algorithm, we take into account the maximum likelihood estimates as initial guess values, and the associated variance–covariance matrix. At the end, we discarded 1000 burn-in samples among the overall 5000 samples generated from the posterior density, and subsequently obtained Bayes estimates, and HPD interval estimates utilizing the technique of Chen and

Shao [7].

All the average estimates for both methods are reported in Table 1 and Table 2. Further, the first row represents the average estimates and interval estimates, and in the second row, associated means square errors (MSEs) and average interval lengths (AILs) with coverage percentages (CPs) are reported. The convergence of MCMC estimation for $\lambda, \alpha, \beta, \gamma, c$ can be showed in figure 3 and figure 4.

From tabulated values it can be noticed that depended on MSEs, higher values of n lead to better estimates. It is also noticed that the performance of the Bayes estimates obtained is better than the MLE estimates. It can also be noticed that the AILs and associated CPs of HPD intervals of Bayes estimates are better than those of MLE estimates.

From tabulated values it can be noticed that depended on MSEs, higher values of n lead to better estimates. It is also noticed that the performance of the MPS estimates obtained is better than the MLE estimates in the previous section.

Table 1. *Estimated values, interval estimates, MSEs, AILs and CPs for MLE and Bayesian (MCMC) for number of simulation 5000.*

Initial: $\lambda = 0.5, \alpha = 0.5, \beta = 2, c = 1.5, \gamma = 1.5$							
n	**parameters**	**MLE**		**MPS**		**Bayesian (MCMC)**	
		Estimate MSE	**ACI AIL/CP**	**Estimate MSE**	**ACI AIL/CP**	**Estimate MSE**	**HPD interval AIL/CP**
200	α	0.78 (0.45)	0.001,1.98 (1.98\95.6)	0.47 (0.21)	0.01,1.37 (1.37\94.6)	0.43 (0.16)	0.003,1.29 (1.29\96.5)
	β	1.77 (0.31)	0.77,2.77 (1.99\95.9)	1.87 (0.25)	0.93,2.81 (1.87\95.6)	1.85 (0.27)	0.91,2.83 (1.91\96.1)
	λ	0.51 (0.04)	0.11,0.90 (0.78\95.5)	0.55 (0.04)	0.16,0.96 (0.79\96.2)	0.63 (0.07)	0.23,1.16 (0.92\96.8)
	γ	2.26 (2.17)	0.002,4.74 (4.74\95.3)	1.54 (0.98)	0.01,3.48 (3.48\93.9)	1.59 (0.88)	0.45,3.69 (3.24\95.8)
	c	1.20 (0.57)	0.001,2.55 (2.55\96.6)	1.66 (0.54)	0.26,3.06 (2.80\97.9)	1.54 (0.51)	0.21,2.77 (2.57\95.8)
300	α	0.71 (0.32)	0.001,1.74 (1.74\94.6)	0.43 (0.16)	0.001,1.21 (1.21\94.90)	0.40 (0.13)	0.03,1.15 (1.12\95.6)
	β	1.82 (0.20)	1.02,2.63 (1.61\96.1)	1.89 (0.1)	1.16,2.62 (1.47\96.7)	1.87 (0.19)	1.18,2.70 (1.53\96.7)
	λ	0.50 (0.03)	0.15,0.86 (0.71\96.1)	0.54 (0.04)	0.19,0.91 (0.72\95.8)	0.62 (0.07)	0.24,1.09 (0.84\96.8)
	γ	2.03 (1.37)	0.001,4.07 (4.07\94.5)	1.42 (0.62)	0.001,2.96 (2.96\94.2)	1.49 (0.57)	0.53,3.13 (2.59\95.6)
	c	1.30 (0.43)	0.07,2.54 (2.47\97.2)	1.72 (0.47)	0.45,2.99 (2.54\97.5)	1.60 (0.45)	0.37,2.89 (2.52\97.7)
500	α	0.67 (0.24)	0.001,1.58 (1.58\94.8)	0.43 (0.12)	0.001,1.11 (1.11\95.2)	0.40 (0.09)	0.04,1.06 (1.02\95.7)
	β	1.86 (0.11)	1.24,2.48 (1.23\97.0)	1.92 (0.09)	1.35,2.49 (1.13\96.3)	1.91 (0.11)	0.04,1.06 (1.03\95.7)
	λ	0.50 (0.02)	0.19,0.81 (0.62\96.4)	0.55 (0.02)	0.23,0.86 (0.62\96.0)	0.62 (0.06)	0.26,1.03 (0.76\96.4)
	γ	1.86 (0.80)	0.25,3.47 (3.22\95.1)	1.39 (0.42)	0.14,2.65 (2.51\94.7)	1.46 (0.38)	0.56,2.76 (2.19\95.5)
	c	1.36 (0.31)	0.30,2.41 (2.10\96.9)	1.68 (0.32)	0.61,2.75 (2.13\97.4)	1.56 (0.32)	0.53,2.73 (2.20\97.6)

ACI: Asymptotic confidence interval, AIL: Average interval length, CP: Coverage probability

Table 2. *Estimated values, interval estimates, MSEs, AILs and CPs for MLE, MPS and Bayesian (MCMC) for number of simulation 5000.*

Initial: $\lambda = 0.5, \alpha = 2, \beta = 2, c = 1.5, \gamma = 1.5$							
n	**Parameters**	**MLE**		**MPS**		**Bayesian (MCMC)**	
		Estimate MSE	**ACI AIL/CP**	**Estimate MSE**	**ACI AIL/CP**	**Estimate MSE**	**HPD interval AIL/CP**
200	α	2.48 (2.84)	0.001,5.65 (5.67\94.90)	1.76 (1.95)	0.001,4.46 (4.48\94.40)	1.55 (1.85)	0.012,4.27 (4.14\95.30)
	β	1.74 (0.58)	0.02,3.15 (2.82\96.70)	2.10 (0.51)	0.71,3.50 (2.79\97.10	2.01 (0.50)	0.69,3.41 (2.71\96.60)
	λ	0.54 (0.06)	0.0001,1.00 (0.91\96.0)	0.47 (0.06)	0.002,0.97 (0.95\95.60)	0.58 (0.09)	0.11,1.19 (1.08\96.30)
	γ	2.14 (1.98)	0.003,4.60 (4.61\95.60)	1.38 (1.10)	0.01,3.44 (3.45\94.0)	1.45 (1.03)	0.28,3.55 (3.27\95.20)
	c	1.37 (0.89)	0.002,3.22 (3.20\95.40)	2.02 (1.45)	0.04,4.16 (4.17\98.70)	1.88 (1.09)	0.33,3.63 (3.29\96.20)
300	α	2.37 (2.24)	0.01,5.21 (5.21/94.5)	1.7 (1.83)	0.02,4.3 (4.3/94.7)	1.51 (1.79)	0.06,4.13 (4.07/95.1)
	β	1.76 (0.47)	0.48,3.03 (2.54/97.1)	2.09 (0.45)	0.79,3.4 (2.6/97)	2.01 (0.43)	0.65,3.16 (2.51/95.6)
	λ	0.53 (0.05)	0.07,1.01 (0.93/95.1)	0.48 (0.06)	0.007,0.96 (0.96/96.7)	0.6 (0.1)	0.1,1.11 (1.01/95.4)
	γ	2.02 (1.53)	0.01,4.22 (4.22/95.9)	1.37 (0.97)	0.01,3.29 (3.29/94.5)	1.44 (0.94)	0.3,3.42 (3.11/95.2)
	c	1.4 (0.82)	0.02,3.18 (3.18/95.1)	1.98 (1.28)	0.01,3.99 (3.99/97.9)	1.82 (1.02)	0.35,3.58 (3.22/96.4)
500	α	2.35 (1.94)	0.01,5 (5/94.5)	1.85 (1.49)	0.01,4.23 (4.23/96)	1.65 (1.47)	0.14,3.84 (3.69/95.2)
	β	1.78 (0.38)	0.64,2.92 (2.28/97.5)	2.07 (0.32)	0.96,3.18 (2.21/98.3)	1.99 (0.31)	0.8,2.93 (2.13/96)
	λ	0.53 (0.04)	0.1,0.95 (0.84/95.7)	0.47 (0.04)	0.06,0.89 (0.83/96)	0.57 (0.07)	0.1,1.04 (0.94/96)
	γ	1.94 (1.18)	0.01,3.88 (3.88/96.4)	1.41 (0.75)	0.01,3.11 (3.11/93.3)	1.47 (0.7)	0.35,3.25 (2.9/95.4)
	c	1.4 (0.7)	0.02,3.03 (3.03/95.8)	1.88 (1.01)	005,3.7 (3.64/98)	1.74 (0.81)	0.33,3.39 (3.05/96.1)

ACI: Asymptotic confidence interval, AIL: Average interval length, CP: Coverage probability

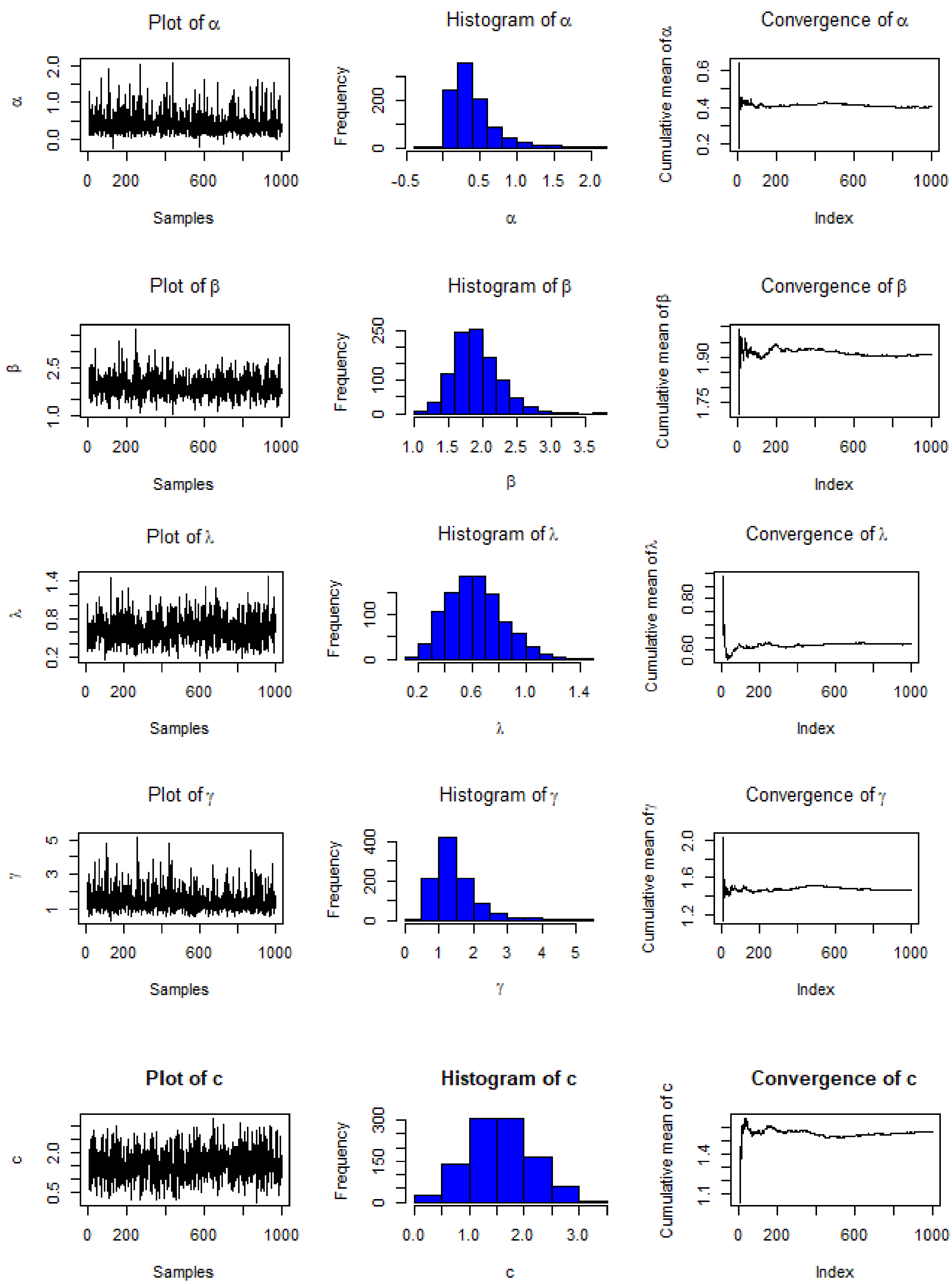

Figure 3. *Convergence of MCMC estimation for* $\lambda = 0.5, \alpha = 0.5, \beta = 2, c = 1.5, \gamma = 1.5$ *when* $n = 500$.

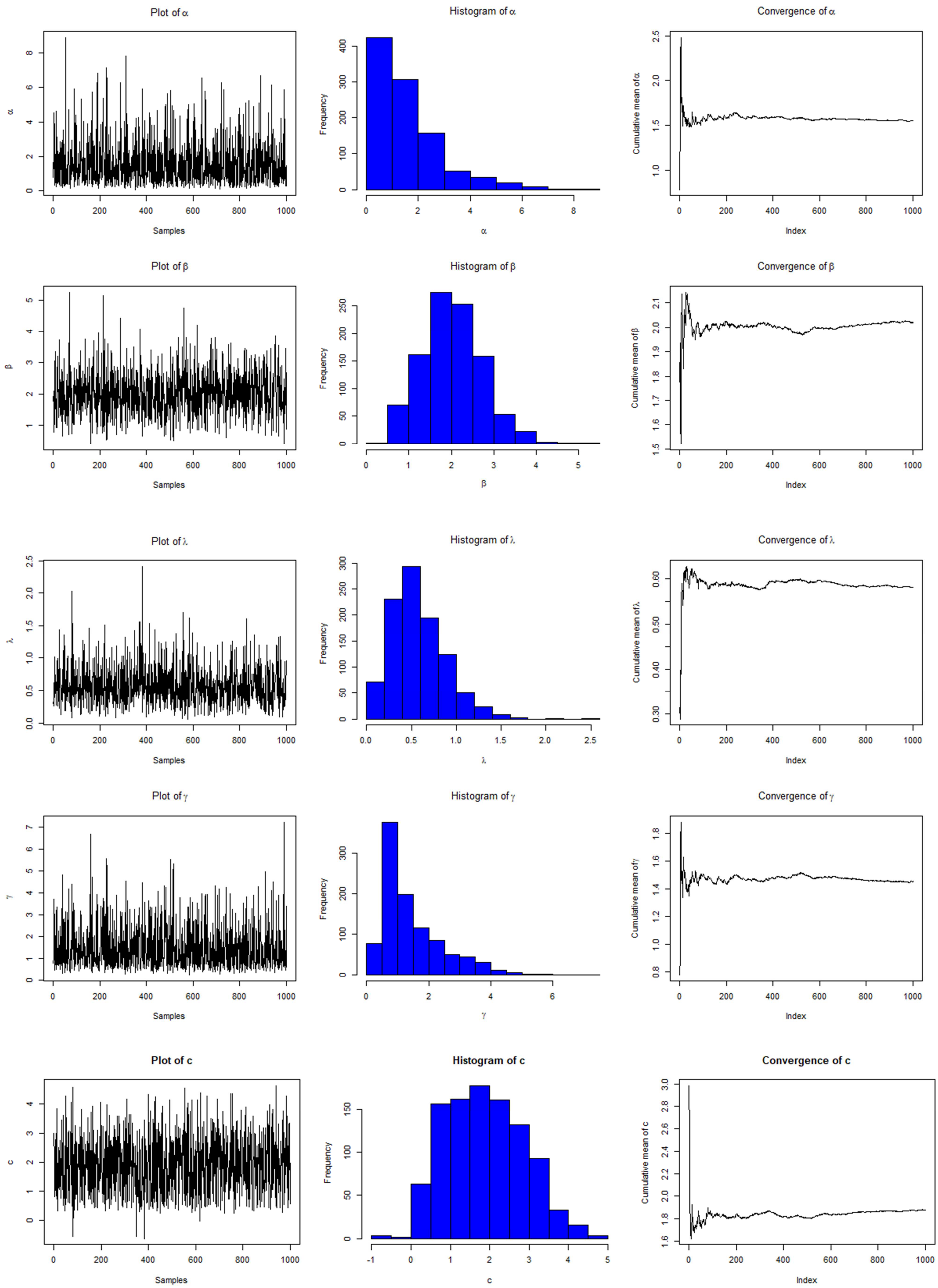

Figure 4. *Convergence of MCMC estimation for* $\lambda = 0.5, \alpha = 2, \beta = 2, c = 1.5, \gamma = 1.5$ *when* $n = 200$.

5.2. Real Data Analysis (Mortality Rates in Egypt)

In order to demonstrate the usefulness of the proposed model in the vital statistics, the data set was taken from the projections of the future population for the total of the Egyptian Arabic Republic for the period 2017-2052, depending on the book which introduced from the central agency for public mobilization and statistics in Feb. (2019).

Table 3 gives calculation of the mortality rates depending on the category ages for 2017 other than infants using number of mortality from vital statistics and number of populations (Males).

Table 3. *Mortality rates depending on the category ages for 2017.*

Categories	Male
Less than 1	17.70
1-4	1.14
5-9	0.50
10-14	.52
15-19	.92
20-24	1.09
25-29	1.20
30-34	1.49
35-39	1.91
40-44	2.91
45-49	4.89
50-54	9.99
55-59	16.73
60-64	25.31
65-69	40.13
70-74	60.21
More than75	111.25

5.3. Goodness of Fit Test

To check the validity of the fitted model, Kolomgrov-Simrnov goodness of fit test is performed for the data set by the following steps,

1) Normality test using the Kolmogorov-Smirnov test.

Goodness of fit test "Do the q_x values follow the POGE-G distribution

2) The test of moderation

This test is performed on the variable: death rates q_x using the Kolmogorov-Smirnov test, which assumes the following assumptions:

H_0: q_x data is not characterized by moderation

H_1: q_x data is characterized by moderation

Test statistics

$$\mathrm{D} = \max\|F0(x) - F1(x)\| = 0.50$$

Where $F0(x)$ is the cumulative frequency distribution at H_0 and $F1(x)$ is the cumulative frequency distribution at H1 and the calculated p-value value is shown as follows

P-value <0.03663

It appears from this test that the calculated value of p-value is less than 5% and thus cannot accept H_0 and accept the H1 meaning that q_x data is characterized by moderation.

5.4. Calculating the Reconciliation Quality of q_x Data

The values of failure rates and Q_x mortality rates using by using the hazard rate (2.3) of the Power Odd Generalized Exponential Gompertz distribution is calculated, the accuracy of the true q_x mortality rates also calculated from the factual data in Table 4.

Table 4. *The calculated values of Q_x and the corresponding values of h (x) are displayed as well as the q_x calculation.*

Age	qx	h(x)	Qx
0	0.01770	0.002504330	0.002501197
1	0.00457	0.002090119	0.002087936
5	0.00198	0.001903248	0.001901438
10	0.00207	0.001807650	0.001806017
15	0.00366	0.001761429	0.001759879
20	0.00435	0.001746986	0.001745461
25	0.00479	0.001755709	0.001754168
30	0.00596	0.001783072	0.001781483
35	0.00760	0.001826695	0.001825027
40	0.01156	0.001885463	0.001883686
45	0.01938	0.001959103	0.001957185
50	0.03920	0.002047965	0.002045870
55	0.06476	0.002152916	0.002150600
60	0.09636	0.002275297	0.002272710
65	0.14858	0.002416928	0.002414009
70	0.21496	0.002580142	0.002576816

5.5. Graphical Description for Mortality Rates

After calculating mortality rates numerically by using (3) and assumed a different values for the parameter of POGE-G distribution it can be describing the q_x and Q_x graphically as follows.

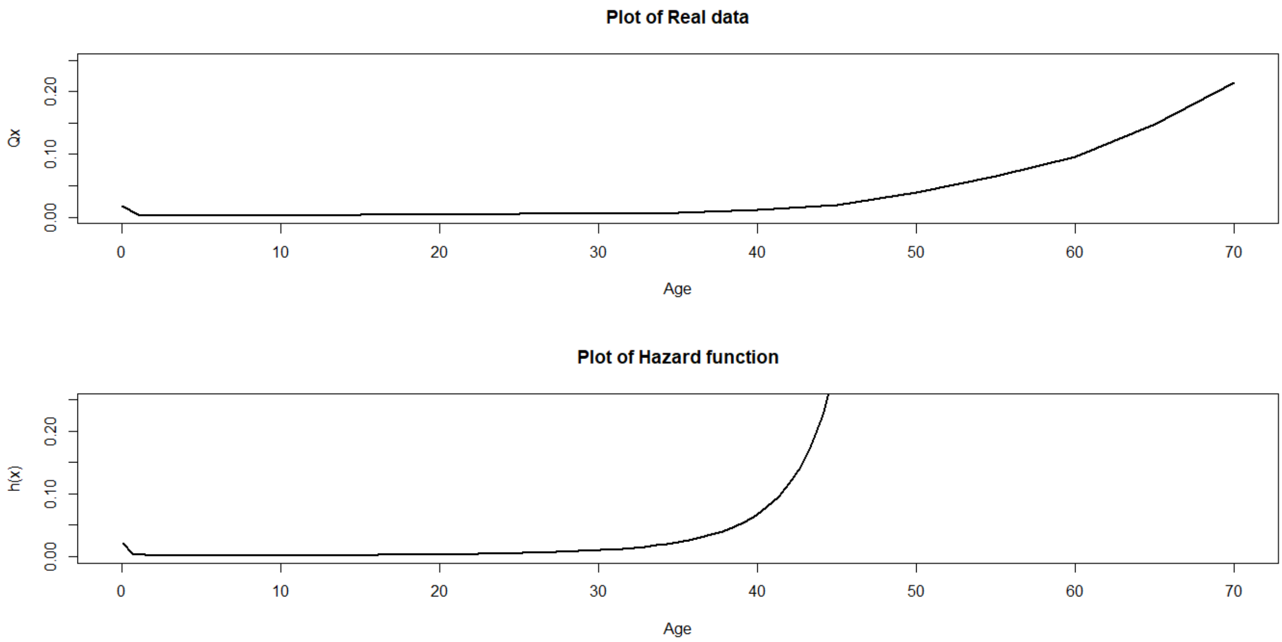

Figure 5. *The failure function and the expected mortality rates for POGE-G distribution at (α = 0.001, β = 0.5, λ = 0.025, γ=1.1, c = 0.035).*

Figure 5 shows that:

1) It is clear that the failure rate function has the property of the U-shape.
2) The failure rate function increasing as the age increasing.

Through this presentation, it can be noted that this distribution can be applied to population data.

5.6. The Estimation of the Parameter Using Different Methods

After checking the validity of this data to the model by using Kolomgrov-Simrnov goodness of fit test (KS) (with its P-value=0.093), it can be use this data to estimate the parameter of the POGE-G distribution by using ML, MPS and Bayesian methods of estimation which considered in chapter three. All the results are listed in Table 5. Depending on that the parameter c is known, it can be noted that the results of the three methods seems to be approximately.

Table 5. *ML, MPS and Bayesian Estimates with its Ks, of the model parameters* (α, λ, β) *for the data.*

Methods	Parameters	Estimates
ML	α	2.511
	β	195.49
	λ	1.475
	γ	0.074
MPS	α	0.093
	β	4.851
	λ	3.898
	γ	0.074
Bayesian	α	0.116
	β	4.197
	λ	3.699
	γ	0.083

6. Conclusion

In this paper, we have studied the problem of estimation of the power odd generalized exponential-Gompertz (POGE-G) distribution from classical and Bayesian viewpoints. We derived maximum likelihood estimates, maximum product spacing and associated asymptotic confidence interval estimates for the unknown parameters of a POGE-G distribution. Then, we calculated Bayes estimates and the corresponding HPD interval estimates under informative priors. Our simulation study indicates that the performance of Bayes estimates is better MLE estimates. Though we have considered squared error loss function under Bayesian set up, yet other loss functions can also be considered. Through this presentation, it can be noted that this distribution can be applied to population data. The present work can also be extended to design of progressive censoring sampling plan and other censoring schemes can also be considered.

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