The Asymptotic Analysis of the Solution of an Elasticity Theory Problem for a Transversely Isotropic Hollow Cylinder with Mixed Boundary Conditions on the Side Surface

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Abstract: The problem of elasticity theory for the transversely isotropic hollow cylinder with mixed conditions on the side surface is considered in the paper. Transcendental equations are obtained regarding the eigenvalues of the problem. The roots of the characteristic equations are studied thoroughly. The study of the eigenvalues allowed to establish the essential characteristics of the stress-strain state of an anisotropic shell in comparison with isotropic shells. Homogeneous solutions were built here.

Keywords: Theory of Elasticity, Transversely Isotropic Hollow Cylinder, Side Surface, Mixed Boundary Conditions, Stress-Strain State, Eigenvalues, Transcendental Equation, Anisotropic Shell

1. Introduction

The modern theory of shells is deeply developed section of the mechanics of a deformable solid. However, the calculation of shells on the basis of three-dimensional equations of the theory of elasticity is associated with considerable mathematical difficulties. Therefore, it is necessary to apply to a variety of approximate methods to simplify the calculation of shells. Many methods of bringing the three-dimensional problem to a two-dimensional one use small shell thickness compared to its other dimensions in the constructions. Among them a special place is occupied by the asymptotic method. The asymptotic methods of integrating the equations of two-dimensional shell theory obtained the great development in A. L. Goldenveiser's papers [1], [2]. V. V. Novozhilov's [3] combination of complex transformation of equations of the shell theory with the asymptotic methods is presented in K. F. Chernykh's works [4], [5]. With regard to the study of three-dimensional stress-strain state of elastic

bodies the development of an asymptotic method belongs to K. Fridrix, L. Dressler [6], [7], A.L. Goldenweiser, I.I.Vorovich [8], [9], [10]. Further development of the asymptotic method went in two directions. In the first one the solution of the elasticity problem for thin bodies is carried out by means of direct integration of elasticity equations with the help of two iterative processes. This direction is developed in the works of A.L. Goldenveiser, M.I. Huseyn-Zade, A.V. Kolos [11], [12] and L.A. Agalovyan [13].

The second approach is based on the investigation of a system of uniform solutions. The above-mentioned direction was developed by I.I. Vorovich [8], [9], D.C. Aksentyan [14], [15], O.S. Malkina [16], [17], N.N. Bazarenko [18], [19], T.A. Vilenskaya [20], Y.A. Ustinov [21], [22], [23] and M.F. Mekhtiev [24], [25].

Thus, the asymptotic method developed by A. L. Goldenweiser, I. I. Vorovich, V. B. Lidski [26] and their followers [27], [28], [29], [13], [30], [31], has made significant contribution to the development of the theory of

plates and shells.

2. Statement of the Problem and Its Solution

Let the cylinder occupies a volume

$$\Gamma = \{r \in [R_1, R_2], \ \varphi \in [0, 2\pi], \ z \in [-l, l]\}$$

The equilibrium equations in displacements are in the form [32]:

$$b_{11}\left(\Delta_0 u_\rho - \frac{u_\rho}{\rho}\right) + \frac{\partial^2 u_\rho}{\partial \xi^2} + (1 + b_{13})\frac{\partial^2 u_\xi}{\partial \rho \partial \xi} = 0$$

$$(1 + b_{13})\frac{\partial}{\partial \xi}\left(\frac{\partial u_\rho}{\partial \rho} + \frac{u_\rho}{\rho}\right) + \Delta_0 u_\xi + b_{33}\frac{\partial^2 u_\xi}{\partial \xi^2} = 0 \qquad (1)$$

here

$$\rho = R_0^{-1} \cdot r, \ \ \xi = R_0^{-1} \cdot z, \ \ u_\rho = R_0^{-1} \cdot u_r, \ \ u_\xi = R_0^{-1} \cdot u_z$$

$R_0 = \frac{1}{2}(R_1 + R_2)$ is the radius of the middle surface of the shell,

$$m\,b_{11} = 2G_0(1 - v_1 v_2), \ \ m\,b_{13} = 2G_0 v_1(1 + v),$$

$$m\,b_{33} = 2G_0(1 - v^2)$$

$$b_{12} = b_{11} - 2G_0, \ \ E_0 = E_1 \cdot E^{-1}, \ \ G_0 = G \cdot G_1^{-1},$$

$$v_2 = E_0 \cdot v_1$$

$m = 1 - v - 2v_1 v_2$ are dimensionless quantities, E, G, v are isotropic material constants, E_1, G_1, v_1 are material constant in a plane perpendicular to the plane of isotropy.

The relations of the generalized Hooke's law are [33]:

$$\begin{aligned}
\sigma_r &= G_1\big(b_{11}\varepsilon_r + b_{12}\varepsilon_\varphi + b_{13}\varepsilon_z\big) \\
\sigma_\varphi &= G_1\big(b_{12}\varepsilon_r + b_{11}\varepsilon_\varphi + b_{13}\varepsilon_z\big) \\
\sigma_z &= G_1\big[b_{12}(\varepsilon_r + \varepsilon_\varphi) + b_{33}\varepsilon_z\big] \\
\tau_{rz} &= G\varepsilon_{rz}
\end{aligned} \qquad (2)$$

where

$$\varepsilon_r = \frac{\partial u_\rho}{\partial \rho}, \ \ \ \varepsilon_\varphi = \frac{u_\rho}{\rho}, \ \ \ \varepsilon_z = \frac{\partial u_\xi}{\partial \xi},$$

$$\varepsilon_{rz} = \frac{\partial u_\rho}{\partial \xi} + \frac{\partial u_\xi}{\partial \rho}$$

Suppose that the boundary conditions are given at the cylinder side surface

$$u_r = 0, \ \ \tau_{rz} = 0 \ \text{for} \ r = r_s \ (s = 1, 2) \qquad (3)$$

The nature of the boundary conditions at the ends of the cylinder is not defined yet, but we assume them so that the cylinder is in equilibrium state.

The solution (1), (3) will be sought in the form of:

$$u_\rho = u(\rho)\frac{dm}{d\xi}, \ \ \ u_\xi = W(\rho)m(\xi) \qquad (4)$$

where the function $m(\xi)$ is subjected to the condition:

$$\frac{d^2 m}{d\xi^2} - \mu^2 m(\xi) = 0 . \qquad (5)$$

Substituting (4) into (1) to (5), we obtain the following boundary value problem

$$b_{11}\left(u' + \frac{u}{\rho}\right)' + \mu^2 u + (1 + b_{13})W' = 0$$

$$(1 + b_{13})\mu^2\left(u' + \frac{u}{\rho}\right) + W'' + \frac{1}{\rho}W' + b_{33}\mu^2 W = 0 \qquad (6)$$

$$u = 0, \ \ \mu^2 u + W' = 0 \ \text{for} \ \rho = \rho_s \qquad (7)$$

The general solution of (6) has the form of:

$$u(\rho) = \big(b_{33}\mu^2 - \alpha_1^2\big)Z_1(\alpha_1\rho) + \big(b_{33}\mu^2 - \alpha_2^2\big)Z_1(\alpha_2\rho),$$

$$W(\rho) = -(b_{13} + 1)\mu^2\big[\alpha_1 Z_0(\alpha_1\rho) + \alpha_2 Z_0(\alpha_2\rho)\big]. \qquad (8)$$

Here $\quad Z_k(\rho) = C_1 J_k(\rho) + C_2 Y_k(\rho) \quad$, the functions $J_k(\rho), \ Y_k(\rho)$ are linearly independent solutions of the Bessel equation, C_1, C_2 are the arbitrary constants.

$\alpha_n = \sqrt{t_n}$, t_n are the roots of a quadratic equation:

$$t^2 - 2q_1\mu^2 t + q_2\mu^4 = 0;$$

$$q_1 = \frac{v_1}{v_2}(1 - v_1 v_2)^{-1}(1 + v)(G_o - v_2)$$

$$q_2 = \frac{v_1}{v_2}(1 - v_1 v_2)^{-1}(1 - v^2), \qquad (9)$$

$$\alpha_n = \mu s_n, \ s_n = \left[q_1 + (-1)^n \sqrt{q_1^2 - q_2}\right]^{1/2}$$

By satisfying the homogeneous boundary conditions (7), we obtain the characteristic equation

$$\Delta(\mu, \rho_1, \rho_2) = b_{33}\mu^4\big(s_2^1 - s_1^2\big)^2 \times$$

$$L_{11}(\alpha_1\rho_1, \ \alpha_1\rho_2)\,L_{11}(\alpha_2\rho_1, \ \alpha_2\rho_2) = 0 \qquad (10)$$

where

$$L_{11}(\alpha x, \alpha y) = J_1(\alpha x) \cdot Y_1(\alpha y) - J_1(\alpha y) \cdot Y_1(\alpha x) .$$

The transcendental equation (10) defines a countable set of roots μ_k, and the corresponding constants $c_{1n}, c_{2n}, c_{3n}, c_{4n}$ are proportional to the cofactors of some row of the determinant of the system. Choosing the cofactors of the elements of the

first row as a system of solutions, the solutions to system (1) can be written as:

$$u_\rho = \sum_{n=1}^{\infty} C_n U_n(\rho)\frac{dm_n}{d\xi}$$

$$u_\xi = \sum_{n=1}^{\infty} C_n W_n(\rho) m_n(\xi) \qquad (11)$$

where C_n are arbitrary constants.

$$U_n(\rho) = \Big[\alpha_1^4\big(b_{33}\mu_n^2 - \alpha_2^2\big)^2 -$$
$$-\alpha_1^2\alpha_2^2\big(b_{33}\mu_n - \alpha_1^2\big)\big(b_{33}\mu_n - \alpha_2^2\big)\Big]\times$$
$$\times L_{11}\big(\alpha_2\rho,\,\alpha_2\rho_2\big) L_{11}\big(\alpha_1\rho_1,\,\alpha_1\rho_2\big) +$$
$$+\Big[\alpha_2^4\big(b_{33}\mu_n^2 - \alpha_1^2\big)^2 -$$
$$-\alpha_1^2\alpha_2^2\big(b_{33}\mu_n - \alpha_1^2\big)\big(b_{33}\mu_n - \alpha_2^2\big)\Big]\times$$
$$\times L_{11}\big(\alpha_1\rho,\,\alpha_1\rho_2\big) L_{11}\big(\alpha_2\rho_1,\,\alpha_2\rho_2\big);$$

$$W_n(\rho) = -(b_{13}+1)\Big\{\alpha_1\Big[\alpha_2^4\big(b_{33}\mu_n - \alpha_1^2\big) -$$
$$-\alpha_1^2\alpha_2^2\big(b_{33}\mu_n - \alpha_2^2\big)\Big]\times$$
$$\times L_{01}\big(\alpha_1\rho,\,\alpha_1\rho_2\big) L_{11}\big(\alpha_2\rho_1,\,\alpha_2\rho_2\big) +$$
$$+\alpha_2\Big[\alpha_1^4\big(b_{33}\mu_n - \alpha_2^2\big) -$$
$$-\alpha_1^2\alpha_2^2\big(b_{33}\mu_n - \alpha_1^2\big)\Big] L_{01}\big(\alpha_2\rho,\,\alpha_2\rho_2\big) L_{11}\big(\alpha_1\rho_1,\,\alpha_1\rho_2\big)\Big\}.$$

As for the stresses, they can be determined by the generalized Hooke's law [33], [34].

3. The Asymptotic Analysis of the Problem

The left side of equation (1.10) as an entire function of the parameter μ has a countable set of zeros with the accumulation point at infinity. For effective study of its zeros we'll assume that the shell is thin-walled.

Let us assume that

$$\rho_1 = 1 - \varepsilon,\ \ \rho_2 = 1 + \varepsilon,$$
$$\varepsilon = (2R_0)^{-1}(R_2 - R_1), \qquad (12)$$
$$\rho = 1 + \varepsilon\eta,\ \ -1 \le \eta \le 1$$

We believe that ε is a small parameter $\mu = 0$. Substituting (12) into (10) we obtain

$$D(\mu,\varepsilon) = \Delta(\mu,\rho_1,\rho_2) = 0 \qquad (13)$$

Equation (13) has one restricted root. From (11) we find that this corresponds to the root of the following decision:

$$u_\rho = 0\ ;\ \ u_\xi = m\big[G_0(1+\nu)\big]^{-1} C_0\xi\ ;$$

$$\sigma_\phi = \sigma_r = G_1\nu_1 C_0\ ;\ \ \sigma_z = (1-\nu)G_1 C_0\ ;\quad (14)$$
$$\tau_{rz} = 0\ ,$$

C_0 is an arbitrary constant.

Stress state corresponding to zero $\mu = 0$ is equivalent to the principal vector P of stresses directed along the axis of the cylinder.

$$P = \pi(1-\nu)G_1 C_0\big(R_2^2 - R_1^2\big)$$

Hence

$$C_0 = P\big[\pi(1-\nu)G_1\big(R_2^2 - R_1^2\big)\big]^{-1} \qquad (15)$$

Let us prove that the characteristic equation (13) at $\varepsilon \to 0$ does not have any other restricted roots. For this purpose, we expand $D(\mu,\varepsilon)$ in a series in ε and confine ourselves to the first terms of the expansion. We get

$$D(\mu,\varepsilon) = 16\mu^4\big(s_2^2 - s_1^2\big)^2 \cdot \pi^{-2}\cdot\varepsilon^2\big[1 + O(\varepsilon)\big] \qquad (16)$$

This shows that the characteristic equation has no other restricted roots besides $\mu = 0$. Thus, all the roots of the characteristic equation tend to infinity as $\varepsilon \to 0$.

In principle, there could be the following limiting cases:
1) $\varepsilon\mu_k \to 0$ at $\varepsilon \to 0$; 2) $\varepsilon\mu_k \to \infty$ at $\varepsilon \to 0$;
3) $\varepsilon\mu_k \to const$ at $\varepsilon \to 0$.

As in [32], we can prove that the cases 1 and 2 are not feasible. In the third case, we seek μ_n in the form:

$$\mu_n = \varepsilon^{-1}\cdot\delta_n + 0(\varepsilon) \qquad (n=1,2,\ldots) \qquad (17)$$

As in the [32], the following cases are possible here:
1) $\mu_{1,2} = \pm s_1\delta_n,\ \ \mu_{3,4} = \pm s_2\delta_n,$

$$q_1 > 0,\ \ q_1^2 - q_2 > 0,\ \ s_i^2 = \delta_n^2\tau_i\ \ (i=1,2)$$

$$s_{1,2} = \sqrt{q_1 \pm \sqrt{q_1^2 - q_2}},\ \ q_1^2 > q_2$$

$$s_{1,2} = \chi + i\beta = \sqrt{q_1 \pm i\sqrt{q_2 - q_1^2}},\ \ q_1^2 < q_2\ .$$

2) The roots of the characteristic equation (9) are multiple.

$$\mu_{1,2} = \mu_{3,4} = \pm\delta_n\cdot p,$$

$$q_1 > 0\ \ q_1^2 - q_2 = 0,\ \ p = \sqrt{q_1}$$

3) $\mu_{1,2} = \pm i s_1\delta_n,\ \ \mu_{3,4} = \pm i s_2\delta_n,$

$$q_1 < 0\ ,\ \ q_1^2 - q_2 \ne 0$$

$$s_{1,2} = \sqrt{|q_1| + \sqrt{q_1^2 - q_2}},\ \ q_1^2 > q_2$$

$$s_{1,2} = \sqrt{|q_1| \pm i\sqrt{q_2 - q_1^2}},\ \ q_1^2 < q_2$$

4) $\mu_{1,2} = \mu_{3,4} = \pm i\,\delta_n p,$

$$q_1 < 0,\quad q_1^2 - q_2 = 0,\quad p = \sqrt{|q_1|}$$

In cases 1 and 2, after substituting (17) into (9) and its transformation using a series expansion in ε we get:

$$\cos(s_2 + s_1)\delta_n \pm \cos(s_2 - s_1)\delta_n = 0 \qquad (18)$$

$$\cos 2p\delta_n \pm 1 = 0 \qquad (19)$$

$$ch2x\delta_n \pm \cos 2\beta\delta_n = 0. \qquad (20)$$

With regard to the cases 3 and 4, their results are obtained from cases 1 and 2 by a formal replacement of s_1, s_2 into is_1, is_2, and of p into $i\,p$. These equations coincide with the equations determining the performance of Saint-Venant's edge effects in an anisotropic elasticity theory for a layer.

The table 1 shows the values of the coefficients for some materials:

***Table 1.** The coefficients for some materials.*

	magnesium	cadmium	zink
q_1	1,276	0,725	0,281
q_2	1,032	0,425	0,378
$q_1^2 - q_2$	0,595	0,101	-0,299

4. Asimptotic Analysis of Stress-Strain State

We now present the first terms of the asymptotic expansions of solutions, co-responding to different groups of roots. For displacements and stresses, in the first approximation, we get two classes of solutions, the first of which corresponds to the zeros

$$\cos(s_2 + s_1)\delta_n + \cos(s_2 - s_1)\delta_n,\quad \cos 2p\delta_n + 1,$$
$$ch2x\delta_n + \cos 2\beta\delta_n,$$

and the second – to the zeros of the function $\cos(s_2 - s_1)\delta_n - \cos(s_2 - s_1)\delta_n,\quad \cos 2p\delta_n - 1,$
$ch2x\delta_n - \cos 2\beta\delta_n$
respectively, we have

$$u_{\rho 0} = \varepsilon \sum_{n=1,3,\ldots}^{\infty} C_n \Big[s_2^2 \big(b_{33} - s_1^2 \big) \cos s_2\delta_n \cdot \cos s_1\delta_n\eta -$$
$$s_1^2 \cos s_1\delta_n \cdot \cos s_2\delta_n\eta + 0(\varepsilon) \Big]\frac{dm_n}{d\xi};$$

$$u_{\xi 0} = (b_{13} + 1)s_1 s_2 \sum_{n=1,3,\ldots}^{\infty} C_n\delta_n \big[s_2 \cos s_2\delta_n \cdot \sin s_1\delta_n\eta -$$
$$s_1 \cos s_1\delta_n \cdot \sin s_2\delta_n\eta + 0(\varepsilon) \big]m_n(\xi);$$

$$\sigma_{r0} = G_1 s_1 s_2 \times$$
$$\sum_{n=1,3,\ldots}^{\infty} C_n\delta_n \Big[s_2 \big(b_{13}^2 + b_{13} - b_{11}b_{33} + b_{11}s_1^2 \big)\cos s_2\delta_n \cdot \sin s_1\delta_n\eta -$$
$$- s_1 \big(b_{13}^2 + b_{13} - b_{11}b_{33} + b_{11}s_2^2 \big)\cos s_1\delta_n \cdot \sin s_2\delta_n\eta + 0(\varepsilon) \Big] \times \qquad (21)$$
$$\times \frac{dm_n}{d\xi};$$

$$\sigma_{\phi 0} = G_1 s_1 s_2 \times$$
$$\sum_{n=1,3,\ldots}^{\infty} C_n\delta_n \Big[s_2 \big(b_{13}^2 + b_{13} - b_{12}b_{33} + b_{12}s_1^2 \big)\cos s_2\delta_n \cdot \sin s_1\delta_n\eta -$$
$$- s_1 \big(b_{13}^2 + b_{13} - b_{12}b_{33} + b_{12}s_2^2 \big)\cos s_1\delta_n \cdot \sin s_2\delta_n\eta + 0(\varepsilon) \Big] \times$$
$$\times \frac{dm_n}{d\xi};$$

$$\sigma_{z0} = G_1 s_1 s_2 \times$$
$$\sum_{n=1,3,\ldots}^{\infty} C_n\delta_n \Big[s_2 \big(b_{13}b_{33} + b_{33} - b_{12}b_{33} + b_{12}s_1^2 \big)\cos s_2\delta_n \cdot \sin s_1\delta_n\eta -$$
$$- s_1 \big(b_{13}b_{33} + b_{33} - b_{12}b_{33} + b_{12}s_2^2 \big)\cos s_1\delta_n \cdot \sin s_1\delta_n\eta + 0(\varepsilon) \Big] \times$$
$$\times \frac{dm_n}{d\xi};$$

$$\tau_{rz0} = G \cdot \frac{1}{\varepsilon} \sum_{n=1,3,\ldots}^{\infty} C_n\delta_n^2 \Big[s_2^2 \big(b_{13}s_1^2 + b_{33} \big)\cos s_2\delta_n \cdot \cos s_1\delta_n\eta -$$
$$- s_1^2 \big(b_{13}s_2^2 + b_{33} \big)\cos s_1\delta_n \cdot \cos s_2\delta_n\eta + 0(\varepsilon) \Big]m_n(\xi)$$

$$u_{\rho 0} = \varepsilon \times$$
$$\sum_{n=1,3,\ldots}^{\infty} B_n \Big\{ p\big[p\delta_n(b_{13}+1)\sin \rho\delta_n - (b_{13}+2)\cos p\delta_n \big]\cos p\delta_n\eta -$$
$$- (b_{13}+1)p^2\delta_n \cdot \eta\cos p\delta_n \cdot \sin p\delta_n\eta + 0(\varepsilon) \Big]\frac{dm_n}{d\xi};$$

$$u_{\xi 0} = (b_{13}+1)p \times$$
$$\sum_{n=1,3,\ldots}^{\infty} \Big[C_n\delta_n \Big\{ \Big[\sin p\delta_n + \frac{\cos p\delta_n}{p\delta_n(b_{13}+1)} \Big]\sin p\delta_n\eta +$$
$$+ \eta\cos p\delta_n \cdot \cos p\delta_n\eta + 0(\varepsilon) \Big\}m_n(\xi);$$

$$\sigma_{r0} = G_1 \sum_{n=1,3,\ldots}^{\infty} B_n \Big\{ \big[(b_{13}+1)p\,\delta_n\big(b_{13} - b_{11}p^2\delta_n \big)\sin p\delta_n +$$
$$+ \big(b_{13} + b_{11}p^2\delta_n \big)\cos p\delta_n \big]\sin p\delta_n\eta +$$
$$+ (b_{13}+1)\big(b_{13} - b_{11}p^2\delta_n \big)\cdot\eta\cos p\delta_n \cos p\delta_n\eta + 0(\varepsilon) \Big\}\frac{dm_n}{d\xi};$$

$$\sigma_{\phi 0} = G_1 \sum_{n=1,3,\ldots}^{\infty} C_n \Big\{ \big[(b_{13}+1)p\,\delta_n\big(b_{13} - b_{12}p^2\delta_n \big)\sin p\delta_n +$$
$$+ \big(b_{13} + b_{12}p^2\delta_n \big) \big]\sin p\delta_n\eta +$$
$$+ (b_{13}+1)\big(b_{13} - b_{12}p^2\delta_n \big)\cdot\eta\cos p\delta_n \cdot \cos p\delta_n\eta + 0(\varepsilon) \Big\}\frac{dm_n}{d\xi};$$

$$\sigma_{z0} = G_1 \sum_{n=1,3,\dots}^{\infty} B_n \Big\{ \big[(b_{13}+1) p\, \delta_n \big(b_{33} - b_{12} p^2 \delta_n \big) \sin p\delta_n +$$

$$+ \big(b_{33} + b_{12} p^2 \delta_n \big) \cos p\delta_n \big] \sin p\delta_n \eta +$$

$$+ (b_{13}+1)\big(b_{33} - b_{12} p^2 \delta_n \big) \cdot \eta \cos p\delta_n \cdot \cos p\delta_n \eta + 0(\varepsilon) \Big\} \frac{dm_n}{d\xi};$$

$$\tau_{rz0} = \frac{2G(b_{13}+1)}{\varepsilon} \times$$

$$\sum_{n=1,3,\dots}^{\infty} B_n \big[\delta_n^3 \big(\sin p\delta_n \cdot \cos p\delta_n \eta - \eta \cos p\delta_n \cdot \sin p\delta_n \eta \big) + 0(\varepsilon) \big] \times \quad (22)$$

$$\times m_n(\xi);$$

Expressions for $n = 2, 4, 6, \dots$ are obtained from the formulas (21), (22) by the replacement of $\cos x$ into $\sin x$ and of $\sin x$ into $-\cos x$ respectively.

$$u_{\rho 0} = \varepsilon \sum_{n=1,3,\dots}^{\infty} D_n \Big\{ \big[(b_{33} + \beta^2 - \chi^2) \cos \beta\delta_n \eta\, ch\chi\delta_n\eta +$$

$$+ 2\chi\beta \sin \beta\delta_n\eta \cdot sh\chi\delta_n\eta \big) \Delta_{1n} -$$

$$- \big[(b_{33} + \beta^2 - \chi^2) \sin \beta\delta_n\eta\, sh\,\delta_n\eta -$$

$$- 2\chi\beta \cos \beta\delta_n\eta \cdot ch\chi\delta_n\eta \big) \Delta_{2n} + 0(\varepsilon) \Big\} \frac{dm_n}{d\xi};$$

$$u_{\xi 0} = (b_{13}+1) \times$$

$$\sum_{n=1,3,\dots}^{\infty} D_n \Big[\big(\beta \sin \beta\delta_n\eta\, ch\chi\delta_n\eta - \chi \cos \beta\delta_n\eta \cdot sh\chi\delta_n\eta \big) \Delta_{1n} +$$

$$+ \big(\chi \sin \beta\delta_n\eta\, ch\chi\delta_n\eta + \beta \cos \beta\delta_n\eta\, sh\chi\delta_n\eta \big) \Delta_{2n} + 0(\varepsilon) \Big] \times$$

$$\times m_n(\xi);$$

$$\sigma_{r0} = G_1 \sum_{n=1,3,\dots}^{\infty} D_n \big[b_{11} F'_{n1}(\eta) + b_{13} F_{n2}(\eta) + 0(\varepsilon) \big] \frac{dm_n}{d\xi};$$

$$\sigma_{\phi 0} = G_1 \sum_{n=1,3,\dots}^{\infty} D_n \big[b_{12} F'_{n1}(\eta) + b_{13} F_{n2}(\eta) + 0(\varepsilon) \big] \frac{dm_n}{d\xi};$$

$$\sigma_{z0} = G_1 \sum_{n=1,3,\dots}^{\infty} D_n \big[b_{12} F'_{n1}(\eta) + b_{33} F_{n2}(\eta) + 0(\varepsilon) \big] \frac{dm_n}{d\xi};$$

$$\tau_{rz0} = \frac{G}{\varepsilon} \sum_{n=1,3,\dots}^{\infty} D_n \big[\delta_n^2 F_{1n}(\eta) + F'_{n2}(\eta) + 0(\varepsilon) \big] m_n(\xi); \quad (23)$$

where

$$F_{n1}(\eta) = \big[(b_{33} + \beta^2 - \chi^2) \cos \beta\delta_n\eta\, ch\chi\,\delta_n\eta +$$

$$+ 2\chi\beta \sin \beta\delta_n\eta\, sh\chi\delta_n\eta \big] \Delta_{1n} -$$

$$- \big[(b_{33} + \beta^2 - \chi^2) \sin \beta\delta_n\eta\, sh\chi\,\delta_n\eta -$$

$$- 2\chi\beta \cos \beta\delta_n\eta\, ch\chi\delta_n\eta \big] \Delta_{2n};$$

$$F_{n2}(\eta) = \big(\beta \sin \beta\delta_n\eta\, ch\chi\delta_n\eta - \chi \cos \beta\delta_n\eta\, sh\chi\delta_n\eta \big) \Delta_{1n} +$$

$$+ \big(\chi \sin \beta\delta_n\eta\, ch\chi\delta_n\eta + \beta \cos \beta\delta_n\eta\, sh\chi\delta_n\eta \big) \Delta_{2n};$$

$$\Delta_{1n} = - \big[b_{33} + (b_{13}+2)(\beta^2 - \chi^2) \big] \sin \beta\delta_n sh\chi\delta_n +$$

$$+ 2\chi\beta(b_{13}+2) \cos \beta\delta_n ch\chi\delta_n;$$

$$\Delta_{2n} = - \big[b_{33} + (b_{13}+2)(\beta^2 - \chi^2) \big] \cos \beta\delta_n ch\chi\delta_n -$$

$$- 2\chi\beta(b_{13}+2) \sin \beta\delta_n sh\chi\delta_n;$$

Expressions for $n = 2, 4, 6, \dots$ are obtained from (23) by simply replacing $ch\chi \leftrightarrow sh\chi$; C_n, B_n, D_n are arbitrary constants.

We note that the solution (23) is characteristic only for anisotropic shells. It disappears completely in the transition to an isotropic shell $(G_0 = 1)$. With regard to the solutions (21) and (22) when $G_0 = 1$ they merge into one, and this solution coincides with the Saint Venant's solution for an isotropic plate.

In [32] a generalized condition of orthogonality of homogeneous solutions for the transverse isotropic hollow cylinder is proved, which allows to accurately satisfy the boundary conditions at the ends on special conditions of the shell edge bearing.

With the help of generalized orthogonality conditions, we consider the following problem: let the condition (3) satisfy on the side surface of the cylinder and the following boundary conditions be defined at the ends:

$$\sigma_z = \lambda(1 - c\eta^2), \quad u_r = 0 \quad \text{when } \xi = \pm l_0,$$

$2l_0$ is the dimensionless height of the cylinder.

According to (21) $u_r, u_z, \sigma_r, \tau_{rz}$ can be written as

$$u_r = \sum_{n=1}^{\infty} C_n u_n(\eta) \frac{dm_n}{dz}; \quad u_z = \sum_{n=1}^{\infty} C_n W_n(\eta) m_n(z);$$

$$\sigma_z = \sum_{n=1}^{\infty} C_n Q_n(\eta) \frac{dm_n}{dz}; \quad \tau_{rz} = \sum_{n=1}^{\infty} C_n T_n(\eta) m_n(z); \quad (24)$$

The summation in the series (24) is taken by the roots μ_n located in the upper half-plane $(\text{Im}\,\mu_n > 0)$. In view of the relations of a generalized orthogonality, the desired constants C_n have the form:

$$C_n = -\lambda \Delta_n^{-1} ch\mu_k \ell_0 \int_{-1}^{1} (1 - c\eta^2) W_n(\eta) d\eta$$

$$\Delta_n = \int_{-1}^{1} \big[u_n(\eta) T_n(\eta) - Q_n(\eta) W_n(\eta) \big] d\eta .$$

In general, the boundary value problem is reduced to solving systems of linear infinite algebraic equations using Lagrange variational principle.

5. Conclusion

The main results obtained in the article, the following:

1) There are obtained simple asymptotic formulas allowing to find strain-deformed state of cylindrical shell with given precision;

2) There is distinguished a class of solution (23) which is characteristic only for anisotropic shells and totally disappear on passage to isotropic case;

3) It is shown that stress-strain state of a cylindrical shell is a sum of interior stress-strain state and countable set of boundary-layer solutions which is localized near the shell edge;

4) For $G_0 = 1$ boundary-layer solutions totally coincide with Saint-Venan solution for anisotropic plate.

By the same method there were investigated various problems some of which we consider [35], [36], [37], [38], [39], [40].

One of our authers (Mekhtiyev M.F.) devoted two monographs to the elaboration of asymptotic method of integrating the equations of anisotropic theory of elasticity for plates and shells of variable thickness [24], [25].

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