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Some identities on the Higher-order Daehee and Changhee Numbers

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Abstract: In this note, we shall give an explicit formula for the coefficients of the expansion of given generating function, when that function has an appropriate form, the coefficients can be represented by the higher-order Daehee and Changhee polynomials and numbers of the first kind. By the classical method of comparing the coefficients of the generating function, we show some interesting identities related to the Higher-order Daehee and Changhee numbers.

Keywords: Higher-order Daehee Numbers, Higher-order Changhee Numbers, Bernoulli Number, Euler Number

1. Introduction

Throughout this paper let $D_n^{(r)}$ be the Daehee numbers of order $r \in \mathbb{N}$ are defined by the generating function[1,2]

$$\left(\frac{\log(1+t)}{t}\right)^r = \sum_{n=0}^{\infty} D_n^{(r)} \frac{t^n}{n!}, \tag{1}$$

where $D_n^{(1)} = D_n$ are called ordinary Daehee numbers, and let $Ch_n^{(r)}$ be the $n-th$ Changhee numbers of order $r \in \mathbb{N}$ are defined by the generating function[3] respectively.

$$\left(\frac{2}{t+2}\right)^r = \sum_{n=0}^{\infty} Ch_n^{(r)} \frac{t^n}{n!} \tag{2}$$

The Cauchy numbers of the first kind of order r denoted by $C_n^{(r)}$ are defined by the generating function[4]

$$\left(\frac{t}{\log(1+t)}\right)^r = \sum_{n=0}^{\infty} C_n^{(r)} \frac{t^n}{n!}. \tag{3}$$

We note that the Cauchy numbers of the first kind of order r, i.e., $C_n^{(r)}$ above are also called Nörlund polynomial with another notation $b_n^{(x)}$, say, $b_n^{(x)} = C_n^{(x)}$. The explicit formula for the $C_n^{(r)}$ (or $b_n^{(r)}$) and further results, readers may refer to [3-6].

The Bernoulli numbers of order $r \in \mathbb{N}$ are defined[7-9] by

$$\left(\frac{t}{e^t-1}\right)^r = \sum_{n=0}^{\infty} B_n^{(r)} \frac{t^n}{n!} \tag{4}$$

and the Euler numbers of order $r \in \mathbb{N}$ are defined[10,11] by

$$\left(\frac{2}{e^t+1}\right)^r = \sum_{n=0}^{\infty} E_n^{(r)} \frac{t^n}{n!}. \tag{5}$$

In this paper, we shall consider several interesting identities related to the Higher-order Daehee and Changhee numbers.

2. Generating Function Theorem

First we state auxiliary theorems which are useful in investigation power series.

Let $f(t)$ be a generating function (a power series) for a sequence $\{A_n\}$, we denote the sequence of coefficients of the expansion of $f(t)^r$ by $A_n^{(r)}$, where r is a fixed real nonzero number. By using the Stirling numbers $s(n,k)$ of the first kind which generated[12] by

$$(t)_n = t(t-1)\cdots(t-n+1) = \sum_{k=0}^{n} s(n,k)t^k \tag{6}$$

or by means of the generating function

$$\left(\log(1+t)\right)^k = k!\sum_{n=k}^{\infty} s(n,k)\frac{t^n}{n!}. \tag{7}$$

we may state the generating function theorem as

Theorem A[13]. Let $0 \neq r \in \mathrm{N}$, and let

$$f(t) = \sum_{n=0}^{\infty} \frac{A_n}{n!}t^n, \tag{8}$$

$$\left(f(t)\right)^r = \sum_{n=0}^{\infty} \frac{A_n^{(r)}}{n!}t^n \tag{9}$$

absolutely convergent in a neighborhood of the origin. Suppose $f(t)$ has a subsidiary generating function $g(t)$ so that

$$f(z) = (1+g(z))^{-1},\ |g(z)|<1, \text{ and } \left(g(t)\right)^n = \sum_{m=M(n)}^{\infty} \frac{\alpha_m^{(n)}}{m!}t^m. \tag{10}$$

We have

$$A_m^{(r)} = \sum_{k=1}^{M^{-1}(m)} \alpha(m,k)r^k, m \geq 1, \tag{11}$$

where $\alpha(m,k) = (-1)^k \sum_{n=k}^{M^{-1}(m)} \frac{1}{n!}s(n,k)\alpha_m^{(n)}$ and $M^{-1}(m)$ indicates the inverse function of m.

The lines of the proof is not difficult by using binomial expansion and base change, by the classical method for obtaining the values of the Riemann zeta-function at even positive integral arguments, i.e., by comparing the coefficients of $\frac{t^n}{n!}$ in the generating function $f(t)^r$, Eq. (11) follows.

By the Daehee polynomials of order $r \in \mathrm{N}$ are defined by the generating function[14]

$$\left(\frac{\log(1+t)}{t}\right)^r (1+t)^x = \sum_{n=0}^{\infty} D_n^{(r)}(x)\frac{t^n}{n!}, \tag{12}$$

and the Cauchy polynomials of the first kind of order r defined (cf.[14]) by

$$\left(\frac{t}{\log(1+t)}\right)^r (1+t)^x = \sum_{n=0}^{\infty} C_n^{(r)}(x)\frac{t^n}{n!}, \tag{13}$$

we have the generating function Theorem B below

Theorem B. Let $0 \neq r \in \mathrm{N}$, we have

$$A_m^{(r)} = \sum_{l=0}^{M^{-1}(m)} \sum_{n=l}^{M^{-1}(m)} \frac{1}{n!}\binom{n}{l} D_l^{(r)}(-r)C_{n-l}^{(r)}\alpha_m^{(n)} \tag{14}$$

$$A_m^{(r)} = \sum_{l=0}^{M^{-1}(m)} \sum_{n=l}^{M^{-1}(m)} \frac{1}{n!}\binom{n}{l} D_l^{(r)}C_{n-l}^{(r)}(-r)\alpha_m^{(n)}. \tag{15}$$

where $A_m^{(r)}, \alpha_m^{(n)}$ and $M^{-1}(m)$ are defined in Theorem A .

Proof of theorem B. By the binomial expansion, we have

$$f(t)^r = (1+g(t))^{-r} = \sum_{n=0}^{\infty} \binom{-r}{n} g(t)^n, \tag{16}$$

substituting Eq.(10) in Eq.(16), and change the order of summation, we obtain

$$f(t)^r = \sum_{n=0}^{\infty} \binom{-r}{n} \sum_{m=M(n)}^{\infty} \frac{\alpha_m^{(n)}}{m!}t^m = \sum_{m=0}^{\infty} \frac{t^m}{m!} \sum_{n=0}^{M^{-1}(m)} \binom{-r}{n}\alpha_m^{(n)} \tag{17}$$

comparing the coefficients of $\frac{t^n}{n!}$ in Eq.(9) and Eq.(17), we have

$$A_m^{(r)} = \sum_{n=0}^{M^{-1}(m)} \binom{-r}{n}\alpha_m^{(n)}. \tag{18}$$

Note that

$$\begin{aligned}
(1+t)^x &= \left(\frac{\log(1+t)}{t}\right)^r (1+t)^x \left(\frac{t}{\log(1+t)}\right)^r \\
&= \left(\sum_{l=0}^{\infty} D_l^{(r)}(x)\frac{t^l}{l!}\right)\left(\sum_{l=0}^{\infty} C_l^{(r)}\frac{t^l}{l!}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}D_l^{(r)}(x)C_{n-l}^{(r)}\right)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}D_l^{(r)}C_{n-l}^{(r)}(x)\right)\frac{t^n}{n!}
\end{aligned}$$

and

$$(1+t)^x = \sum_{n=0}^{\infty} (x)_n \frac{t^n}{n!},$$

we deduce

$$\binom{x}{n} = \frac{1}{n!}\sum_{l=0}^{n}\binom{n}{l}D_l^{(r)}(x)C_{n-l}^{(r)} = \frac{1}{n!}\sum_{l=0}^{n}\binom{n}{l}D_l^{(r)}C_{n-l}^{(r)}(x) \tag{19}$$

Replacing x by $-x$, and substituting $\binom{-x}{n}$ in Eq.(18)

We have

$$A_m^{(r)} = \sum_{n=0}^{M^{-1}(m)} \frac{1}{n!}\sum_{l=0}^{n}\binom{n}{l} D_l^{(r)}(-r)C_{n-l}^{(r)}\alpha_m^{(n)},$$

$$A_m^{(r)} = \sum_{n=0}^{M^{-1}(m)} \frac{1}{n!}\sum_{l=0}^{n}\binom{n}{l} D_l^{(r)}C_{n-l}^{(r)}(-r)\alpha_m^{(n)}.$$

whence, by changing the order of summation, we have Eq.(14) and Eq.(15).

We shall derive some explicit formulas of Daehee and Changhee numbers by using the argument of the interesting generating function theorems A and B.

3. The Explicit Formulas of Daehee and Changhee Numbers

In this section, we assume that t is in some neighborhood of origin, and $s(n,k)$ denotes the Stirling numbers of the first kind defined by Eq.(6) or Eq.(7) .

3.1. Identities Related to the Higher-order Daehee Numbers

From Eq. (1) and Eq.(4), we have

$$\sum_{n=0}^{\infty} D_n^{(r)}\frac{t^n}{n!} = \left(\frac{\log(1+t)}{t}\right)^r = \left(\frac{\log(1+t)}{e^{\log(1+t)}-1}\right)^r$$
$$= \sum_{n=0}^{\infty} B_n^{(r)}\frac{(\log(1+t))^n}{n!} = \sum_{n=0}^{\infty} B_n^{(r)}\sum_{n=m}^{\infty} s(n,m)\frac{t^n}{n!}$$
$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} B_m^{(r)} s(n,m)\right)\frac{t^n}{n!}$$

By comparing the coefficients, one has

$$D_n^{(r)} = \sum_{m=0}^{n} B_m^{(r)} s(n,m) . \tag{20}$$

We note that $f(t) = \frac{\log(1+t)}{t}$ and $g(t) = \frac{t}{\log(1+t)} - 1$,

Recall the Cauchy numbers $C_n^{(r)}$, See Eq.(3),

$$(g(t))^r = \left(\frac{t}{\log(1+t)} - 1\right)^r = \sum_{k=0}^{r}\binom{r}{k}\left(\frac{t}{\log(1+t)}\right)^k (-1)^{r-k}$$
$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{r}(-1)^{r-k}\binom{r}{k}C_n^{(k)}\right)\frac{t^n}{n!} \tag{21}$$

whence $\alpha_n^{(r)} = \sum_{k=0}^{r}(-1)^{r-k}\binom{r}{k}C_n^{(k)}$.

Theorem A gives

$$D_n^{(r)} = \sum_{k=1}^{n}\sigma(n,k)r^k \tag{22}$$

where $\sigma(n,k) = (-1)^k\sum_{j=k}^{n} s(j,k)\frac{1}{j!}\sum_{m=0}^{j}(-1)^{j-k}\binom{j}{m}C_n^{(m)}$.

and Theorem B gives

$$A_m^{(r)} = \sum_{l=0}^{m}\sum_{n=l}^{m}\frac{1}{n!}\binom{n}{l}D_l^{(r)}(-r)C_{n-l}^{(r)}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}C_n^{(k)} \tag{23}$$

$$A_m^{(r)} = \sum_{l=0}^{m}\sum_{n=l}^{m}\frac{1}{n!}\binom{n}{l}D_l^{(r)}C_{n-l}^{(r)}(-r)\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}C_n^{(k)} . \tag{24}$$

Note that

$$1 = \left(\frac{\log(1+t)}{t}\right)^r\left(\frac{t}{\log(1+t)}\right)^r = \left(\sum_{n=0}^{\infty} D_n^{(r)}\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty} C_n^{(r)}\frac{t^n}{n!}\right)$$
$$= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}D_l^{(r)}C_{n-l}^{(r)}\right)\frac{t^n}{n!}$$

We have the recurrence

$$C_0^{(r)} = D_0^{(r)} = 1, \sum_{l=0}^{n}\binom{n}{l}D_l^{(r)}C_{n-l}^{(r)} = 0, n \ge 1 . \tag{25}$$

From the generating function $f(t)^r$, and Eq.(7),

$$\sum_{n=0}^{\infty} D_n^{(r)}\frac{t^n}{n!} = \frac{1}{t^r}(\log(1+t))^r = r!\sum_{n=0}^{\infty} s(n+r,r)\frac{t^n}{(n+r)!} \tag{26}$$

comparing the coefficients of $\frac{t^n}{n!}$ in Eq.(26), we obtain

$$D_n^{(r)} = \frac{s(n+r,r)}{\binom{n+r}{r}} . \tag{27}$$

Rewrite $f(t)$ as

$$f(t) = \frac{\log(1+t)}{t} = 1 + \frac{1}{t}(\log(1+t) - t) ,$$

we have

$$(f(t))^r = \left(1 + \frac{1}{t}(\log(1+t) - t)\right)^r = \sum_{m=0}^{r}\binom{r}{m}\frac{1}{t^m}(\log(1+t) - t)^m$$

Using the associated Stirling numbers $d(n,k)$ defined by (cf.[5]),

$$(\log(1+t) - t)^k = k!\sum_{n=2k}^{\infty}(-1)^{n-k}d(n,k)\frac{t^n}{n!} , \tag{28}$$

Applying the binomial expansion, we have

$$\left(f(t)\right)^r=\sum_{m=0}^{r}\binom{r}{m}\frac{1}{t^m}m!\sum_{n=2m}^{\infty}(-1)^{n-m}d(n,m)\frac{t^n}{n!}$$
$$=\sum_{m=0}^{r}\binom{r}{m}\sum_{n=m}^{\infty}(-1)^n\frac{d(n+m,m)}{\binom{n+m}{m}}\frac{t^n}{n!}$$
$$=\sum_{n=0}^{\infty}(-1)^n\sum_{m=0}^{n}\binom{r}{m}\frac{d(n+m,m)}{\binom{n+m}{m}}\frac{t^n}{n!}$$

therefore by comparing the coefficients of $\frac{t^n}{n!}$ above gives

$$\mathrm{D}_{\mathrm{n}}^{(\mathrm{r})}=(-1)^n\sum_{m=0}^{n}\binom{r}{m}\frac{d(n+m,m)}{\binom{n+m}{m}}. \tag{29}$$

3.2. Identities Related to the Higher-order Changhee Numbers

Similarly, From Eq.(2) and Eq.(5), we have

$$\sum_{n=0}^{\infty}Ch_n^{(r)}\frac{t^n}{n!}=\left(\frac{2}{t+2}\right)^r=\left(\frac{2}{e^{\log(1+t)}+1}\right)^r$$
$$=\sum_{n=0}^{\infty}E_n^{(r)}\frac{\left(\log(1+t)\right)^n}{n!}=\sum_{n=0}^{\infty}E_n^{(r)}\sum_{n=m}^{\infty}s(n,m)\frac{t^n}{n!}$$
$$=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}E_m^{(r)}s(n,m)\right)\frac{t^n}{n!}$$

comparing the coefficients of $\frac{t^n}{n!}$, we have

$$Ch_n^{(r)}=\sum_{m=0}^{n}E_m^{(r)}s(n,m). \tag{30}$$

In the case of Eq.(2), $f(t)=\frac{2}{2+t}$, one has

$$f(t)^r=\left(1+\frac{1}{2}t\right)^{-r}=\sum_{m=0}^{\infty}\binom{-r}{m}\left(\frac{1}{2}t\right)^m, \tag{31}$$

then by Eq.(6), we may change the base,

$$\binom{-r}{m}=\frac{1}{m!}\sum_{k=1}^{m}s(m,k)(-r)^k \tag{32}$$

Substituting Eq.(32) in Eq.(31), we obtain

$$\sum_{n=0}^{\infty}Ch_n^{(r)}\frac{t^n}{n!}=\sum_{m=0}^{\infty}\sum_{k=1}^{m}(-1)^k\frac{s(m,k)}{2^m}r^k\frac{t^m}{m!} \tag{33}$$

comparing the coefficients of $\frac{t^n}{n!}$ in Eq.(33),

$$Ch_n^{(r)}=2^{-n}\sum_{k=1}^{n}(-1)^k s(n,k)r^k. \tag{34}$$

From Eq.(31), say, $g(t)=\frac{1}{2}t$ and $g(t)^m=\left(\frac{1}{2}t\right)^m$, which show $\alpha_m^{(m)}=\frac{m!}{2^m},\alpha_j^{(m)}=0, j\neq m$, in Eq.(10), Theorem A also gives the Eq.(34).

Similarly, Theorem B gives

$$Ch_n^{(r)}=2^{-n}\sum_{l=0}^{n}\binom{n}{l}D_l^{(r)}(-r)C_{n-l}^{(r)} \tag{35}$$

$$Ch_n^{(r)}=2^{-n}\sum_{l=0}^{n}\binom{n}{l}D_l^{(r)}C_{n-l}^{(r)}(-r)n. \tag{36}$$

3.3. Bernoulli and Euler Numbers Related to the Higher-order Daehee and Changhee Numbers

By the Eq.(4), $f(t)=\frac{t}{e^t-1}$ and $g(t)=\frac{1}{t}(e^t-1-t)$, and for the associated Stirling numbers $b(n,k)$ we have the expansion

$$(e^t-1-t)^r=r!\sum_{m=2r}^{\infty}b(m,k)\frac{t^m}{m!} \tag{37}$$

therefore,

$$g(t)^n=\left(\frac{1}{t}(e^t-1-t)\right)^n=\sum_{m=n}^{\infty}\frac{b(m+n,n)}{\binom{m+n}{n}}\frac{t^m}{m!} \tag{38}$$

where $M(n)=n$ and $\alpha_m^{(n)}=\frac{b(m+n,n)}{\binom{m+n}{n}}$.

Theorem B gives

$$B_m^{(r)}=\sum_{l=0}^{m}\sum_{n=l}^{m}\frac{1}{n!}\binom{n}{l}D_l^{(r)}(-r)C_{n-l}^{(r)}\frac{b(m+n,n)}{\binom{m+n}{n}} \tag{39}$$

$$=\sum_{l=0}^{m}\sum_{n=l}^{m}\frac{1}{n!}\binom{n}{l}D_l^{(r)}C_{n-l}^{(r)}(-r)\frac{b(m+n,n)}{\binom{m+n}{n}}. \tag{40}$$

In the case of Eq.(5), $f(t)=\frac{2}{e^t+1}=\frac{2}{\left(e^t-1\right)+2}$, from Eq.(2) we have

$$\sum_{n=0}^{\infty}E_n^{(r)}\frac{t^n}{n!}=\left(\frac{2}{\left(e^t-1\right)+2}\right)^r=\sum_{n=0}^{\infty}Ch_n^{(r)}\frac{\left(e^t-1\right)^n}{n!}, \tag{41}$$

Using the Stirling number of the second kind defined[5] by

$$x^n = \sum_{k=0}^{n} S(n,k)(x)_k \tag{42}$$

or by generating function

$$(e^t - 1)^k = k! \sum_{n=k}^{\infty} S(n,k) \frac{t^n}{n!} \tag{43}$$

We have

$$\sum_{n=0}^{\infty} E_n^{(r)} \frac{t^n}{n!} = \sum_{n=0}^{\infty} Ch_n^{(r)} \frac{\left(e^t - 1\right)^n}{n!} = \sum_{n=0}^{\infty} Ch_n^{(r)} \sum_{m=n}^{\infty} S(m,n) \frac{t^m}{m!},$$

Changing the order of summation, we have

$$E_n^{(r)} = \sum_{k=0}^{n} Ch_n^{(r)} S(k,n), \tag{44}$$

we rewrite $f(t) = \frac{2}{e^t + 1} = \frac{2}{\left(e^t - 1\right) + 2}$,say, $g(t) = \frac{1}{2}(e^t - 1)$,

By Eq.(7) and Eq.(36) and Theorem A gives

$$E_n^{(r)} = \sum_{k=1}^{n} (-1)^k \sum_{j=k}^{n} \frac{1}{2^j} S(n,j) s(n,j) r^k. \tag{45}$$

4. Remarks

In the cases of Eq. (3), $C_n^{(r)}$ the Cauchy numbers of the first kind of order r, is called Nörlund polynomial of the second kind and denoted by $b_n^{(r)}$, has well treated by several authors, a great deal of identities of $C_n^{(r)}$ (or $b_n^{(r)}$) has been derived. For example, readers may refer to Liu[6].

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