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On Existence of Eigen Values of Several Operator Bundles with Two Parameters

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Abstract: For the several operator bundles with two parameters when the number of equation is greater than the number of parameters in the Hilbert spaces is given the criterion of existence of the common point of spectra. In the special case the common point of spectra is the common eigen value. In the proof of the theorem the authors use the results of the spectral theory of multiparameter systems.

Keywords: Operator, Space, Resultant, Criterion, Eigenvector, Bundle

1. Introduction

Spectral theory of multiparameter systems arose as a result of studying questions related with the solutions of partial differential equations and equations of mathematical physics. Founder of the spectral theory of multiparameter systems is F.V Atkinson. Atkinson [1] studied the available at the time fragmentary results for multiparameter symmetric differential systems, built multiparameter spectral theory of multiparameter systems of operators in finite-dimensional Euclidean spaces. Further, by taking the limit Atkinson built the spectral theory of multiparameter systems with self-adjoint compact operators in infinite-dimensional Hilbert spaces.

Later, many of his followers have studied spectral problems of self-adjoint multiparameter systems.

2. Necessary Definitions and Notions

Consider the two parameter system

$$A_i(\lambda,\mu)x_i=(A_{0,i}+\lambda A_{1,i}+\ldots+\lambda^{m_1}A_{m_1}+\mu A_{m_1+1,i}+\ldots+\mu^{n_1}A_{m_1+n_1,i})x_i=0 \qquad (1)$$
$$i=1,2,\ldots,N$$

in the Hilbert spaces H_i ;

$H=H_1\otimes H_2\otimes\ldots\otimes H_N$ is the tensor product of spaces $H_1,H_2,\ldots,H_N$

We consider the case when the number of operator equations in (1) is greater than the number of parameters. Let the number of equations in (1) is equal to $N>2$. The case when $N=2$ is considered in the work [9].

In (1) linear operators $A_{k,i}(k=0,1,\ldots,m_i+n_i)$ act in the Hilbert space H_i .

Let $f_1\otimes f_2\otimes\ldots\otimes f_N\in H$ and $g_1\otimes g_2\otimes\ldots\otimes g_N\in H$ be two decomposable tensors .The inner product of these elements in the space $H_1\otimes H_2\otimes\ldots\otimes H_N$ is defined by means of the formulae

$$[f_1\otimes\ldots\otimes f_N,g_1\otimes\ldots\otimes g_N]_H=(f_1,g_1)_{H_1}\cdot(f_2,g_2)_{H_2}\cdots(f_N,g_N)_{H_N}$$

This definition is spread on other elements of the tensor product space $H=H_1\otimes H_2\otimes\ldots\otimes H_N$ on linearity and continuity

Let's reduce a series of known positions concerning the spectral theory of multiparameter system.

Definition1. ([1, 3]) $(\lambda,\mu)\in C^2$ is an eigen value of the system (1) if there are non-zero elements $x_i \in H_i$, $i=1,2,...,N$ such that (1) is fulfilled. A decomposable tensor $z = x_1 \otimes ... \otimes x_N$ is named an eigenvector of the system (1), corresponding to the eigenvalue $(\lambda,\mu)\in C^2$.

*Definition2. The o*perator $A^+_{k,i}$ is induced by an operator $A_{k,i}$, acting in the space H_i, into the tensor space $H = H_1 \otimes H_2 \otimes ... \otimes H_N$, if on each decomposable tensor $x = x_1 \otimes x_2 \otimes ... \otimes x_N$ of tensor product space $H = H_1 \otimes H_2 \otimes ... \otimes H_N$ we have $A^+_{k,i} = E_1 \otimes ... \otimes E_{i-1} \otimes A_{k,i} \otimes E_{i+1} \otimes ... \otimes E_N$ and on all the other elements of the space H operator $A^+_{k,i}$ is defined on linearity and continuity. E_i is an identical operator in H_i, $i=1,2,...,N$

Definition3.[5]. Let be two polynomial bundles

$$A(\lambda) = A_0 + \lambda A_1 + ... + \lambda^m A_m,$$

$$B(\lambda) = B_0 + \lambda B_1 + ... + \lambda^n B_n, \tag{2}$$

depending on the same parameter λ and acting, generally speaking, in various Hilbert spaces H_1, H_2, accordingly.

The operator $\operatorname{Re}s(A(\lambda),B(\lambda))$ is given with help of matrix

$$\operatorname{Re}s(A(\lambda),B(\lambda)) = \\ = \otimes \begin{pmatrix} A_0^+ & A_1^+ & A_2^+ & \cdots & A_m^+ & 0 & \cdots & 0 \\ 0 & A_0^+ & A_1^+ & \cdots & A_{m-1}^+ & A_m^+ & \cdots & 0 \\ . & . & . & \cdots & . & . & \cdots & . \\ 0 & 0 & 0 & \cdots A_0^+ & A_1^+ & A_2^+ & \cdots & A_m^+ \\ B_0^+ & B_1^+ & B_2^+ & \cdots & B_n^+ & 0 & \cdots & 0 \\ 0 & B_0^+ & B_1^+ & \cdots & B_{m-1}^+ & B_n^+ & \cdots & 0 \\ . & . & . & \cdots & . & . & \cdots & . \\ 0 & 0 & 0 & \cdots B_0^+ & B_1^+ & B_2^+ & \cdots & B_n^+ \end{pmatrix} \tag{3}$$

In [4,11] this operator is called an abstract analog of an Resultant for polynomial bundles (2).

In definition of a Resultant (3) of bundles (2) lines with operators A_i repeated n times, and lines with operators B_i repeated exactly m times Numbers m, n are the greatest degrees of parameter λ in bundles $A(\lambda)$ and $B(\lambda)$ accordingly. Thus, the Resultant is an operator, acting in space $(H_1 \otimes H_2)^{m+n}$ that is a direct sum $m+n$ of copies of tensor product spaces $H_1 \otimes H_2$. Value of $\operatorname{Re}s(A(\lambda),B(\lambda))$ is equal to its formal expansion when each term of this expansion is tensor product of operators. Let operator A_m, or B_n is invertible. If the greatest degrees of parameter λ in bundles of $A(\lambda)$ and $B(\lambda)$ coincide (see [11]) or if the greatest degrees of parameter λ in bundles of $A(\lambda)$ and $B(\lambda)$, generally speaking, can not coincide (see [4]) bundles (2) have a common eigen values in the only case when the Resultant of these bundles has non-zero kernel. In fact, that in case of when bundles act in finite-dimensional spaces, common point of spectra of these bundles is a common eigen value.

When in (1) the number $N=2$ the following result is proved in the work [9]:

Theorem1. Let operators $A_i (i=0,1,...,m_1+n_1)$ also $B_i (i=0,1,...,m_2+n_2)$ act in finite-dimensional spaces H_1 and H_2, accordingly, and one of three following conditions is fulfilled:

a) $\max(m_1n_2, m_2n_1) = m_1n_2$, $KerA_{m_1} = \{\theta\}, KerB_{m_2+n_2} = \{\theta\}$; $A_{m_1}, B_{m_2+n_2}$ are selfadjoint operators everyone in their space

b) $\max(m_1n_2, m_2n_1) = m_2n_1$, $KerB_{m_2} = \{\theta\}, KerA_{m_1+n_1} = \{\theta\}$ operators $B_{m_2} A_{m_1+n_1}$ are selfadjoint everyone in their spaces

c) $m_1n_2 = m_2n_1$, $Ker(A_{m_1}^{n_2} \otimes B_{m_2+n_2}^{n_1} + (-1)^{n_1n_2} A_{m_1+n_1}^{n_2} \otimes B_{m_2}^{n_1}) = \{\theta\}$, $A_{m_1}, B_{m_2}, A_{m_1+n_1}, B_{m_2+n_2}$ are the self-adjoint operators, acting everyone in its finite-dimensional space.

Then the $\max(m_1n_2, m_2n_1)$ -multiple basis of system of eigen and associated vectors of (2) takes place

This paper is devoted to the investigation of the system (1) when the N>2. The goal of our investigations is the finding the criterion of existing of common point of spectra of the system (1).

For proving we use the notion of criterion of existence of common point of spectra of several bundles with the same one parameter.

3. Necessary and Sufficient Conditions of Existence of Eigen Values of Several Operator Polynomials

Consider [7,10]

$$\begin{cases} B_i(\lambda) = B_{0,i} + \lambda B_{1,i} + ... + \lambda^{k_i} B_{k_i,i} \\ i = 1,2,...,n \end{cases} \tag{4}$$

when $B_i(\lambda)$ is an operator bundle with a discrete spectrum, acting in Hilbert space H_i ($i=1,2,...,n$). Without loss of generality, we assume that. $k_1 \ge k_2 \ge ... \ge k_n$. $H^{k_1+k_2}$ is the direct sum of k_1+k_2 copies of the tensor spaces $H = H_1 \otimes ... \otimes H_n$.

Introduce the operators R_i $(i=1,2,...,n-1)$ with help of the operator matrices (5)

$$R_{i-1} = \\ = \begin{pmatrix} B^+_{0,1} & B^+_{1,1} & \cdots & B^+_{k_1,1} & 0 & 0 & \cdots & 0 \\ 0 & B^+_{0,1} & \cdots & B^+_{k_1-1,1}, & B^+_{k_1,1} & 0 & \cdots & 0 \\ . & . & \cdots & . & . & . & \cdots & . \\ . & . & \cdots & B^+_{0,1} & B^+_{1,1} & . & \cdots & B^+_{k_1,1} \\ B^+_{0,i} & B^+_{1,i} & \cdots & B^+_{k_i-1,i} & B^+_{k_i-1,i} & 0 & \cdots & 0 \\ 0 & B^+_{0,i} & \cdots & B^+_{k_i-2,i} & B^+_{k_i-1,i} & B^+_{k_i,i} & \cdots & 0 \\ . & . & \cdots & . & . & . & \cdots & . \\ 0 & 0 & 0 & 0 & B^+_{0,i} & B^+_{1,i} & \cdots & B^+_{k_i,i} \end{pmatrix} \qquad (5)$$

Rows with the operators $B^+_{i,1}$ $(i = 0,1,...,k_1)$ are repeated k_2 times and rows with the operators $B^+_{s,i}(s = 0,1,...,k_2)$, $B_{k_i+1,i} = ... = B_{k_2,i} = 0$ are repeated $\max(m_1 n_2, m_2 n_1) = m_2 n_1$ times. The operators $B^+_{i,s}(i = 1,2,...,k_s; s = 2,...,k_i)$ are induced by an operator $B_{i,s}$, acting in the space $B_{i,s}$, into the space $H_1 \otimes \cdots \otimes H_N$ by the formulae

$$B^+_{i,s} = E_1 \otimes \cdots \otimes E_{s-1} \otimes B_{i,s} \otimes E_{s+1} \otimes \cdots \otimes E_n$$

Denote $\sigma_p\left(B_i(\lambda)\right)$ the set of eigen values of operator $B_i(\lambda)$.

Theorem 2.[10]. Let all operators $B_{j,k}$ are bounded in the corresponding spaces H_k, the operator $B_{k_1,1}$ has an inverse. Spectrum of each operator pencil $B_i(\lambda)$ contains only eigen values.

Then $\bigcap_{i=1}^{n} \sigma_p(B_i(\lambda)) \neq \{\vartheta\}$ if and only if

$$\bigcap_{i=2}^{n} Ker R_i \neq \{\theta\},\ (Ker B_{k_1} = \{\theta\}). \qquad (6)$$

This result is obtained in [10].

4. Criterion of Existence of Common Eigen Values of Two Polynomial Bundles with Two Parameters

We fix in the all operators (1) the parameter λ. Let be $\lambda = \lambda_0$. Then we have N operators, depending on the same parameter μ. Without loss of generality we suppose $n_1 \geq n_2 \geq ... \geq n_N$.

Then the system (1) turns out to the system

$$\begin{aligned} & A_i(\mu(\lambda_0))x_i = (A_{0,i} + \lambda_0 A_{1,i} + ... + \lambda_0^{m_1} A_{m_1} + \\ & + \mu A_{m_1+1,i} + ... + \mu^{n_1} A_{m_1+n_1,i})x_i = 0 \\ & i = 1,2,...,N \end{aligned} \qquad (7)$$

So the parameter λ is fixed arbitrarily we missed the index 0 in the λ_0. Denote though $A_i^+(\mu(\lambda))$ the operator in the space H, induced into H by the operator $A_i(\mu(\lambda))$ acting in the space H_i.

Let all operators $A^{-1}_{m_i+n_i,i}$, $i = 1,2,...,N$ exist and bounded.

Introduce the notations

$$\begin{aligned} & \tilde{A}_i(\lambda) = A_{0,i} + \lambda A_{1,i} + ... + \lambda^{m_i} A_{m_i,i} \\ & i = 1,2,...,N \end{aligned}$$

Construct the resultants of operators $A_1(\mu(\lambda))$ and $A_i(\mu(\lambda))$, $i = 2,...,N$.

$$R_{i-1} = \begin{pmatrix} \tilde{A}_1^+(\lambda) & A^+_{m_1+1,1} & \cdots & A^+_{m_1+n_1,,1} & 0 & 0 & \cdots & 0 \\ 0 & \tilde{A}_1^+(\lambda) & \cdots & A^+_{m_1+n_1-1,1} & A^+_{m_1+n_1,,1} & 0 & \cdots & 0 \\ . & . & \cdots & . & . & . & \cdots & . \\ . & . & \cdots & \tilde{A}_1^+(\lambda) & A^+_{m_1+1,1} & . & \cdots & A^+_{m_1+n_1,1} \\ \tilde{A}_i^+(\lambda) & A^+_{m_i+1,i} & \cdots & A^+_{m_i+n_i-1,i} & A^+_{m_i+n_i,i} & 0 & \cdots & 0 \\ 0 & \tilde{A}_i^+(\lambda) & \cdots & A^+_{m_i+n_i-2,i} & A^+_{m_i+n_i-1,i} & A^+_{m_i+n_i,i} & \cdots & 0 \\ . & . & \cdots & . & . & . & \cdots & . \\ 0 & 0 & 0 & 0 & \tilde{A}_i^+(\lambda) & A^+_{m_i+1,i} & \cdots & A^+_{m_i+n_2,i} \end{pmatrix} \qquad (8)$$

$$i = 2,...,N$$

The rows in (8) with the operators $\tilde{A}_1^+(\lambda), A^+_{m_1+1,1},..., A^+_{m_1+n_1,1}$ are repeated n_2 time, rows with the operators $\tilde{A}_i^+(\lambda), A^+_{m_i+1,i},..., A^+_{m_i+n_1,i}$ are repeated n_1 time for all meanings of i. The operators R_i act in the direct sum of $n_1 + n_2$ copies of tensor product space $H = H_1 \otimes ... \otimes H_N$. If $Ker A_{m_i+n_i,i} = \{\vartheta\}$ $i = 1,2,...,N$, $, Ker R_i = \{\vartheta\}$, all operators $A^{-1}_{m_1+n_1,1}$, $A^{-1}_{m_i+n_i,i}$ $\tilde{A}_i(\lambda), A_{m_i+1,i},..., A_{m_i+n_i,i}$ are bounded for all meanings of i. Suppose that the eigen and associated vectors of the bundle $A_1(\mu(\lambda))$ for each fixed λ form the basis of the space H_1. Consider the operators R_i. It is known they act in the direct sum of $n_1 + n_2$ copies of tensor product space $H = H_1 \otimes ... \otimes H_N$. From the results of the spectral theory of multiparameter system the first component of the element of the kernel of R_i is the linear combination of elements of an aspect

$$U_{i,1} \otimes E_2 \otimes ... \otimes E_{i-1} \otimes U_{0,i} \otimes ... \otimes E_N + U_{i-1,1} \otimes E_2 \otimes ... \otimes U_{1,i} \otimes ... \otimes E_N + ... + U_{0,1} \otimes ... \otimes U_{i,i} \otimes ... \otimes E_N, \tag{9}$$

when $U_{0.1}, U_{1,1}, ..., U_{s,1}$ ((accordingly, $U_{0,i}, U_{1,i}, ..., U_{s,i}$) there is a restricted chain of eigen and associated vectors of an operator $A_1(\mu(\lambda))$ (accordingly, $A_j(\mu(\lambda))$) , corresponding to some common eigenvalue $\mu(\lambda)$ of both operators. It is clear if $KerR_i \neq \{\vartheta\}$ and all other conditions are fulfilled the element of the type $\sum_{i+j=s} U_{0,i} \otimes E_2 \otimes ... \otimes E_{i-1} \otimes U_{0,j} \otimes ... \otimes E_N$, $s = 0,1,...$ form the first component of the element entering the kernel of the resultant of operator bundles $A_1(\mu(\lambda))$ and $A_j(\mu(\lambda))$

If the condition $Ker \bigcap R_i(\lambda) \neq \{\vartheta\}$ is fulfilled and the element $Z \in Ker \bigcap R_i(\lambda)$, $0 \neq Z \in H^{n_1+n_2}$ then this element $0 \neq Z \in H^{n_1+n_2}$ enters the kernel of resultant of operator bundles $A_1(\mu(\lambda))$ and $A_j(\mu(\lambda))$ for the first component of the element $0 \neq Z \in H^{n_1+n_2}$ expression (9) are fulfilled. From [12] it follows that element of the space H entering the $Ker \bigcap R_i(\lambda) \neq \{\vartheta\}$ is the linear combination of elements of the aspect

$$\sum_{i_1+...+i_N=s} U_{i_1,1} \otimes U_{i_2,2} \otimes ... \otimes U_{i_N,N} \tag{10}$$

when $U_{0.1}, U_{1,1}, ..., U_{s,1}$ (accordingly, $U_{0,i}, U_{1,i}, ..., U_{s,i}$) there is a restricted chain of e.a. vectors of an operator $A_1(\mu(\lambda))$ and operator $A_i(\mu(\lambda))$ $i = 2,...,N$, correspondingly ,

Naturally, when $s = 0$ in (10) decomposable tensor $U_{0,1} \otimes ... \otimes U_{0,i} \otimes ... \otimes U_{0,N}$ is the first component of the element of $Ker \bigcap R_i(\lambda) \neq \{\vartheta\}$ which has the form

($U_{0,1} \otimes ... \otimes U_{0,i} \otimes ... \otimes U_{0,N}$, $\mu U_{0,1} \otimes ... \otimes U_{0,N}$,... .., $\mu^{N-1} U_{0,1} \otimes ... \otimes U_{0,N}$). Element $U_{0,1} \otimes ... \otimes U_{0,i} \otimes ... \otimes U_{0,N}$ is the eigen vector of the system (1), corresponding to the eigen value $(\lambda, \mu(\lambda))$.

Denote though $\tilde{R}_i$ the decompositions of the resultants R_i. In fact, $\tilde{R}_i$ $(i = 1,2,...,N-1)$ are the operator bundles, acting in the space $H = H_1 \otimes ... \otimes H_N$.

We have the $N-1$ operator bundles with one parameter λ. The greatest degree of parameter λ in the operator bundle $\tilde{R}_i$ $(i = 1,2,...,N-1)$ is equal to $\max(m_1 n_2, m_{i+1} n_1)$. Moreover, we have the following result:

Let be all operators $A_{i,k}$ bounded in the corresponding space H_k , $KerA_{m_1+n_1,1} = \{\vartheta\}, KerA_{m_i+n_i,i} = \{\vartheta\}$ then the N operator bundles (7), depending on the parameter λ have the common point of spectra if and only if

$$\bigcap_{i=1}^{N-1} KerR_i(\lambda) \neq \{\theta\}.$$

Remark1. If spectrum of each operator bundle in (7) at each fixed meaning of the parameter λ contains only eigen values then the common point of spectra of these operator bundles is their common eigenvalue.

Remark 2. If the spaces H_k are finite dimensional spaces, then this common point of spectra of (7) is the common eigenvalue of (7).

So we have the $N-1$ operator bundles R_i $(i = 1,2,...,N-1)$, depending on one parameter λ and acting in the space H.

Without loss of generality we suppose

$$\begin{aligned} &\max(m_1 n_2, m_2 n_1) \geq \max(m_1 n_2, m_3 n_1) \geq ... \geq \\ &\geq \max(m_1 n_2, m_N n_1) \end{aligned} \tag{11}$$

Denote the operator coefficients of the parameter λ^s in the $\tilde{R}_i$ though $\ddot{\tilde{A}}_{s,i}$. $\max(m_1 n_2, m_{i+1} n_1) = M_i, i = 1,2,...,N-1$

All operators $\ddot{\tilde{A}}_{s,i}$ act in the space H.

Thus we have $N-1$ operator bundles, acting in the space H and depending on one same parameter:

$$\tilde{R}_i = \dddot{A}_{0,i} + \lambda \dddot{A}_{1,i} + \cdots + \lambda^{M_i} \dddot{A}_{M_i,i} \tag{12}$$

$$i = 1,...,N-1$$

Further, the proof is spent by analogy of the proof the *Theorem2.* Construct the resultants of the obtained operator bundles $\tilde{R}_1$ and $\tilde{R}_2$, $\tilde{R}_1$ and $\tilde{R}_3$,…, $\tilde{R}_1$ and $\tilde{R}_{N-1}$.

The resultants have the form

$$\tilde{\tilde{R}}_{i-1} =$$

$$= \begin{pmatrix} \dddot{A}_{0,1} & \dddot{A}_{1,1} & \cdots & \dddot{A}_{M_1 1} & 0 & 0 & \cdots & 0 \\ 0 & \dddot{A}_{0,1} & \cdots & \dddot{A}_{M_1-1,1} & \dddot{A}_{M_1,1} & 0 & \cdots & 0 \\ . & . & \cdots & . & . & . & \cdots & . \\ . & . & \cdots & \dddot{A}_{0,1} & \dddot{A}_{1,1} & . & \cdots & \dddot{A}_{M_1,1} \\ \dddot{A}_{0,i} & \dddot{A}_{1,i} & \cdots & \dddot{A}_{M_i-1,i} & \dddot{A}_{M_i,i} & 0 & \cdots & 0 \\ 0 & \dddot{A}_{0,i} & \cdots & \dddot{A}_{M_i-2,i} & \dddot{A}_{M_i-1,i} & \dddot{A}_{M_i,i} & \cdots & 0 \\ . & . & \cdots & . & . & . & \cdots & . \\ 0 & 0 & 0 & 0 & \dddot{A}_{0,i} & \dddot{A}_{1,i} & \cdots & \dddot{A}_{M_{i,i}} \end{pmatrix}$$

$i = 2,...,N-1$

In the resultant $\tilde{\tilde{R}}_{i-1}$ the rows with the operators $\dddot{A}_{0,1}, \dddot{A}_{1,1}, ..., \dddot{A}_{M_1,1}$ are repeated $\max(m_1 n_2, m_3 n_1)$ times, and the rows with the operators $\dddot{A}_{0,i}, \dddot{A}_{1,i}, ..., \dddot{A}_{M_i,i}$ are repeated $\max(m_1 n_2, m_2 n_1)$ times.

Theorem 3. The system of operator bundles (1) have a common points of spectra if and only if $Ker\bigcap_{i=1}^{N-2} \tilde{\tilde{R}}_i \neq \{\vartheta\}$ $Ker\bigcap_{i=1}^{N-1} R_i \neq \{\vartheta\}$, $Ker\dddot{A}_{M_i,i1} = \{\vartheta\}$, $KerA_{m_i+n_i,i} = \{\vartheta\}$, all operators, forming the system (1) and $A^{-1}_{m_i+n_i,i}$, $\dddot{A}^{-1}_{M_i,i}$ are bounded in corresponding spaces.

Proof of the *Theorem 3.*

Necessity. Let the system (1) has a common point (λ, μ) of spectra. Then for fixed first component of (λ, μ) the spectrum of each operator bundle contains only eigen values . From the Theorem1 it follows that $\bigcap_{i=1}^{N-1} KerR_i \neq \{\theta\}$. Last means that the system (12) has the eigen value λ and there is the non-zero decomposable tensor of the space H that following equations (13)

$$\tilde{R}_i z = (\dddot{A}_{0,i} + \lambda \dddot{A}_{1,i} + \cdots + \lambda^{M_i} \dddot{A}_{M_i,i}) z = 0 \qquad (13)$$

$i = 1, ..., N-1, \quad z = x_1 \otimes x_2 \otimes ... \otimes x_N$ are satisfied.

So (13) is fulfilled then the result of theorem2 demands the fulfilling of the condition $Ker\bigcap_{i=1}^{N-2} \tilde{\tilde{R}}_i \neq \{\vartheta\}$.

Sufficiency. Let be $\bigcap_{i=1}^{N-1} KerR_i \neq \{\theta\}$, $Ker\bigcap_{i=1}^{N-2} \tilde{\tilde{R}} \neq \{\vartheta\}$ $Ker\dddot{A}_{M_i,i1} = \{\vartheta\}$, $KerA_{m_i+n_i,i} = \{\vartheta\}$, operators $A^{-1}_{m_i+n_i,i}$ and $\dddot{A}^{-1}_{M_i,i}$ are bounded.

Let be $\bigcap_{i=1}^{N-1} KerR_i \neq \{\theta\}$. Last means there is non-zero element of the space $H^{n_1+n_2}$ (direct sum of the $n_1 + n_2$ copies of the space H), entering R_i . Each operator R_i is the resultant of the bundles $A_1(\mu(\lambda))$ and $A_{i+1}(\mu(\lambda))$ From $\bigcap_{i=1}^{N-1} KerR_i \neq \{\theta\}$ it follows the (12) are satisfied. Therefore , each equation in (13) means that the first operator bundle with the any other operator bundle has the common eigen value $\mu(\lambda)$ and common for all operator bundles eigenvector. $\mu(\lambda)$ is the common eigen value of the system(1) for this fixed λ.

All operators $\tilde{\tilde{R}}_i$ act in the same Hilbert space- direct sum of $\max(m_1 n_2, m_2 n_1) + \max(m_1 n_2, m_3 n_1)$ copies of the space H.

The condition $Ker\bigcap_{i=1}^{N-1} \tilde{\tilde{R}}_i \neq \{\vartheta\}$ means that the results of the theorem2 take place for the system (1). Consequently, the system (1) has a common eigen value.

Theorem3 is proved.

5. Nonlinear Algebraic System of Equations with Two Unknown Variables

Consider the algebraic system with two variables, when the number of equations is greater than 2.

$$\begin{aligned} &a_i(x, y) = a_{0,i} + a_{1,i}x + ... + a_{m_1,i}x^{m_i} + a_{m_1+1,i}y + ... + a_{m_1+n_1,i}y^{n_i} = 0 \\ &i = 1, 2, ..., N \end{aligned} \qquad (14)$$

Instead of the spaces H_i we adopt the space R and the variables x, y play the role of the parameters λ, μ , correspondingly. Numbers $a_{i,k}$ are the bounded operators in the space R .

The operators R_i and $\tilde{\tilde{R}}_i$ turn out to the form

$$R_{i-1} =$$

$$= \begin{pmatrix} a_1(x) & a_{m_1+1,1} & \cdots & a_{m_1+n_1,,1} & 0 & 0 & \cdots & 0 \\ 0 & a_1(x) & \cdots & a^{+}_{m_1+n_1-1,1} & a^{+}_{m_1+n_1,,1} & 0 & \cdots & 0 \\ . & . & \cdots & . & . & . & \cdots & . \\ . & . & \cdots & a_1(x) & a_{m_1+1,1} & . & \cdots & a_{m_1+n_1,1} \\ a^{+}_i(x) & a^{+}_{m_i+1,i} & \cdots & a_{m_i+n_i-1,i} & a_{m_i+n_i,i} & 0 & \cdots & 0 \\ 0 & a_i(x) & \cdots & a_{m_i+n_i-2,i} & a_{m_i+n_i-1,i} & a_{m_i+n_i,i} & \cdots & 0 \\ . & . & \cdots & . & . & . & \cdots & . \\ 0 & 0 & 0 & 0 & a_i(x) & a_{m_i+1,i} & \cdots & a_{m_i+n_2,i} \end{pmatrix}$$

$$i = 2, ..., N$$

when $a_i(x) = a_{0,i} + a_{1,i}x + ... + a_{m_1,i}x^{m_i}$, $i = 1, 2, ..., N$

Denote the coefficients at x^s in decomposition R_i though

$\tilde{a}_{s,i}$. So we obtain $N-1$ polynomials

$$\begin{aligned} R_i &= \tilde{a}_{0,i} + \tilde{a}_{1,i}x + \cdots + \tilde{a}_{M_i,i}x^{M_i} \\ i &= 1,2,...,N-1 \end{aligned} \qquad (14)$$

Accord to our notations we construct the operators

$$\tilde{\tilde{R}}_{i-1} = \\ = \begin{pmatrix} \tilde{a}_{0,1} & \tilde{a}_{1,1} & \cdots & \tilde{a}_{M_1 1} & 0 & 0 & \cdots & 0 \\ 0 & \tilde{a}_{0,1} & \cdots & \tilde{a}_{M_1-1,1} & \tilde{a}_{M_1,1} & 0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \tilde{a}_{0,1} & \tilde{a}_{1,1} & \cdot & \cdots & \tilde{a}_{M_1,1} \\ \tilde{a}_{0,i} & \tilde{a}_{1,i} & \cdots & \tilde{a}_{M_i-1,i} & \tilde{a}_{M_i,i} & 0 & \cdots & 0 \\ 0 & \tilde{a}_{0,i} & \cdots & \tilde{a}_{M_i-2,i} & \tilde{a}_{M_i-1,i} & \tilde{a}_{M_i,i} & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & 0 & \tilde{a}_{0,i} & \tilde{a}_{1,i} & \cdots & \tilde{a}_{M_{i,i}} \end{pmatrix} \\ i = 2,...,N$$

Operator $\tilde{\tilde{R}}_i$ is the resultant of polynomials, obtaining as result of decompositions of the R_1 and R_i.

Criterion of existence of solution of the system (14) is defined by the

Theorem 4. The system of polynomials (12) has a common solutions if and only if $\tilde{a}_{M_i,i} \neq 0$, $a_{m_i+n_i,i} \neq 0$

$$Ker\bigcap_{i=1}^{N-2} \tilde{\tilde{R}}_i \neq \{\vartheta\}, Ker\bigcap_{i=1}^{N-1} R_i \neq \{\vartheta\}.$$

6. Conclusion

In this paper the necessary and sufficient conditions of existence of the common eigenvalue of the two-parameter system of operators in the Hilbert space in the case when the number of operators is greater than the number of parameters are proved. As corollary of this result for the non-linear algebraic system of equations is solved the problem of existence of solutions.

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