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# On the explicit parametric equation of a general helix with first and second curvature in Nil 3-space

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**Abstract:** Nil geometry is one of the eight geometries of Thurston's conjecture. In this paper we study in Nil 3-space and the Nil metric with respect to the standard coordinates (x,y,z) is $g_{Nil_3}=(dx)^2+(dy)^2+(dz-xdy)^2$ in $IR^3$. In this paper, we find out the explicit parametric equation of a general helix. Further, we write the explicit equations Frenet vector fields, the first and the second curvatures of general helix in Nil 3-Space. The parametric equation the Normal and Binormal ruled surface of general helix in Nil 3-space in terms of their curvature and torsion has been already examined in [12], in Nil 3-Space.

**Keywords:** Nil Space, Helix, Curvatures

## 1. Introduction

In mathematics, Thurston's conjecture proposed a complete characterization of geometric structures on three-dimensional manifolds. The conjecture was proposed by William Thurston (1982), and implies several other conjectures, such as the Poincaré conjecture and Thurston's elliptization conjecture. Thurston's geometrization conjecture states that; Certain three-dimensional topological spaces each have a unique geometric structure that can be associated with them. It is an analogue of the uniformization theorem for two-dimensional surfaces, which states that every simply-connected Riemann surface can be given one of three geometries (Euclidean, spherical, or hyperbolic). In three dimensions, it is not always possible to assign a single geometry to a whole topological space. Instead, the geometrization conjecture states that every closed 3-manifold can be decomposed in a canonical way into pieces that each have one of eight types of geometric structure. Thurston's conjecture is that, after you split a three-manifold into its connected sum and the Jaco-Shalen-Johannson torus decomposition, the remaining components each admit exactly one of the following geometries

Euclidean geometry,
Hyperbolic geometry,
Spherical geometry,
The geometry of S2 × R,
The geometry of H2 × R,
The geometry of the universal cover SL_2R~ of the Lie group SL_2R,
Nil geometry,
Sol geometry.
For more detail see [13].

A nilmanifold is a differentiable manifold which has a transitive nilpotent group of diffeomorphisms acting on it. In the Riemannian category, there is also a good notion of a nilmanifold. A Riemannian manifold is called a homogeneous nilmanifold if there exist a nilpotent group of isometries acting transitively on it. The requirement that the transitive nilpotent group acts by isometries leads to the following rigid characterization: every homogeneous nilmanifold is isometric to a nilpotent Lie group with left-invariant metric (see [4]).

The two-parameter family of metrics first appeared in the works of Bianchi, Cartan and Vranceanu, these spaces are often referred to as Bianchi-Cartan-Vranceanu spaces, or BCV-spaces for short. Some well-known examples of BCV-spaces are the Riemannian product spaces $S^2\times R$, $H^2\times R$ and the 3-dimensional Heisenberg group [5]. Let κ and τ be real numbers, with $\tau\geq 0$. The Bianchi-Cartan-Vranceanu spaces, (BCV-spaces) $M^3(\kappa,\tau)$ is defined as the set

$$\{(x,y,z)\in IR^3:1+(\kappa/4)(x^2+y^2)>0\}$$

equipped with metric

$$ds^2=((dx^2+dy^2)/((1+(\kappa/4)(x^2+y^2))^2))$$
$$+(dz+\tau((ydx-xdy)/(1+(\kappa/4)(x^2+y^2))))^2.$$

* if $\kappa = \tau = 0$, then $M^3(\kappa,\tau) \cong IE^3$

* if $\kappa = 0$ and $\tau \neq 0$, then $M^3(\kappa,\tau) \cong Nil_3$.

More details can be found in [4 ] and [1].

In [5], it is restricted to the 3-dimensional Heisenberg group coming from $R^2$ with the canonical symplectic form $\omega((x,y),(x_1,y_1))=xy_1-x_1y$, i.e., they consider $R^3$ with the group operation

$$(x,y,z)*(x_1,y_1,z_1)=(x+x_1,y+y_1,z+z_1+((xy_1)/2)-((x_1y)/2)).$$

For every non-zero number $\tau$ the following Riemannian metric on $(R^3,*)$ is left invariant:

$$ds^2=dx^2+dy^2+4\tau^2(dz+((ydx-xdy)/2))^2.$$

After the change of coordinates $(x, y, 2\tau z)\to(x,y,z)$, this metric is expressed as

$$ds^2=dx^2+dy^2+(dz+\tau(ydx-xdy))^2.$$

By some authors the notation Nil 3-space is only used if $\tau = (1/2)$. We will use the notation $Nil_3$ in short. It is well known that Nil space is isometric to Heisenberg space. The geometry of Nil is the three dimensional Lie group of all real 3 triangular matrices of the form

$$\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}.$$

Let $(IR^3, g_{Nil_3})$ denote Nil space, where the metric with respect to the standard coordinates $(x,y,z)$ in $IR^3$ can be written [14] as

$$g_{Nil_3}=(dx)^2+(dy)^2+(dz-xdy)^2.$$

Hence we get the symetric tensor field $g_{Nil_3}$ on $Nil_3$ by components.

$$g_{ij}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1+x^2 & -x \\ 0 & -x & 1 \end{bmatrix}.$$

Note that the Nil metric can also be written as:

$$ds^2=\sum\nolimits^3_{i=1}\omega i\otimes\omega i,$$

where $\omega^1=dx$, $\omega^2=dy$, $\omega^3=dz-xdy$, and the orthonormal basis dual to the 1-forms is

$$E_1=(\partial/(\partial x)),\ E_2=(\partial/(\partial y))+x(\partial/(\partial z)),\ E_3=(\partial/(\partial z)).$$

With respect to this orthonormal basis, the Levi-Civita connection and the Lie brackets can be easily computed as:

$$\nabla_{E_1}E_1 = 0, \quad \nabla_{E_1}E_2=(1/2)E_3, \quad \nabla_{E_1}E_3=((-1)/2)E_2$$

$$\nabla_{E_2}E_1 = ((-1)/2)E_3, \quad \nabla_{E_2}E_2=0, \quad \nabla_{E_2}E_3=(1/2)E_1$$

$$\nabla_{E_3}E_1 = ((-1)/2)E_2, \quad \nabla_{E_3}E_2=(1/2)E_1, \quad \nabla_{E_3}E_3=0.$$

$$[E_1,E_2]=E_3, \qquad [E_2,E_3]=0, \qquad [E_1,E_3]=0.$$

Hence

$$\begin{bmatrix} 0 & \frac{1}{2}E_3 & \frac{-1}{2}E_2 \\ \frac{-1}{2}E_3 & 0 & \frac{1}{2}E_1 \\ \frac{-1}{2}E_2 & \frac{1}{2}E_1 & 0 \end{bmatrix}$$

is the matrix with (i,j)- element in the table equals $\nabla_{E_i}E_j$ for the basis $\{E_1,E_2,E_3\}$. See for more details [14].

# 2. The Parametric Equation of General Helix in Nil 3-Space

### *2.1. Riemannian Structure of Nil Space*

Helix is one of the fascinating curve in science and nature. In this section, we study on the general helices in the $Nil_3$. We characterize the general helices in terms of their curvature and torsion. A curve of constant slope or general helix is defined by the property that the tangent makes a constant angle with a fixed straight line (the axis of the helix). A classical result stated by M. A. Lancret in 1802 and first proved by B. de Saint Venant in 1845 (see [7] and [2] for details) is: A necessary and sufficient condition that a curve be a helix is that the ratio of curvature to torsion be constant. Helices are examined in [9] and [6]. Let $\alpha$ be a helix that lies on the cylinder. A helix which lies on the cylinder is called cylindrical helix or general helix. Assume that $\{T,N,B,\kappa,\tau\}$ be the Frenet apparatus along the curve $\alpha$. It has been known that the curve $\alpha$ is a cylindrical helix if and only if $((\kappa/\tau))$ is constant, then $((\kappa/\tau))'=0$ where $\kappa$ and $\tau$ are the curvatures of $\alpha$. If the curve is a general helix, the ratio of the first curvature of the curve to the torsion of the curve must be constant. We call a curve a circular helix if both $\tau\neq0$ and $\kappa$ are constant. Then, the Frenet frame satisfies the following Frenet-Serret equations

$$\nabla_T T = \kappa N,$$

$$\nabla_T N = -\kappa T+\tau B,$$

$$\nabla_T B = -\tau N.$$

With respect to the orthonormal basis $\{E_1, E_2, E_3\}$, we can write

$$T = T_1E_1+T_2E_2+T_3E_3,$$

$$N = N_1E_1+N_2E_2+N_3E_3,$$

$$B = T\times N=B_1E_1+B_2E_2+B_3E_3.$$

Parametric equations of general helices in the sol space $Sol_3$ are examined in [3]. Normal ruled surfaces of general helices in the Sol space $Sol_3$ are examined in [8].

Normal and Binormal ruled surfaces of general helices in Nil 3-space with the Riemannian Structure of Nil 3-space are examined in [12].

Parametric equation of general helix and all the Frenet apparatus are examined as in the following theorems.

### *2.2. The parametric equation of General Helices in Nil Space $Nil_3$*

*Theorem:*

Let $\alpha:I\to Nil_3$ be a unit speed non-geodesic general helix. Then, the equation of a unit speed non-geodesic general helix $\alpha$, with respect to the orthonormal basis, $\{E_1, E_2, E_3\}$,

$$\alpha(s)= \left(\frac{\sin\beta}{C_1}\sin D+C_3\right)E_1+\left(\frac{-\sin\beta}{C_1}\cos D+C_4\right)E_2$$

$$+\left(\frac{\sin^2\beta}{4C_1^2}\sin 2D-\frac{C_4\sin\beta}{C_1}\sin D\right.$$

$$\left.+\left(\frac{\sin^2\beta}{2C_1}+\cos\beta\right)s-C_3C_4+C_5\right)E_3,$$

where we take $D=C_1s+C_2$, $C_1,C_2\in IR$.

*Proof:*

Assume that $\alpha: I\to Nil_3$ be a unit speed non-geodesic general helix. So, without loss of generality, we take its axis as parallel to the vector $E_3$. Then

$$g_{nil_3}(T,E_3)=T_3=\cos\beta,$$

where $\beta$ is constant angle. On the other hand the tangent vector T is an unit vector, so the following condition is satisfied $T_1^2+T_2^2=1-\cos^2\beta$. Since $\cos^2\beta+\sin^2\beta=1$, we have the general solution of $T_1^2+T_2^2=\sin^2\beta$ can be written in the following form

$$T_1=\sin\beta\cos D,$$

$$T_2=\sin\beta\sin D,$$

$$T_3=\cos\beta$$

Also, without loss of generality, where we take $D=C_1s+C_2$ where $C_1,C_2\in IR$. So, substituting the components $T_1$, $T_2$ and $T_3$ in the equation, we have the following equation

$$T=\sin\beta\cos DE_1+\sin\beta\sin DE_2+\cos\beta E_3.$$

Definition of the tangent vector field T, give us;

$$\frac{dx}{ds} = \sin\beta\cos D$$

$$\frac{dy}{ds} = \sin\beta\sin D$$

$$\frac{dz}{ds} = x\sin\beta\sin D+\cos\beta.$$

Integrating both sides, we have

$$\Rightarrow x(s)=\frac{\sin\beta}{C_1}\sin D+C_3$$

$$\Rightarrow y(s)=\frac{-\sin\beta}{C_1}\cos D+C_4$$

$$\Rightarrow z(s)=\left(-\frac{\sin^2\beta}{4C_1^2}\sin 2D\right.$$

$$-\frac{C_3\sin\beta}{C_1}\cos D$$

$$\left.+\left(\frac{\sin^2\beta}{2C_1}+\cos\beta\right)s\right)+C_5$$

where $C_3$, $C_4$, $C_5$ are constant of integration. Substituting all them in $\alpha(s)$, this proves our assertion. Thus, the proof of theorem is completed.

## 3. The Curvatures of the General Helix in Nil 3-Space

### *3.1. First Curvature of the General Helix in Nil Space $Nil_3$*

*Theorem:*

The first curvature (curvature ) of the general helix in Nil Space $Nil_3$ is

$$\kappa = (\cos\beta - C_1)\sin\beta, \quad (\cos\beta - C_1)\sin\beta > 0$$

*Proof*:

Assume that $\alpha: I\to Nil_3$ be a unit speed non-geodesic general helix with

$$T= \sin\beta\cos DE_1+\sin\beta\sin DE_2+\cos\beta E_3.$$

The Levi-Civita connection and Lie brackets can be easily computed as:

$$\nabla_T T = T'_1E_1-(1/2)T_1T_2E_3+((-1)/2)T_1T_3E_2$$

$$+T'_2E_2+(1/2)T_2T_1E_3+(1/2)T_2T_3E_1$$

$$+T'_3E_3-(1/2)T_3T_1E_2+(1/2)T_3T_2E_1$$

$$= (T'_1+T_2T_3)E_1+(T'_2-T_1T_3)E_2+(T'_3)E_3$$

By substituting $T_1$, $T_2$, $T_3$ and derivates, we get

$$\nabla_T T = (\cos\beta-C_1)(\sin\beta\sin DE_1-\sin\beta\cos DE_2)$$

$$= (\cos\beta-C_1)\sin\beta\,(\sin DE_1-\cos DE_2).$$

Since $\kappa=g_{Nil3}(\nabla_T T, N)$ and $N=(1/\kappa)\nabla_T T$ we have

$$\kappa =g_{Nil3}(\nabla_T T, (1/\kappa)\nabla_T T)\ \kappa^2 =gNil3(\nabla_T T, \nabla_T T)$$

$$\kappa^2 = (C_1\sin\beta-\sin\beta\cos\beta)^2$$

$$= (\cos\beta - C_1)^2\sin^2\beta$$

and also for $(\cos\beta - C_1)\sin\beta > 0$, first curvature is

$$\kappa = (C_1-\cos\beta)\sin\beta.$$

#### *3.1.1. The Normal Vector Fields of the General Helix*

The following theorem gives us the explicit parametric equation of normal vector fields in $Nil_3$.

*Theorem*:

Let $\alpha:I\to Nil_3$ be a unit speed non-geodesic general helix. Then, the normal vector field of the general helix is

$$N= (\sin D,-\cos D,0)$$

where we take $D=C_1s+C_2$, $C_1,C_2\in IR$.

*Proof*:

Let $\alpha:I\to Nil_3$ be a unit speed non-geodesic general helix. By the use of Frenet formula $\nabla_T T =\kappa N$,

$$\kappa N = (\cos\beta-C_1)(\sin\beta\sin DE_1-\sin\beta\cos DE_2)$$

$$= (\cos\beta-C_1)\sin\beta\,(\sin DE_1-\cos DE_2).$$

Hence the normal vector field of the general helix is

$$N = (1/\kappa)((-C_1\sin\beta + ((\sin 2\beta)/2))\ \sin DE_1$$
$$+(C_1\sin\beta - ((\sin 2\beta)/2))\ \cos DE_2)$$

where we take $D=C_1s+C_2$, where $C_1,C_2\in IR$. Also we know that; $\kappa = (C_1-\cos\beta)\sin\beta$, so

$$N = (1/\kappa)\ \nabla_T T$$

$N= 1/(\sin\beta(\cos\beta-C_1)))(\cos\beta-C_1)(\sin\beta\sin DE_1-\sin\beta\cos DE_2)$

$= \sin DE_1-\cos DE_2$

$N = (\sin D, -\cos D, 0)$,

or substituting

$$E_1=(\partial/(\partial x)),\ E_2=(\partial/(\partial y))+x(\partial/(\partial z)),\ E_3=(\partial/(\partial z))$$

In

$$N = \sin DE_1-\cos DE_2$$
$$N = \sin D(\partial/(\partial x))-\cos D((\partial/(\partial y))+x(\partial/(\partial z)))$$

$N=\sin D(\partial/(\partial x))-\cos D(\partial/(\partial y))-(((\sin\beta)/(C_1))\sin D-C_3)\cos D(\partial/(\partial z))$.

### *3.2. Second Curvature (Torsion) of the General Helix in Nil Space $Nil_3$*

*Theorem*:

The second curvature (torsion) of the general helix in Nil Space $Nil_3$ is

$$\tau = (C_1^2-C_1\cos\beta+ (1/4))^{1/2}$$

*Proof*:

With the Levi-Civita connection and Lie brackets can be easily computed as:

$$\nabla_T N = (N'_1+ (1/2)N_2T_3+(1/2)N_3T_2)E_1$$
$$+ (N'_2+((-1)/2)N_1T_3+((-1)/2)N_3T_1)E_2$$
$$+ (N'_3+(1/2)N_2T_1+((-1)/2)N_1T_2)E_3.$$

Also for $N= \sin D\ E_1 - \cos D\ E_2$, we know that

$$N_1 = \sin D;\ N'_1=C_1\cos D$$
$$N_2 = -\cos D;\ N'_2=C_1\sin D$$
$$N_3 = 0,\ N'_3=0.$$

Now it is easy to say that for

$$\nabla_T N= ((C_1-(1/2)\cos\beta)\cos DE_1$$
$$+ (C_1-(1/2)\cos\beta)\sin DE_2$$
$$+ ((-1)/2)\sin\beta)E_3$$

$$\nabla_T N = (1/2)((2C_1-\cos\beta)\cos D,(2C_1-\cos\beta)\sin D,-\sin\beta)$$

It is well known that Binormal vector field of a curve is $B=(1/\tau)(\nabla_T N +\kappa T)$. Also torsion is

$$\tau = g_{Nil_3}(\nabla_T N,B)$$
$$\tau = g_{Nil_3}(\nabla_T N,\ (1/\tau)(\nabla_T N +\kappa T))$$
$$\tau^2=g_{Nil_3}(\nabla_T N,\ \nabla_T N +\kappa T))$$
$$\tau^2=g_{Nil_3}(\nabla_T N,\nabla_T N))+g_{Nil_3}(\nabla_T N,\kappa T))$$
$$\tau^2=(1/4)(((2C_1-\cos\beta)\cos D)^2+((2C_1-\cos\beta)\sin D)^2+\sin^2\beta)$$
$$\tau^2 = (1/4)((2C_1-\cos\beta)^2+\sin^2\beta) \text{ or } \tau^2 = C_1^2-C_1\cos\beta+(1/4).$$

#### *3.2.1. The Binormal Vector Field of the General Helix in Nil Space $Nil_3$*

*Theorem*:

Let $\alpha : I\rightarrow Nil_3$ be a unit speed non-geodesic general helix. Then, the binormal vector field of the general helix is

$$B= (1/ (C_1^2-C_1\cos\beta+(1/4))^{1/2})$$
$$((C_1-(1/2)\cos\beta+\sin^2\beta\cos\beta-\sin^2\beta C_1)\cos DE_1$$
$$+ (C_1-(1/2)\cos\beta+\sin^2\beta\cos\beta-\sin^2\beta C_1)\sin DE_2$$
$$+ (\cos^2\beta-\cos\beta C_1-(1/2))\sin\beta E_3)]$$

where we take $D=C_1s+C_2$, where $C_1,C_2\in IR$.

*Proof*:

With the Levi-Civita connection and Lie brackets can be easily computed as:

$$\nabla_T N = (1/2)((2C_1-\cos\beta)\cos D,(2C_1-\cos\beta)\sin D,-\sin\beta).$$

And, using the Frenet-Serret equation $\nabla_T N = -\kappa T+\tau B$, we have $B = (1/\tau)\ (\ \nabla_T N +\kappa T)$

$$B= (1/\tau)((((2C_1-\cos\beta)/(2\kappa)))(\cos\beta-C_1)+\kappa)\sin\beta\cos DE_1$$
$$+ ((((2C_1-\cos\beta)/(2\kappa)))(\cos\beta-C_1)+\kappa)\sin\beta\sin DE_2$$
$$+ (\kappa\cos\beta-((\sin^2\beta)/(2\kappa))(\cos\beta-C_1))E_3.$$

By substituting $\kappa$ and $\tau$, we complete the proof.

*Example*:

Assume that $\alpha:I\rightarrow Nil_3$ be a unit speed non-geodesic general helix and its axis as parallel to the vector $E_3$. Also $g_{Nil_3}(T,E_3)=T_3=\cos(\pi/6)$, where $\beta=(\pi/6)$ is constant angle. Hence the explicit parametric equation of helix is

$$\alpha(s) = ((1/2)\sin s+1)E_1+ (((-1)/2)\cos s+1)E_2$$
$$+((1/(16))\sin 2s-(1/2)\sin s+(((1+2\sqrt{3})/4))s)E3.$$

Hence; the first curvature is

$$\kappa=(1/2)(((\sqrt{3})/2)-1)=((\sqrt{3}-2)/4)$$

and the second curvature is

$$\tau=(5-2\sqrt{3})^{1/2}/2.$$

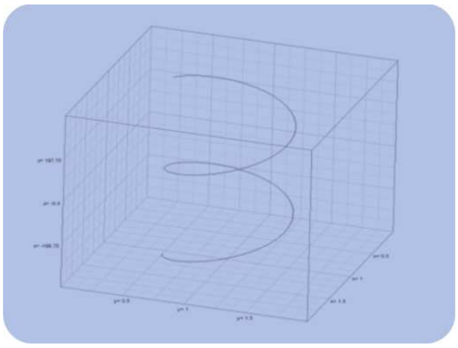

***Figure 1.*** *The figure of Helix in Example.*

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