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Connection forms of an orthonormal frame field in the Minkowski space

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Abstract: In this work, connection formulas and forms of an orthonormal frame field in the Minkowski space IR_1^3 were introduced and then the variation of connection forms was studied. In addition, the relation between the matrix of connection forms and the transition matrix of an orthonormal basis of tangent space were established, and an example was illustrated.

Keywords: Minkowski Space, One-Form, Connection Forms

1. Introduction

It is well-know that Euclidean geometry is a very useful tool in classical mechanics. On the other hand, Riemannian geometry has tremendous amount of applications in general relativity. Therefore, differential geometry has always given rise to new branches of physics [3,7]. Over the years, differential forms have generated a considerable amount of interest not only because they are interesting, but also important as they influenced the research direction both in Euclidean and Lorentzian geometries. Some of the works in this direction are given in [2,4,6] where the authors studied connections forms. In particular, they investigated covariant derivatives of frame elements as connected to this frame and they obtained connection forms and their matrices.

In the spirit of this study, we investigated the connection formulas and forms of an orthonormal frame field in the Minkowski space IR_1^3. This paper is organized as follows: subsequently, we provided the background material concerning the basic concepts and definitions. Then, we study the variation of connection forms. In particular, we establish the relationship between the matrix of connection forms and the transition matrix of an orthonormal basis of the tangent space, and we present an example. In the final section, we summarize our results.

2. Preliminaries

We consider the Minkowski 3-space IR_1^3 with the scalar product

$$\left\langle \vec{X},\vec{Y}\right\rangle = x_1y_1 + x_2y_2 - x_3y_3$$

where $\vec{X}=(x_1,x_2,x_3)$ and $\vec{Y}=(y_1,y_2,y_3)$ are vectors in IR_1^3. $\vec{X}$ and $\vec{Y}$ are called perpendicular if $\left\langle \vec{X},\vec{Y}\right\rangle=0$. The norm of $\vec{X}$ is defined by $\|\vec{X}\|=\sqrt{\left|\left\langle \vec{X},\vec{X}\right\rangle\right|}$. $\vec{X}$ is called space-like if $\left\langle \vec{X},\vec{X}\right\rangle>0$ or $\vec{X}=\vec{0}$, time-like if $\left\langle \vec{X},\vec{X}\right\rangle<0$ and light-like(null) if $\left\langle \vec{X},\vec{X}\right\rangle=0$ and $\vec{X}\neq\vec{0}$ [5]. The cross product of $\vec{X}$ and $\vec{Y}$ is defined by [1]

$$\vec{X}\Lambda\vec{Y} = (x_3y_2 - x_2y_3,\, x_1y_3 - x_3y_1,\, x_1y_2 - x_2y_1)$$

Let $\vec{\alpha}: I \to IR_1^3,\ I \subset IR$, be a regular curve in IR_1^3, and consider the tangent vector $\dot{\vec{\alpha}}(s)$, $s \in I \subset IR$. Then in [5],

1) $\vec{\alpha}$ is a space-like curve if $\left\langle \dot{\vec{\alpha}}(s),\dot{\vec{\alpha}}(s)\right\rangle > 0$,
2) $\vec{\alpha}$ is a time-like curve if $\left\langle \dot{\vec{\alpha}}(s),\dot{\vec{\alpha}}(s)\right\rangle < 0$,
3) $\vec{\alpha}$ is a null curve if $\left\langle \dot{\vec{\alpha}}(s),\dot{\vec{\alpha}}(s)\right\rangle = 0$.

Let $\vec{\alpha}(s)$ be a space-like curve of unit speed in IR_1^3 with the natural curvature $\kappa(s)$ and torsion $\tau(s)$. Let us consider the Frenet frame $\{\vec{t},\vec{n},\vec{b}\}$ of $\vec{\alpha}(s)$ where $\vec{t}$, $\vec{n}$ and $\vec{b}$ are the space-like unit tangent vector, time-like unit principal normal vector and space-like unit binormal vector, respectively. Then scalar and cross product of $\vec{t}$, $\vec{n}$ and $\vec{b}$ are given by

$$\left\langle \vec{t},\vec{t}\right\rangle = -\left\langle \vec{n},\vec{n}\right\rangle = \left\langle \vec{b},\vec{b}\right\rangle = 1,\quad \left\langle \vec{t},\vec{n}\right\rangle = \left\langle \vec{t},\vec{b}\right\rangle = \left\langle \vec{n},\vec{b}\right\rangle = 0,$$

$$\vec{t}\Lambda\vec{n}=-\vec{b},\ \vec{n}\Lambda\vec{b}=-\vec{t},\ \vec{b}\Lambda\vec{t}=\vec{n}.$$

Finally, Frenet formulas are given by [8]

$$\dot{\vec{t}}=\kappa(s)\vec{n},\quad \dot{\vec{n}}=\kappa(s)\vec{t}+\tau(s)\vec{b},\quad \dot{\vec{b}}=\tau(s)\vec{n}.$$

3. Main Results

Let $\{E_1,E_2,E_3\}$ be an orthonormal frame field in the Minkowski space in which E_3 is a time-like vector. Consider a tangent vector $v_p\in T_p(IR_1^3)$ at any $P\in IR_1^3$. Let D be the Levi Civita connection on IR_1^3. Inspired by Frenet formulas, we can consider the covariant derivative of vector fields E_i, $1\le i\le 3$, with respect to the tangent vector v_p as connected to this frame field. Since $D_{v_p}E_i\in T_p(IR_1^3)$, and $\{E_1(p),E_2(p),E_3(p)\}$ is an orthonormal basis of tangent space $T_p(IR_1^3)$ with $w_{ij}(v_p)\in IR$, the covariant derivative of E_i, $1\le i\le 3$, with respect to v_p can be written by

$$(3.1)\ D_{v_p}E_i=\sum_{j=1}^{3}w_{ij}(v_p)E_j(p)$$

where

$$(3.2)\ w_{ij}(v_p)=\left\langle D_{v_p}E_i,E_j(p)\right\rangle\varepsilon_j \text{ with } \varepsilon_j=\begin{cases}1 & ,\ j\neq 3\\ -1 & ,\ j=3\end{cases}$$

From a geometric point of view, this equation extracts the number $w_{ij}(v_p)$ that is component of the variation of vector E_i with respect to $E_j(p)$ where the tangent vector v_p is the velocity vector along a curve.

Let v_p, $u_p\in T_p(IR_1^3)$ and let $a,b\in IR$. Since

$$w_{ij}(av_p+bu_p)=aw_{ij}(v_p)+bw_{ij}(u_p)$$

the transformation $(w_{ij})_p:T_p(IR_1^3)\to IR$, defined by

$$(w_{ij})_p(v_p)=w_{ij}(v_p)$$

is linear. Thus w_{ij} corresponds to a linear transformation from the tangent space $T_p(IR_1^3)$ to IR for all P in the Minkowski space IR_1^3. In this case, note that $(w_{ij})_p$ is an element of the cotangent space $T_p^*(IR_1^3)$, that is, a one-form in IR_1^3.

Theorem 3.1: One-forms w_{ij} of the orthonormal frame field $\{E_1,E_2,E_3\}$ in which E_3 is a time-like vector are given by $w_{ij}=-\varepsilon_i\varepsilon_j w_{ji}$, $1\le i,j\le 3$.

Proof: Since $\left\langle E_i,E_j\right\rangle:IR_1^3\to IR$ is a constant function, we have $v_p\left[\left\langle E_i,E_j\right\rangle\right]=0$ for all $v_p\in T_p(IR_1^3)$. On the other hand, the equation

$$v_p\left[\left\langle E_i,E_j\right\rangle\right]=\left\langle D_{v_p}E_i,E_j(p)\right\rangle+\left\langle E_i(p),D_{v_p}E_j\right\rangle$$

Implies that

$$\left\langle D_{v_p}E_i,E_j(p)\right\rangle=-\left\langle E_i(p),D_{v_p}E_j\right\rangle$$

Thus

$$\begin{aligned}w_{ij}(v_p)&=\varepsilon_j\left\langle D_{v_p}E_i,E_j(p)\right\rangle\\&=-\varepsilon_j\left\langle D_{v_p}E_j,E_i(p)\right\rangle\\&=-\varepsilon_j\varepsilon_i(\varepsilon_i\left\langle D_{v_p}E_j,E_i(p)\right\rangle)\\&=-\varepsilon_j\varepsilon_i w_{ji}(v_p).\end{aligned}$$

Since the equation is true for all $v_p\in T_p(IR_1^3)$, we obtain

$$w_{ij}=-\varepsilon_j\varepsilon_i w_{ji}.$$

Definition 3.1: One-forms w_{ij} are called connection forms of the orthonormal frame field $\{E_1,E_2,E_3\}$.

Definition 3.2: The equation given in (3.1) is called connection formulas of the orthonormal frame field $\{E_1,E_2,E_3\}$.

Using Theorem 3.1, we can obtain the matrix $W=[w_{ij}]_{3\times 3}$ of one-forms as

$$W=\begin{bmatrix}0 & w_{12} & w_{13}\\ -w_{12} & 0 & w_{23}\\ w_{13} & w_{23} & 0\end{bmatrix}.$$

Note that W is a skew-adjoint matrix in the sense that $W^t=-\varepsilon W\varepsilon$, where ε is the signature matrix given by

$$\varepsilon=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{bmatrix}.$$

Let $\left\{\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2},\frac{\partial}{\partial x_3}\right\}$ be the natural frame field in the Minkowski space IR_1^3 in which $\frac{\partial}{\partial x_3}$ is a time-like vector. Then the vector E_i, $1\le i\le 3$, can be written as

$$(3.3)\ E_i=\sum_{j=1}^{3}a_{ij}\frac{\partial}{\partial x_j},$$

where $a_{ij}:IR_1^3\to IR$ is a differentiable function. Let $A=[a_{ij}]_{3\times 3}$ be the transition matrix between the orthonormal bases $\{E_1(p),E_2(p),E_3(p)\}$ and $\left\{\frac{\partial}{\partial x_1}(p),\frac{\partial}{\partial x_2}(p),\frac{\partial}{\partial x_3}(p)\right\}$ of the tangent space $T_p(IR_1^3)$. The equation given in (3.3) can be written as

$$(3.4)\quad \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} = A \begin{bmatrix} \dfrac{\partial}{\partial x_1} \\ \dfrac{\partial}{\partial x_2} \\ \dfrac{\partial}{\partial x_3} \end{bmatrix}.$$

Note that A is an orthogonal matrix, that is, $A^{-1} = \varepsilon A^t \varepsilon$. Now we are ready to state the relation between the matrices A and W.

Theorem 3.2: The skew-adjoint matrix W and the orthogonal matrix A satisfy $W = dA.A^{-1}$.

Proof: Let v_p be an element of $T_p(IR_1^3)$. Then,

$$D_{v_p} E_i = D_{v_p}\left(\sum_{k=1}^{3} a_{ik}\frac{\partial}{\partial x_k}\right)$$

$$= \sum_{k=1}^{3} D_{v_p}(a_{ik}\frac{\partial}{\partial x_k})$$

$$= \sum_{k=1}^{3}\left(v_p\,[a_{ik}]\frac{\partial}{\partial x_k}(p) + a_{ik}(p) D_{v_p}\left[\frac{\partial}{\partial x_k}\right]\right)$$

$$= \sum_{k=1}^{3} v_p[a_{ik}]\frac{\partial}{\partial x_k}(p)$$

and therefore

$$\begin{aligned} w_{ij}(v_p) &= \left\langle D_{v_p} E_i, E_j(p)\right\rangle \varepsilon_j \\ &= \left\langle \sum_{k=1}^{3} v_p[a_{ik}]\frac{\partial}{\partial x_k}(p), \sum_{l=1}^{3} a_{jl}\frac{\partial}{\partial x_l}(p)\right\rangle \varepsilon_j \\ &= [v_p[a_{i1}]a_{j1}(p) + v_p[a_{i2}]a_{j2}(p) - v_p[a_{i3}]a_{j3}(p)]\varepsilon_j \\ &= [(da_{i1})(v_p)a_{j1}(p) + (da_{i2})(v_p)a_{j2}(p) - (da_{i3})(v_p)a_{j3}(p)]\varepsilon_j \\ &= \left[da_{i1}a_{j1} + da_{i2}a_{j2} - da_{i3}a_{j3}\right]\varepsilon_j(v_p). \end{aligned}$$

Since this equation is correct for all $v_p \in T_p(IR_1^3)$, we obtain

$$\varepsilon_j w_{ij} = da_{i1}a_{j1} + da_{i2}a_{j2} - da_{i3}a_{j3}.$$

Finally, the above equation along with the identity $A^{-1} = \varepsilon A^t \varepsilon$ imply that $W = dA.A^{-1}$.

Example: Let (r, θ, z) be the usual cylindrical coordinates in IR_1^3 as indicated in Figure 1. There the coordinate functions are well defined and an inverse mapping exists given by

$$x = r\cos\theta,\;\; y = r\sin\theta,\;\; z = z.$$

Consider the natural frame field $\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\}$ in IR_1^3. The cylindrical frame field $\left\{\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial z}\right\}$ is given by

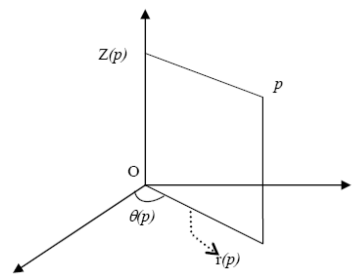

Figure 1. The Cylindrical Coordinate System

$$\frac{\partial}{\partial r} = \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y},$$

$$\frac{\partial}{\partial \theta} = -r\sin\theta\frac{\partial}{\partial x} + r\cos\theta\frac{\partial}{\partial y},$$

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial z}.$$

It follows from the above equations the that the transition matrix A is given by

$$A = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Finally, applying Theorem 3.2 we obtain the matrix of connection forms of the cylindrical frame as

$$W = dA\,A^{-1}$$

$$W = \begin{bmatrix} -\sin\theta d\theta & \cos\theta d\theta & 0 \\ -r\cos\theta d\theta - \sin\theta dr & -r\sin\theta d\theta + \cos\theta dr & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \cos\theta & -r^{-1}\sin\theta & 0 \\ \sin\theta & r^{-1}\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$W = \begin{bmatrix} 0 & r^{-1}d\theta & 0 \\ -rd\theta & r^{-1}dr & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

4. Conclusions

In this paper, we studied the connection forms of an orthonormal frame in the Minkowski space IR_1^3. In contrast to IR^3, we observed that w_{ij}, the component of E_i in the direction of $E_j(p)$ along a curve with velocity vector v_p, depends on indices; however, the this dependence does not change the relation between the skew-adjoint matrix W and the orthogonal matrix A. We believe that our results will provide a base for further studies, in particular for connection forms in the Minkowski space.

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