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On involute and evolute of the curve and curve-surface pair in Euclidean 3-space

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Absract: In this paper, the involute and evolute of the curve is studied in type of the curve-surface pair at first time. In additional whenβ is considered evolute and involute of the curveα, involute and evolute curve-surface pairs (called as strip) and strip of the curve$\alpha$is shown as(α, M) are given with depending on the constant angleφthat is between in$\vec{\eta}$ and $\vec{n}$ in Euclidean 3-Space E^3.

Keywords: Curve-Surface Pair,Evolute, Involute.2010 Mathematics Subject Classification. 53A04, 53A05

1. Introduction

The evolute and involute of the curve pair is well known by the mathematicians especially the differential geometry scientists. The evolute of any curve is defined as the locus of the centers of the curvature of the curve. These type of curves are defined after as the involute of the evolute of the curves. This subject is about the curve corresponding to curve. In other words it is the subject on curves. There are a lot of literatures by Hacısalihoglu and Sabuncuoglu. Similarly the curve-surface pair was observe by Hacısalihoglu, Keles and Sabuncuoglu. They observed the curve-surface pair, found the curvatures, relations between the curvatures of the curve and the curve-surface pair, also this curve-surface pair was studied the subject on curves on the same surface or the curves on different surfaces. Now the mix of these two spesific subject, their frenet vectors, curvatures relations between their special features are studied similar as the differential book of Sabuncuoglufirstly in this paper in Euclidean 3-Space E^3.

2. The Curve-Surface Pair (Strip)

Definition :

Let M and α be a surface in E^3 and a curve in $M \subset E^3$. We define a surface element of M is the part of a tangent plane at the neighbour of the point. The locus of these surface element along the curve α is called a curve-surface pair and is shown as (α, M).

Definition:

Let $\{\vec{t}, \vec{n}, \vec{b}\}$ and $\{\vec{\xi}, \vec{\eta}, \vec{\zeta}\}$ be the curve and curve-surface pair's vector fields. The curve-surface pair's tangent vector field, normal vector field and binormal vector field is given by

$\vec{t} = \vec{\xi}$, $\vec{\zeta} = \vec{N}$ ($\vec{N} = \vec{n}$) and $\vec{\eta} = \vec{\zeta} \wedge \vec{\xi}$ ([1,2,3]).

2.1. Curvatures of the Curve-Surface Pair and Curvatures of the Curve α

Let $k_n = -b, k_g = c, t_r = a$ and $\{\vec{\xi}, \vec{\eta}, \vec{\zeta}\}$ be the normal curvature, the geodesic curvature, the geodesic torsion of the strip and the curve-surface pair's vector fields on α (1,2,3]).

Then we have

$$\begin{aligned} \vec{\xi}' &= c\vec{\eta} - b\vec{\zeta} \\ \vec{\eta}' &= -c\vec{\xi} + a\vec{\zeta} \\ \vec{\zeta}' &= b\vec{\xi} - a\vec{\eta}. \end{aligned} \tag{1}$$

We know that a curve α has two curvatures κ and τ. A curve has a strip and a strip has three curvatures k_n, k_g and t_r. Let k_n, k_g and t_r be the $-b, c$ and a.

From last equations we have $\vec{\xi}' = c\vec{\eta} - b\vec{\zeta}$. If we substitude

$\vec{\xi}=\vec{t}$ in last equation, we obtain

$$\vec{\xi}'=\kappa\vec{n}$$

and

$$b=-\kappa\sin\phi$$
$$c=\kappa\cos\phi$$

([1,2,3]).

Fromlasttwoequations we obtain,

$$\kappa^2=b^2+c^2\ .$$

This equation is a relation between the curvature κ of a curve α and normal curvature and geodesic curvature of a curve-surface pair.

By using similar operations, we obtain a new equation as follows

$$\tau=a+\frac{b'c-bc'}{b^2+c^2}$$

This equation is a relation between τ (torsion or second curvature of α and curvatures of a curve-surface pair that belongs to the curve α). And also we can write([1,2,3]).

$$a=\phi'+\tau$$

The special case:

If ϕ is constant, then $\phi'=0$. So the equation is $a=\tau$. That is, if the angle is constant, then torsion of the curve-surface pair is equal to torsion of the curve.

Definition:

Let α be a curve in $M\subset E^3$. If the geodesic curvature (torsion) of the curve α is equal to zero, then the curve-surface pair (α,M) is called a curvature curve-surface pair (strip) [1,2,3].

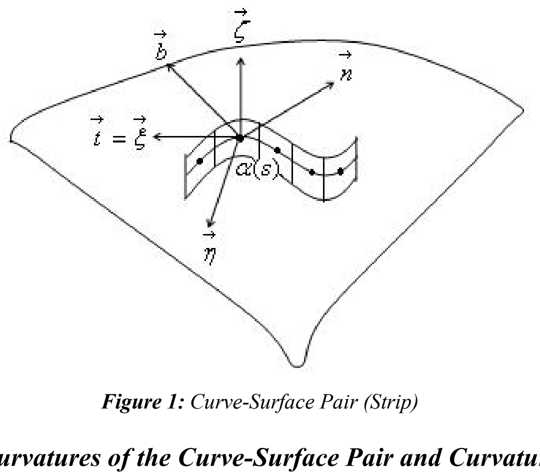

Figure 1: *Curve-Surface Pair (Strip)*

2.2. Curvatures of the Curve-Surface Pair and Curvatures of the Curve α^*

Let α^* is involute of α and $k_n=-b^*, k_g=c*, t_r=a^*$ and $\left\{\vec{\xi}^*,\vec{\eta}^*,\vec{\zeta}^*\right\}$ be the normal curvature, the geodesic curvature, the geodesic torsion of the strip and the curve-surface pair's vector fields on α^* (1,2,3].

Then we have

$$\vec{\xi}^{*\prime}=c^*\vec{\eta}^*-b^*\vec{\zeta}^*$$
$$\vec{\eta}^{*\prime}=-c^*\vec{\xi}^*+a^*\vec{\zeta}^*$$
$$\vec{\zeta}^{*\prime}=b^*\vec{\xi}^*-a^*\vec{\eta}^*.$$

We know that a curve α has two curvatures κ^* and τ^*. A curve has a strip and a strip has three curvatures k_n,k_g and t_r. Let k_n,k_g and t_r be the $-b^*,c^*$ and a^*. From last equations we have $\vec{\xi}^{*\prime}=c^*\vec{\eta}^*-b^*\vec{\zeta}^*$. If we substitude $\vec{\xi}^*=\vec{t}^*$ in last equation, we obtain

$$\vec{\xi}^{*\prime}=\kappa^*\vec{n}^*$$

And

$$b^*=-\kappa^*\sin\phi$$
$$c=\kappa^*\cos\phi$$

([1,2,3]).

From last two equations we obtain,

$$\kappa^{*2}=b^{*2}+c^{*2}\ .$$

This equation is a relation between the curvature κ^* of a curve α^* and normal curvature and geodesic curvature of a curve-surface pair.

By using similar operations, we obtain a new equation as follows

$$\tau^*=a^*+\frac{b^{*\prime}c^*-b^*c^{*\prime}}{b^{*2}+c^{*2}}$$

This equation is a relation between τ^* (torsion or second curvature of α^* and curvatures of a curve-surface pair that belongs to the curve α^*).

And also we can write

$$a^*=\phi'+\tau^*\ .$$

Thespecialcase:

If ϕ is constant, then $\phi'=0$. So the equation is $a^*=\tau^*$. That is, if the angle is constant, then torsion of the curve-surface pair is equal to torsion of the curve[1,2,3].

Definition:

Let α^* be a curve in $M\subset E^3$. If the geodesic curvature (torsion) of the curve α^* is equal to zero, then the curve-surface pair (α^*,M) is called a curvature curve-surface pair (strip) [1,2,3].

3. Involute and Evolute Curve-Surface Pairs

3.1.Involute of α

Definition:

Let $\alpha : I \to R^3$ be a unit speed curve and $\alpha^* : I \to R^3$ is given with the same interval. (α,M) and (α^*,M) be the curve-surface pairs in E^3 and $I \subset E^3$. If the tangent of point $\alpha(s)$ of (α,M) is tangent point of $\alpha^*(s)$ of (α^*,M) for every $\forall s \in I$ and

$$\langle \xi^*(s), \xi(s) \rangle = 0 ,$$

so, (α^*,M) is involute of (α,M).

Theorem 3.1[4]]:

Let (α^*,M) be involute of (α,M). Let $\left\{ \vec{\xi}^*, \vec{\eta}^*, \vec{\zeta}^* \right\}$ be the Frenet vector fields of (α^*,M) ,then we write

$$\vec{\xi}^* = \vec{\zeta}$$

$$\vec{\zeta}^* = \frac{-(b^2+c^2)}{\sqrt{(b^2+c^2)+(a+\frac{b'c-bc'}{b^2+c^2})^2}}\vec{\xi} + \frac{a+\frac{b'c-bc'}{b^2+c^2}}{\sqrt{(b^2+c^2)+(a+\frac{b'c-bc'}{b^2+c^2})^2}}\vec{\eta}$$

$$\vec{\eta}^* = \frac{a+\frac{b'c-bc'}{b^2+c^2}}{\sqrt{(b^2+c^2)+(a+\frac{b'c-bc'}{b^2+c^2})^2}}\vec{\xi} + \frac{(b^2+c^2)}{\sqrt{(b^2+c^2)+(a+\frac{b'c-bc'}{b^2+c^2})^2}}\vec{\eta}.$$

Proof:

We have from [4],

$$\begin{aligned}(\alpha^*)'(s) &= \alpha'(s) - \xi(s) + (-s+k)\xi'(s)\\ &= \xi(s) - \xi(s) + (-s+k)K(s)\zeta(s)\\ &= (-s+k)\kappa(s)\zeta(s)\end{aligned}$$

So, we obtain,

$$\|(\alpha^*)'(s)\| = |(-s+k)\kappa(s)| .$$

We have

$$\xi^* = \frac{(\alpha^*)'(s)}{\|(\alpha^*)'(s)\|} = \frac{(-s+k)\kappa(s)\zeta(s)}{|(-s+k)\kappa(s)|}$$

The vectors ξ^* and ζ are the unit lenght vectors so,

$$\xi^* = \zeta \text{ or } \xi^* = -\zeta$$

Now we consider $\xi^* = \zeta$ and do the operations by this.

Let x be the coordinate function on R. We know $x(s) = s$ for $\forall s \in I$.

Now

$$(\alpha^*)'(s) = (-x+k)\kappa(s)\zeta(s)$$

$x' = 1$ and we have $(\alpha^*)''$, $(\alpha^*)'''$ and $(\alpha^*)'' \times (\alpha^*)'''$ from [4],

We calculate

$$(\alpha^*)' \times (\alpha^*)'' = (-x+k)^2\kappa^2\tau\xi + (-x+k)^2\kappa^3\eta$$
$$\|(\alpha^*)' \times (\alpha^*)''\| = (-x+k)^2\kappa^2\sqrt{\kappa^2+\tau^2}$$

From [4] and the all differential geometry books we know

$$B = \frac{\alpha' \times \alpha''}{\|\alpha' \times \alpha''\|}$$

So if we adjust to our curve-surface pair vectors, we write

$$\vec{\eta}^* = \frac{(\alpha^*)' \times (\alpha^*)''}{\|(\alpha^*)' \times (\alpha^*)''\|} = \frac{\tau\xi + \kappa\eta}{\sqrt{\kappa^2+\tau^2}} = \frac{a+\frac{b'c-bc'}{b^2+c^2}}{\sqrt{(b^2+c^2)+(a+\frac{b'c-bc'}{b^2+c^2})^2}}\vec{\xi} + \frac{(b^2+c^2)}{\sqrt{(b^2+c^2)+(a+\frac{b'c-bc'}{b^2+c^2})^2}}\vec{\eta}$$

From $\vec{\zeta}^* = \vec{\eta}^* \times \vec{\xi}^*$,

$$\vec{\zeta}^* = \frac{\tau\vec{\xi} + \kappa\vec{\eta}}{\sqrt{\kappa^2+\tau^2}} \times \vec{\zeta} = \frac{\tau\vec{\eta} - \kappa\vec{\xi}}{\sqrt{\kappa^2+\tau^2}} = \frac{-(b^2+c^2)}{\sqrt{(b^2+c^2)+(a+\frac{b'c-bc'}{b^2+c^2})^2}}\vec{\xi} + \frac{a+\frac{b'c-bc'}{b^2+c^2}}{\sqrt{(b^2+c^2)+(a+\frac{b'c-bc'}{b^2+c^2})^2}}\vec{\eta}$$

We obtain the proof of the theorem from equations.

Theorem 3.2:

Let (α^*,M) be involute of (α,M). Let $\left\{ \vec{\xi}^*, \vec{\eta}^*, \vec{\zeta}^* \right\}$ be the Frenet vector fields of (α^*,M) and if ϕ is constant, then $\phi' = 0$. So the equation is $a = \tau$. Then we have,

$$\vec{\xi}^* = \vec{\eta}$$

$$\vec{\eta}^* = \frac{-(b^2+c^2)}{\sqrt{a^2+b^2+c^2}}\vec{\xi} + \frac{a}{\sqrt{a^2+b^2+c^2}}\vec{\zeta}$$

$$\vec{\zeta} = \frac{a}{\sqrt{a^2+b^2+c^2}}\vec{\xi} + \frac{(b^2+c^2)}{\sqrt{a^2+b^2+c^2}}\vec{\zeta}.$$

Proof:

It is obviosly from Theorem 3.1's proof.

Theorem 3.3[4]:

Let (α^*,M) be involute of (α,M) and let κ^* and τ^* be the curvature and torsion of a curve α^* and also k is a constant number, we can write κ^* and τ^* with using the curvatures of (α,M). Then we have

$$\kappa^*(s)=\frac{\sqrt{(b^2+c^2)+(a+\frac{b'c-bc'}{b^2+c^2})^2(s)}}{|(-s+k)|(b^2+c^2)}$$

$$\eta^*(s)=\frac{((\sqrt{b^2+c^2})(a+\frac{b'c-bc'}{b^2+c^2})'-(\sqrt{b^2+c^2})'(a+\frac{b'c-bc'}{b^2+c^2}))}{(-s+k)(\sqrt{b^2+c^2})((\sqrt{b^2+c^2})^2+(a+\frac{b'c-bc'}{b^2+c^2})^2)(s)}(s).$$

Proof:

It is obvioslyfrom the proof of theTheorem 3.1.

3.2.The Evolute of α

Definition:

Let $\alpha:I\to R^3$ be a unit speed curve and $\alpha^*:I\to R^3$ is given with the same interval. (α,M) and (α^*,M) be the curve-surface pairs in E^3 and $I\subset E^3$. If the tangent of point $\alpha(s)$ of (α,M) is tangent point of $\alpha^*(s)$ of (α^*,M) for every $\forall s\in I$ and

$$\langle \xi^*(s),\xi(s)\rangle=0,$$

so, (α^*,M) is evolute of (α,M).

Theorem 3.4[4]:

Let $\alpha:I\to R^3$ be a unit speed curve and $\alpha^*:I\to R^3$ be the curve in the same interval as α. Similarly (α^*,M) be theevoluteof (α,M). Then we have

$$\vec{\xi}^*=\cos(\phi+c)\vec{\eta}-\sin(\phi+c)\vec{\zeta}$$
$$\vec{\eta}^*=-\vec{\xi}$$
$$\vec{\zeta}^*=\sin(\phi+c)\vec{\eta}+\cos(\phi+c)\vec{\zeta}.$$

Proof:

It is obviosly from the proof of theTheorem 3.1.

Theorem 3.5[4]:

Let $\alpha^*:I\to R^3$ be evolute of the unit speed curve $\alpha:I\to R^3$ and κ^* and τ^* be the curvature and torsion of a curve α^*. Similarly (α^*,M) be the evoluteof (α,M). So we can write,

$$\kappa^*=\frac{(\sqrt{b^2+c^2})^3\cos^3(\phi+c)}{(\sqrt{b^2+c^2})(a+\frac{b'c-bc'}{b^2+c^2})\sin(\phi+c)-(\sqrt{b^2+c^2})'\cos(\phi+c)}$$

$$\tau^*=\frac{-(\sqrt{b^2+c^2})^3\sin(\phi+c)\cos^2(\phi+c)}{(\sqrt{b^2+c^2})(a+\frac{b'c-bc'}{b^2+c^2})\sin(\phi+c)-(\sqrt{b^2+c^2})'\cos(\phi+c)}$$

Proof:

It is proved by the similar way of Theorem 3.1.

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