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The Recillas's conjecture on Szegö kernels associated to Harish-Chandra Modules

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Abstract: The solution of the field equations that involves non-flat differential operators (curved case) can be obtained as the extensions $\Phi+$Szegö operators in *G/K* with *G*, a non-compact Lie group with *K*, compact. This could be equivalent in the context of the Harish-Chandra modules category to the obtaining of extensions in certain sense (*Cousin complexes of sheaves of differential operators to their classification*) of Verma modules as classifying spaces of these differential operators and their corresponding integrals through of geometrical integral transforms.

Keywords: Curved Differential Operators, Deformed Category, Extended Functor, Generalized Verma Modules, Harish-Chandra Category, Recillas's Conjecture

1. Introduction

The classification problems in conformal geometry begins with the preservation problem of the harmonic functions under the action of linear differential operators [1], where these belong to a category whose sheaves of differential operators has a Lie algebra structure as conformal algebra isomorphic $\mathfrak{so}(1, n+1)$, to many TFT, and QFT, phenomena.

Standard group-theoretic methods are applied to derive generating functions and integral representations for the separated solutions. After, and using the recall flat conformal geometry study, the integral representations comes given by certain composition between operators as the Penrose transform and the Szegö operator 1 on $n-$ dimensional sphere2.

There are some methods as the construct of CR-invariant differential operators which define *homomorphisms* of $G-$ modules and can be applied the results of representation theory. Taking the jets at, we may reduce the study of invariant differential operators to the one for the homomorphisms between generalized Verma modules [2].

In more general context and using the flag domains that will be necessary to define many quantum phenomena [3] (as quantum gravity, field torsion, non-Abelian electromagnetism, etc) and equivalences inside of the string theory (for example heterotic strings, $D-$branes and other phenomena) to give solution to extensions of the wave equation on observables of curvature, boson and fermion equations, Schmid equation [4], and classify their differential operators on the same base of vector bundles, but now through holomorphic vector bundles that are $G-$ invariants[6].

But this only covers some aspects of brane theory as those outside of the homogeneous space G/H, when $H=K$. For example are acknowledges the cases to heterotic strings that can develop it on $D-$modules that are $D_{G/C(T)}-$modules when $G/C(T)$, is a flag manifold [7]. In this case the integral operator cohomology $H^{\bullet}(M,\mathcal{O})$, [8, 9] give on such complex submanifolds of a complex maximum torus. What happen

1 In several complex variables the Szegő kernel is an integral kernel that gives rising to a reproduced kernel on a natural Hilbert space of holomorphic functions.

2 In this way, the flat model of conformal geometry is the sphere $S^n=G/P$, with $G=SO_0(n+1,1)$, and P, the stabilizer of a point in $\mathbb{R}^{n+2}$. A classification of all linear conformally invariant differential operators on the sphere is known [5].

when these flags are complex domains or their equivalents Lagrangian submanifolds?

Conjecture (F. Recillas3) 1. 1. The equations with non-flat differential operators can be solved by the corresponding Szegö kernels associated with Harish-Chandra modules [10], of corresponding spherical functions on homogeneous space G/K. The Szegö operator4 completes in some points of the Lie algebra $\mathfrak{so}(1, n+1)$, [11] to the Penrose transform to the case G/K, with K, compact.

Proof. Some results of representation theory obtained by the seminar of representation theory of real reductive Lie groups IM/UNAM (2000-2007) [10].

2. Extension of Differential Operators Classification

Through of application of the *Penrose-Schmid transform* [12, 13], to the cohomology groups of the mirror symmetry (the corresponding of the first and second Hodge numbers to the sheaf $\mathcal{O}_Y$)5 and from the natural scheme of derived categories discussed and given in [6], we can translates these obstruction groups [15], explained by

$$H^{\bullet}_{D^x}(Hom(Mod_{good}(D_X), Mod_{RS(V)}(D_Y)) \supset \supset H^{\bullet}_{D^x}(\mathcal{N} \quad ,0\ ) \supset H^{\bullet}_{D^x}(\underline{\mathcal{N}\quad},0\ ) \tag{2.1}$$

Then the differential operators are not flat [16], but by the theorems in [6], these operators can be re-written for the conformal differential operators under a generalized conformal structure given by vector spaces $\{V_m \subset T_m M | m \in M\}$, and that can be generated by the involutive conic submanifolds as given in the definition [6], for example, to calculate Weyl curvature (remember that the Weyl curvature is the non-symmetrical component of the general curvature tensor [17]) and all the operators used to the G - invariant - $D_{G/H}$ - modules to flag domains [3] to quantizing transforms are *conformal operators*. To Weyl curvature is necessary a conformal sensor sufficiently subtly that exhibits and detects according to the theory of integrability a *obstruction via deformation sheaves of* $D_{\mathrm{P}}-$ *modules*, that is to say, that obtains the isomorphism classes that establishes the equivalences of the solutions to the field theory.

A detailed classification comes given by the classification of *homomorphisms of Verma modules* in [18] is in non-singular case, both integral and non-integral. The singular case can deduce from this by translation. This has been proved to flat case in certain degree of generalization by [19].

Precisely, a careful analysis shows that translations from non-singular to singular submanifolds cover all the homomorphisms of Verma modules. One problem that is beginner in this study via Verma modules was that only case of even dimension of a Verma modules are those irreducible and so there are no non-trivial invariant operators between bundles with fractional conformal weight. In the odd dimensional case, all Verma module which are not (*half* $f-$), integral likewise integral. Those which have half-integral conformal weight give rising to precisely one invariant [16]. These include the invariant Laplacians [6].

Definition 2. 1 (*Verma module to space-time*). The Verma modules of the space-time are to Lie algebra of the Lie group $SO(1,n+1)$, where elements of algebra $\mathfrak{so}(1, n+1)$, , are differential operators. The Verma modules of our interests are to the classification via generating differential operators by Penrose transform in the version Radon-Penrose transform. The group $SO(1,n+1)$, , helps to identify the embedding $\mathbb{R}^n \to S^n$, [3, 12].

3. Consequences in the Field Equations Resolution and Geometrical Langlands Program

By the theorem 4. 2, in [21] and the Recillas's conjecture applied to the obtaining of ramifications of a Hecke category category $\mathcal{H}_G$, $\forall\lambda \in \mathfrak{h}*$ (for example $\mathcal{H}_{G,\lambda}$) on the flag variety G/B, with weight corresponding to twisted differential operators on $\mathrm{Bun}_{G,y}$, we can characterize to the *generalized Verma modules* as the modules $\mathcal{M}\,\mathcal{T}\rho^{\mu}(\mathcal{V})$, obtained from the functor ${}^L\Phi^{\mu}$, being this resulted of a version of the generalized Penrose transform [21].

All integrals to the differential operators of $\mathfrak{sl}(n,n+1)$, are the solution classes $H^{n,n}_{\bullet}(M,\mathcal{L}_{\lambda})$, and to the case of the differential operators to all space-time $\mathfrak{so}(n,n+1)$, we have integrals in $H^{n,m}_{\bullet}(M,\mathcal{O}), m = n+1, n.$

The generalized Verma modules in this case are the cohomological spaces $H^{n,n}_{\bullet}(M,\mathcal{L}_{\lambda})$, and $H^{n,m}_{\bullet}(M,\mathcal{O})$. Their moduli stacks are, for one side, homogeneous line bundles (with tensorization with a form space) in the case of $H^{n,n}_{\bullet}(M,\mathcal{L}_{\lambda})$, and Higgs bundles to $H^{n,m}_{\bullet}(M,\mathcal{O})$.

Example 3. 1. Consider $\mathcal{H}_G = \mathcal{M}(\mathcal{D}^{G/H}_{\lambda})\ \ \forall\ \lambda \in \dot{\mathfrak{h}_{\mathbb{Z}}}$, then $\mathcal{H}_G = \mathcal{D}(B \backslash G/B)$, from the group $G = G^{\mathbb{C}}$. $B-$ equivariant

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3 Prof. Dr. Félix Recillas-Juárez (1918-2010), obtained his PhD, in IAS-Princeton, New Jersey, USA, in 1948, and after he was researcher and Chair professor in Institute of Mathematics UNAM (IM/UNAM), Mexico. He collaborates in several works of algebraic geometry and topological groups with prestigious mathematicians Cartan, Chern, Hurewics, etc, and more mathematicians of the mathematical European school and AMS. He directed the more important research seminar of representation theory of real reductive Lie groups inside the IM/UNAM until his death in 2010. He was our friend and master.

4 Let G/L be the quotient of a semisimple Lie Group G, by the centralizer L of a torus. The author constructs an explicit intertwinig operator from derived functor modules, realized in the Langlands classification, into Dolbeault cohomology on G/L. This operator produces strongly harmonic forms. This work is generalized by [14].

5 The image of functor $\underline{\Psi}_S$ + additional geometrica l hypothesis , given by the Penrose transform that is CY threefold, can be interpreted under the additional geometrical hypothesis, using the Leray-Serre spectral sequence for $\upsilon: Y \to \mathbb{P}^1$, gives an exact sequence

$$H^1(\mathbb{P}^1, \upsilon_* \mathcal{O}_Y) \to H^1(Y, \mathcal{O}_Y) \to H^0(\mathbb{P}^1, R^1\upsilon_* \mathcal{O}_Y),$$

where is the image of right functor on sheaf $\mathcal{O}_Y$. Then their corresponding Hodge numbers [20] are $h^1(\mathcal{O}_Y) = h^2(\mathcal{O}_Y) = 0$.

$D-$module on the flag manifold $X = G/\mathrm{B}$, provide *integral kernels* and thus integral transforms, to know

$$\mathrm{H}^0(X, \mathrm{L}_\lambda) \cong \ker(\tilde{\mathrm{U}}, Q_{BRST}), \tag{3.1}$$

where $Q_{BRST} = \partial + \Phi_1^i\theta_1 - \Phi_2^i\theta_1$, such that $Q_{BRST}^2\phi = 0$, where Φ_1^i, and Φ_2^i, are *Higgs fields* on either side of the open string [22].

Precisely the equivalences given in (3. 1) shape a classification given of the homogeneous vector bundles of lines [17, 23] to a certain sector of differential operators obtaining an *classification by Verma modules* that haves that to see with the Recillas conjecture mentioned in the introduction and the *Szegö kernels* associated with Harish-Chandra modules [24].

For other way, the Penrose transform is evidenced in the determination of the generalized Verma modules when we consider Speh representations realized as spaces of smooth sections on G/K, via Szegö maps. One result given in [14] is on the holomorphic extension realized by the Szegö kernels to connected components of the sheaf O , which provides a realization of the Speh representations in a holomorphic setting.

Studies realized on the generalizations of the Radon transform on $D-$ modules and their images on dual flag manifolds have exhibited some properties of the images under integral transforms of real manifolds as submanifolds of the corresponding extended complex manifolds. These obtained cycles can be identified as ramifications of connections corresponding of loop groups of non-flat operators. These operators stay described as decompositions of the connections (*ramifications*) of holomorphic sheaf of the corresponding holomorphic homogeneous vector bundle. Likewise, if s , is a Szegö kernel, and is the section of a holomorphic homogeneous vector bundle $\mathrm{E} \rightarrow G/K,$ we could want extend s , holomorphically to a domain in $G_{\mathrm{C}}/K_{\mathrm{C}}$. Then these extensions stay described as the Penrose transform whose composition with the corresponding Szegö map extends the Kernel Szegö on the holomorphic homogeneous vector bundle $\mathrm{E} \rightarrow G/K$, and using a homogeneous bundle of lines $\tilde{\mathrm{L}}_\lambda$, provided of the structure of complex flag manifolds, the Penrose transform have of form

$$\mathrm{P} : H_{\bullet}^{n,n}(G/H, \tilde{\mathrm{L}}_\lambda) \rightarrow \mathrm{Hol}(M, \mathrm{O}_{\mathrm{C}}), \tag{3.2}$$

being P , the complex Penrose transform of the real Penrose transform from the cohomology to space $C^\infty(G/K, \mathrm{E})$.

4. Conclusions

The intertwining operators acting on Harish-Chandra modules (as the principal series representations) of real forms of group G, or on any category of $D-$modules on flag variety G/B, are realized as integral operators through Hecke correspondences. One example of this are the basic integral transforms associated to pairs of flags in a fixed relative position and indexed by an element of the Weyl group of G . As was mentioned in the section 2, of this paper, there are closely related to the Radon and Penrose transforms of algebraic analysis. With major generality, all $\mathrm{B}-$equivariant $D-$ modules on the flag variety G/B, provide integral kernels (as obtained by Bergman, Szegö, between others) and thus their integral transforms (integral operators). In particular our attention on the Szegö operators is in the obtaining of classifying spaces as Verma modules to the classification via generating differential operators by Penrose transform in the version Radon-Penrose transform. The interpreting of integral transforms obtained from Szegö kernels is the extension of functor $\Phi+$ Szegö operators, which establish equivalences between Hecke categories.

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