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Classification, α-Inner Derivations and α-Centroids of Finite-Dimensional Complex Hom-Trialgebras

Bouzid Mosbahi[1, *], Ahmed Zahari[2], Imed Basdouri[3]

[1]Department of Mathematics, Faculty of Sciences, University of Sfax, Sfax, Tunisia
[2]IRIMAS-Department of Mathematics, Faculty of Sciences, University of Haute Alsace, Mulhouse, France
[3]Department of Mathematics, Faculty of Sciences, University of Gafsa, Gafsa, Tunisia

Email address:

mosbahi.bouzid.etud@fss.usf.tn (Bouzid Mosbahi), abdou-damdji.ahmed-zahari@uha.fr (Ahmed Zahari), basdourimed@yahoo.fr (Imed Basdouri)
[*]Corresponding author

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Abstract: This article focuses on the classification, α-inner-derivations and α-centroids of complex Hom-trialgebras up to dimension three. The initial research on these algebras was conducted by Loday and Ronco, and this paper builds upon their work by utilizing computer algebra software (Mathematica) to analyze the equations that define the structure constants. Furthermore, we explore the concept of α-inner-derivations and α-centroids of complex Hom-trialgebras. The findings reveal that for 2- and 3-dimensional algebras, there is only one trivial α-inner-derivation. However, there exist 23 non-isomorphic α-inner-derivations for 2- and 3-dimensional algebras. Regarding α-centroids, we identify trivial isomorphism classes for 2- and 3-dimensional Hom-trialgebras. Additionally, there are 11 non-isomorphic classes for 2-dimensional Hom-trialgebras and 19 for 3-dimensional algebras. The range of dimensions for both α-inner-derivations and α-centroids spans from 0 to 3.

Keywords: Hom-Associative Trialgebra, Classification, α-Inner-derivation, α-Centroid

1. Introduction

Notably, a Hom-associative trialgebra $(\mathcal{A}, \dashv, \vdash, \perp, \alpha)$ consists of a vector space, two multiplications and a linear self map. It may be perceived as a deformation of an associative algebra [5, 6], where the associativity condition is twisted by a linear map α, such that when $\alpha = id$ [9], the Hom-associative trialgebras degenerate to exactly triassociative algebras. We aim in this paper to examine the structure of Hom-associative trialgebras. Let $\mathcal{A}$ be an n-dimensional $\mathbb{C}$-linear space and $\{e_1, e_2, \cdots, e_n\}$ be a basis of $\mathcal{A}$. A Hom-associative trialgebra structure on $\mathcal{A}$ with products γ, δ and ξ is determined by $3n^3$ structure constants $\gamma_{ij}^k, \delta_{ij}^k$ and ξ_{ij}^k, where $e_i \dashv e_j = \sum_{k=1}^n \gamma_{ij}^k e_k, \quad e_i \vdash e_j = \sum_{k=1}^n \delta_{ij}^k e_k$ and $e_i \perp e_j = \sum_{k=1}^n \xi_{ij}^k e_k$ and by α which is given by n^2 structure constants a_{ij}, where $\alpha(e_i) = \sum_{j=1}^n a_{ji} e_j$. Requiring the algebra structure to be a Hom-associative trialgebra gives rise to sub-variety $\mathcal{TH}$ of $K^{3n^3+n^2}$. Basic changes in $\mathcal{A}$ yield the natural transport of structure action of $GL_n(k)$ on $\mathcal{TH}$. As a matter of facts, isomorphism classes of n-dimensional Hom-associative trialgebras are one-to-one correspondence with the orbits of the action of $GL_n(k)$ on $\mathcal{TH}$. Classification problems of the Hom-associative trialgebras using the algebraic and geometric technique have drawn much interest in the α-derivations and α-centroids of Hom-associative trialgebras. The associative trialgebras introduced by Loday [4, 8] with a motivation to provide dual dialgebras [3, 7, 11–14], have been further investigated with regard to several areas in mathematics and physics. The classification of Hom-associative algebras was set forward elaborated by [2, 10].

This paper involves around examining the α-Inner-derivations and α-centroids of finite dimensional associative trialgebras. The algebra of α-Inner-derivations and α-centroids are highly needed and extremely useful in terms of algebraic and geometric classification problems of algebras.

The current paper is organized as follows. In the fist section, we identify the topic alongside with some previously obtained results. The chief objective of this paper is to specify and classify α-Inner-derivations as well as α-centroids of Hom-associative trialgebras [1, 15, 16]. In section 2, we tackle the structure of Hom-associative trialgebras. In section 3, we handle the algebraic varieties of Hom-associative trialgebras, and we depict classifications, up to isomorphism, of two-dimensional and three-dimensional Hom-associative trialgebras. We focus upon the classification of the α-Inner-derivations. Eventually, in Section 4, we present the classification of the α-centroids. In this case, the concept of α-derivations and α-centroids is notably inspired and whetted significant scientific concern from that of finite-dimensional algebras. The algebra of α-centroids plays an outstanding role in the classification problems as well as in different applications of algebras. As far as our work is concerned, we elaborated classification results of two and three-dimensional Hom-associative trialgebras. All considered algebras and vectors spaces are supposed to be over a field $\mathbb{C}$ of characteristic zero.

2. Hom-associative Trialgebras

Definition 2.1. A Hom-associative trialgebra is a 5-truple $(\mathcal{A}, \dashv, \vdash, \perp, \alpha)$ consisting of a linear space $\mathcal{A}$ linear maps $\dashv, \vdash, \perp : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ and $\alpha : \mathcal{A} \longrightarrow \mathcal{A}$ satisfying, for all $x, y, z \in \mathcal{A}$, the following conditions :

$$
\begin{aligned}
(x \dashv y) \dashv \alpha(z) &= \alpha(x) \dashv (y \dashv z), && (1)\\
(x \dashv y) \dashv \alpha(z) &= \alpha(x) \dashv (y \vdash z) = \alpha(x) \dashv (y \perp z), && (2)\\
(x \vdash y) \dashv \alpha(z) &= \alpha(x) \vdash (y \dashv z), && (3)\\
(x \dashv y) \vdash \alpha(z) &= \alpha(x) \vdash (y \vdash z) = (x \perp y) \vdash \alpha(z), && (4)\\
(x \vdash y) \vdash \alpha(z) &= \alpha(x) \vdash (y \vdash z), && (5)\\
(x \perp y) \dashv \alpha(z) &= \alpha(x) \perp (y \dashv z), && (6)\\
(x \dashv y) \perp \alpha(z) &= \alpha(x) \perp (y \vdash z), && (7)\\
(x \vdash y) \perp \alpha(z) &= \alpha(x) \vdash (y \perp z), && (8)\\
(x \perp y) \perp \alpha(z) &= \alpha(x) \perp (y \perp z), && (9)
\end{aligned}
$$

for all $x, y, z \in \mathcal{A}$.

Remark 2.1. In addition, α is an endomorphism with respect to $\dashv, \vdash$ and $\perp$. Then, $\mathcal{A}$ is said to be a multiplicative Hom-associative trialgebra

$$
\alpha(x \dashv y) = \alpha(x) \dashv \alpha(y) \quad ; \quad \alpha(x \vdash y) = \alpha(x) \vdash \alpha(y), \quad \alpha(x \perp y) = \alpha(x) \perp \alpha(y) \quad ,
$$

for all $x, y \in \mathcal{A}$.

Definition 2.2. A morphism of Hom-associative trialgebra is a linear map

$$
f : (\mathcal{A}, \dashv, \vdash, \perp, \alpha) \to (\mathcal{A}', \dashv', \vdash', \perp', \alpha')
$$

such that

$$
\alpha' \circ f = f \circ \alpha,
$$

and

$$
f(x \dashv y) = f(x) \dashv' f(y), \quad f(x \vdash y) = f(x) \vdash' f(y),
$$

$$
f(x \perp y) = f(x) \perp' f(y)
$$

for all $x, y \in \mathcal{A}$.

Remark 2.2. A bijective homomorphism is an isomorphism of $\mathcal{A}_1$ and $\mathcal{A}_2$.

Proposition 2.1. Let $(\mathcal{A}, \dashv, \vdash, \perp, \alpha)$ be a Hom-associative trialgebra. Therefore, $(\mathcal{A}, \dashv, \perp, *, \alpha)$ is a Hom-associative trialgebra, where for any $x, y, z \in \mathcal{A}$, $x * y = x \vdash y + x \perp y$.

Proof We prove only one axiom, as others are proven similarly. For any $x, y, z \in \mathcal{A}$,

$$
\begin{aligned}
(x * y) \dashv \alpha(z) &= (x \vdash y + x \perp y) \dashv \alpha(z)\\
&= (x \vdash y) \dashv \alpha(z) + (x \perp y) \dashv \alpha(z)\\
&= \alpha(x) \vdash (y \dashv z) + \alpha(x) \perp (y \dashv z)\\
&= \alpha(x) * (y \dashv z),
\end{aligned}
$$

which ends the proof.

Proposition 2.2. Let $(\mathcal{A}, \dashv, \vdash, \perp, \alpha)$ be a Hom-associative trialgebra and $x * y = x \vdash y + x \dashv y + x \perp y$. From this perspective, $(\mathcal{A}, *, \alpha)$ is a Hom-associative algebra.

Proof For any $x, y, z \in \mathcal{A}$,

$$
\begin{aligned}
&(x * y) * \alpha(z) - \alpha(x) * (y * z)\\
&= (x \vdash y + x \dashv y + x \perp y) * \alpha(z) - \alpha(x) * (y \vdash z + y \dashv z + y \perp z)\\
&= (x \vdash y) * \alpha(z) + (x \dashv y) * \alpha(z) + (x \perp y) * \alpha(z) - \alpha(x) * (y \vdash z) - \alpha(x) * (y \dashv z) - \alpha(x) * (y \perp z)\\
&= (x \vdash y) \vdash \alpha(z) + (x \vdash y) \dashv \alpha(z) + (x \vdash y) \perp \alpha(z) + (x \dashv y) \vdash \alpha(z) + (x \dashv y) \dashv \alpha(z) + (x \dashv y) \perp \alpha(z)\\
&+ (x \perp y) \vdash \alpha(z) + (x \perp y) \dashv \alpha(z) + (x \perp y) \perp \alpha(z) - \alpha(x) \vdash (y \vdash z) - \alpha(x) \dashv (y \vdash z) - \alpha(x) \perp (y \vdash z)\\
&- \alpha(x) \vdash (y \dashv z) - \alpha(x) \dashv (y \dashv z) - \alpha(x) \perp (y \dashv z) - \alpha(x) \vdash (y \perp z) - \alpha(x) \dashv (y \perp z) - \alpha(x) \perp (y \perp z).
\end{aligned}
$$

This corroborates that $(\mathcal{A}, *, \alpha)$ is a Hom-associative algebra.

Corollary 2.1. Let $(\mathcal{A}, \dashv, \vdash, \perp, \alpha)$ be a Hom-associative trialgebra. Then, $\mathcal{A}$ is a Hom-associative algebra with respect to the multiplicative $* : \mathcal{A} \otimes \mathcal{A} \longrightarrow \mathcal{A}$

$$
x * y = x \dashv y + x \vdash y - x \perp y \quad \text{for any} \quad x, y \in \mathcal{A}.
$$

Proposition 2.3. Let $(\mathcal{A}, \dashv, \vdash, \perp, \alpha)$ be a Hom-associative trialgebra. Then $(\mathcal{A}, \dashv, \vdash', \alpha)$ is a Hom-associative trialgebra, where $x \vdash' y = x \vdash y + x \dashv y$, for $x, y \in \mathcal{A}$.

Proof The proof is straightforward by calculation by using Definition 2.1.

Proposition 2.4. Let $(\mathcal{A}, \dashv, \vdash, \perp, \alpha)$ be a Hom-associative trialgebra. Then $(\mathcal{A}, \dashv^{op}, \vdash^{op}, \perp^{op}, \alpha)$ is also a Hom-associative trialgebra, with $x \dashv^{op} y = y \vdash x, \quad x \vdash^{op} y =$

$y \dashv x, \quad x\perp^{op}y = y\perp x$, for any $x, y \in \mathcal{A}$.

Proof It comes immediately from Definition 2.1.

Definition 2.3. A Hom-Leibniz Poisson algebra is the triplet $(\mathcal{A}, \cdot, [\cdot,\cdot], \alpha)$ consisting of a linear space $\mathcal{A}$, two linear maps $\cdot, [\cdot,\cdot] : \mathcal{A}\times\mathcal{A} \longrightarrow \mathcal{A}$ and a linear map $\alpha : \mathcal{A} \longrightarrow \mathcal{A}$ satisfying the following statements

i) $(\mathcal{A}, \cdot, \alpha)$ is a Hom-associative trialbra,

ii) $(\mathcal{A}, [\cdot,\cdot], \alpha)$ is a Hom-Leibniz algebra ,

iii) $[x{\cdot}y, \alpha(z)] = \alpha(x){\cdot}[y, z]{+}[x, z]{\cdot}\alpha(y)$ for all $x, y, z \in \mathcal{A}$

Proposition 2.5. Let $(\mathcal{A}, \dashv, \vdash, \perp, \alpha)$ be a Hom-associative trialgebra. Define now binany operations by

$$\begin{aligned} x * y &= x \dashv y - y \vdash x \\ [x, y] &= x\perp y - y\perp x. \end{aligned}$$

Then $(\mathcal{A}, *, [\cdot,\cdot]\,, \alpha)$ becomes a Hom-associative trialgebra.

Proof By definition, for any $x, y, z \in \mathcal{A}$, we have

$$[x, y] * \alpha(z) = [x, y] \dashv \alpha(z) - \alpha(z) \vdash [x, y] = (x \vdash y - y\perp x) \dashv \alpha(z) - \alpha(z) \vdash (x\perp y - y\perp y).$$

and

$$\begin{aligned} &[x * z, \alpha(y)] + [\alpha(x), y * z] \\ &= [x \dashv z - z \vdash, \alpha(y)] + [\alpha(x), y \dashv z - z \vdash y] \\ &= (x \vdash z - z \vdash x)\perp\alpha(y) - \alpha(y)\perp(x \dashv z - z \vdash x) + \alpha(x)\perp(y \dashv z - z \vdash y) - (y \dashv z - z \vdash y)\perp\alpha(x). \end{aligned}$$

Since

$$\begin{aligned} (x\perp y) \dashv \alpha(z) &= \alpha(x)\perp(y \vdash z), \\ (y\perp x) \dashv \alpha(z) &= \alpha(y)\perp(x \vdash z), \\ \alpha(z) \vdash (x\perp y) &= (z \vdash x)\perp\alpha(y), \\ \alpha(z) \vdash (y\perp x) &= (z \vdash y)\perp\alpha(x), \\ (x\perp z)\perp\alpha(y) &= \alpha(x)\perp(z \vdash y), \\ \alpha(y)\perp(z \vdash x) &= (y \dashv z)\perp\alpha(x), \end{aligned}$$

in a Hom-associative trialgebra, we get $[x, y] * \alpha(z) = [x * z, \alpha(y)] + [\alpha(x), y * z]$.

This ends the proof.

Proposition 2.6. Let $(\mathcal{A}, \dashv, \vdash, \perp, \alpha)$ be a Hom-trialgebra. Therefore, $(*, [-,-]\,, \alpha)$ is a non-commutative Hom-Leibniz-Poisson algebra with respect to the operations $x * y = x\perp y, \quad [x, y] = x \dashv y - x \vdash y$, for all $x, y, z \in \mathcal{A}$.

Proof If following from a straightforward computation.

Corollary 2.2. Let $(\mathcal{A}, \dashv, \vdash, \perp, \alpha)$ be a Hom-associative trialgebra. Hence, $(\mathcal{A}, *, [-,-]\,, \alpha)$ is a non-commutative Hom-Leibniz-Poisson algebra with

$$x * y = x \dashv y + x \vdash y - x\perp y$$

and

$$[x, y] = x * y - y * x \quad \text{for any} \quad x, y \in \mathcal{A}.$$

2.1. Classification of Finite-dimensional Complex Hom-associative Trialgebras

In this section, Hom-associative trialgebras are classified in low dimension. Note that $\mathcal{TH}_n^m$ denotes m^{th} isomorphism class of Hom-associative trialgebra in dimension n.

Let $(\mathcal{A}, \dashv, \vdash, \perp, \alpha)$ be an n-dimensionnal Hom-triagebra of $\mathcal{A}$. For any $i, j \in \mathbb{N}$, $1 \leq i,\ j \leq n$, let us put

$$e_i \dashv e_j = \sum_{k=1}^{n} \gamma_{ij}^k e_k, e_i \vdash e_j = \sum_{k=1}^{n} \delta_{ij}^k e_k, e_i\perp e_j = \sum_{k=1}^{n} \xi_{ij}^k e_k, \alpha(e_i) = \sum_{j=1}^{n} a_{ji}e_j.$$

The axioms in Definition 2.1 are ,respectivly, equivalent to

$$\sum_{p=1}^{n}\sum_{q=1}^{n} \gamma_{ij}^p a_{qk}\gamma_{pq}^r = \sum_{p=1}^{n}\sum_{q=1}^{n} a_{pi}\gamma_{jk}^q\gamma_{pq}^r, \tag{10}$$

$$\sum_{p=1}^{n}\sum_{q=1}^{n} \gamma_{ij}^p a_{qk}\gamma_{pq}^r = \sum_{p=1}^{n}\sum_{q=1}^{n} a_{pi}\delta_{jk}^q\gamma_{pq}^r = \sum_{p=1}^{n}\sum_{q=1}^{n} a_{pi}\xi\phi_{jk}^q\gamma_{pq}^r, \tag{11}$$

$$\sum_{p=1}^{n}\sum_{q=1}^{n} \delta_{ij}^p a_{qk}\gamma_{pq}^r = \sum_{p=1}^{n}\sum_{q=1}^{n} a_{pi}\gamma_{jk}^q\delta_{pq}^r, \tag{12}$$

$$\sum_{p=1}^{n}\sum_{q=1}^{n} \gamma_{ij}^p a_{qk}\delta_{pq}^r = \sum_{p=1}^{n}\sum_{q=1}^{n} a_{pi}\gamma_{jk}^q\delta_{pq}^r = \sum_{p=1}^{n}\sum_{q=1}^{n} \xi_{ij}^p a_{qk}\delta_{pq}^r, \tag{13}$$

$$\sum_{p=1}^{n}\sum_{q=1}^{n} \delta_{ij}^p a_{qk}\delta_{pq}^r = \sum_{p=1}^{n}\sum_{q=1}^{n} a_{pi}\delta_{jk}^q\delta_{pq}^r, \tag{14}$$

$$\sum_{p=1}^{n}\sum_{q=1}^{n}\xi i j^{p} a_{qk}\gamma_{pq}^{r}=\sum_{p=1}^{n}\sum_{q=1}^{n}a_{pi}\gamma_{jk}^{q}\xi_{pq}^{r}, \tag{15}$$

$$\sum_{p=1}^{n}\sum_{q=1}^{n}\gamma_{ij}^{p}a_{qk}\xi_{pq}^{r}=\sum_{p=1}^{n}\sum_{q=1}^{n}a_{pi}\delta_{jk}^{q}\xi_{pq}^{r}, \tag{16}$$

$$\sum_{p=1}^{n}\sum_{q=1}^{n}\delta_{ij}^{p}a_{qk}\xi_{pq}^{r}=\sum_{p=1}^{n}\sum_{q=1}^{n}a_{pi}\xi_{jk}^{q}\delta_{pq}^{r}, \tag{17}$$

$$\sum_{p=1}^{n}\sum_{q=1}^{n}\xi_{ij}^{p}a_{qk}\xi_{pq}^{r}=\sum_{p=1}^{n}\sum_{q=1}^{n}a_{pi}\xi_{jk}^{q}\xi_{pq}^{r}. \tag{18}$$

where $(a_{ij})_{1\leq i,j\leq n}$ refers to the matrix of α and (γ_{ij}^{k}), (δ_{ij}^{k}) and (ξ_{ij}^{k}) stand for the structure constants of $\mathcal{A}$. The axioms in Remark 2.1 are,respectively, equivalent to

$$\sum_{k=1}^{n}\gamma_{ij}^{k}a_{qk}=\sum_{k=1}^{n}\sum_{p=1}^{n}a_{ki}a_{pj}\gamma_{kp}^{q},\quad \sum_{k=1}^{n}\delta_{ij}^{k}a_{qk}=\sum_{k=1}^{n}\sum_{p=1}^{n}a_{ki}a_{pj}\delta_{kp}^{q},\quad \sum_{k=1}^{n}\xi_{ij}^{k}a_{qk}=\sum_{k=1}^{n}\sum_{p=1}^{n}a_{ki}a_{pj}\xi_{kp}^{q}.$$

Theorem 2.1. Any 2-dimensional complex Hom-associative trialgebra is either associative or isomorphic to one of the following pairwise non-isomorphic Hom-associative trialgebras:

Table 1. *2-dimensional complex Hom-associative trialgebras.*

$\mathcal{TH}_2^m$	$\dashv$	$\vdash$	$\perp$	α
$\mathcal{TH}_2^1$	$e_1\dashv e_2=e_1,$ $e_2\dashv e_1=e_1,$ $e_2\dashv e_2=e_1$	$e_1\vdash e_2=e_1,$ $e_2\vdash e_1=e_1,$ $e_2\vdash e_2=e_1,$	$e_1\perp e_2=e_1,$ $e_2\perp e_1=e_1,$ $e_2\perp e_2=e_1,$	$\alpha(e_2)=e_1.$
$\mathcal{TH}_2^2$	$e_1\dashv e_2=e_1,$ $e_2\dashv e_1=e_1,$ $e_2\dashv e_2=e_1$	$e_1\vdash e_2=e_1,$ $e_2\vdash e_1=e_1,$ $e_2\vdash e_2=e_1,$	$e_2\perp e_2=e_1,$	$\alpha(e_2)=e_1.$
$\mathcal{TH}_2^3$	$e_1\dashv e_2=e_1,$ $e_2\dashv e_1=e_1,$ $e_2\dashv e_2=e_1,$	$e_1\vdash e_2=e_1,$ $e_2\vdash e_1=e_1,$	$e_2\perp e_2=e_1,$	$\alpha(e_2)=e_1.$
$\mathcal{TH}_2^4$	$e_2\dashv e_2=e_1,$	$e_2\vdash e_2=e_1,$	$e_2\perp e_2=e_1,$	$\alpha(e_1)=e_1,$ $\alpha(e_2)=e_1+e_2.$
$\mathcal{TH}_2^5$	$e_2\dashv e_1=e_1,$ $e_2\dashv e_2=e_1+e_2,$	$e_2\vdash e_1=e_1,$ $e_2\vdash e_2=e_1+e_2,$	$e_2\perp e_1=e_1$ $e_2\perp e_2=e_1+e_2,$	$\alpha(e_1)=e_1,$ $\alpha(e_2)=e_1+e_2.$
$\mathcal{TH}_2^6$	$e_1\dashv e_2=e_1,$ $e_2\dashv e_2=e_1+e_2,$	$e_1\vdash e_2=e_1,$ $e_2\vdash e_2=e_1+e_2,$	$e_1\perp e_2=e_1$ $e_2\perp e_2=e_1+e_2,$	$\alpha(e_1)=e_1,$ $\alpha(e_2)=e_1+e_2.$
$\mathcal{TH}_2^7$	$e_1\dashv e_2=e_1,$ $e_2\dashv e_2=e_2,$	$e_2\vdash e_1=e_1,$ $e_2\vdash e_2=e_2,$	$e_1\perp e_1=e_1$ $e_1\perp e_2=e_1$ $e_2\perp e_1=e_1$ $e_2\perp e_2=e_2,$	$\alpha(e_1)=e_1,$ $\alpha(e_2)=e_2.$
$\mathcal{TH}_2^8$	$e_1\dashv e_2=e_1,$ $e_2\dashv e_2=e_2,$	$e_2\vdash e_1=e_1,$ $e_2\vdash e_2=e_2,$	$e_1\perp e_2=e_1,$ $e_2\perp e_1=e_1,$ $e_2\perp e_2=e_2,$	$\alpha(e_1)=e_1,$ $\alpha(e_2)=e_2.$
$\mathcal{TH}_2^9$	$e_1\dashv e_1=-e_1,$ $e_1\dashv e_2=e_2,$ $e_2\dashv e_1=e_2,$	$e_1\vdash e_1=-e_1,$ $e_1\vdash e_2=e_2,$ $e_2\vdash e_1=e_2,$	$e_1\perp e_1=-e_1,$ $e_1\perp e_2=e_1,$ $e_2\perp e_2=e_2,$	$\alpha(e_1)=e_1,$ $\alpha(e_2)=-e_2.$
$\mathcal{TH}_2^{10}$	$e_1\dashv e_1=-e_1,$ $e_1\dashv e_2=e_2,$	$e_1\vdash e_1=-e_1,$ $e_1\vdash e_2=e_2,$	$e_1\perp e_1=-e_1,$ $e_1\perp e_2=e_2,$ $e_2\perp e_1=e_2,$	$\alpha(e_1)=e_1,$ $\alpha(e_2)=-e_2.$
$\mathcal{TH}_2^{11}$	$e_1\dashv e_1=e_1,$ $e_2\dashv e_2=e_2,$	$e_1\vdash e_1=e_1,$ $e_2\vdash e_2=e_2,$	$e_1\perp e_1=e_1,$ $e_2\perp e_2=e_2,$	$\alpha(e_2)=e_2.$
$\mathcal{TH}_2^{12}$	$e_2\dashv e_2=e_2,$	$e_2\vdash e_2=e_2,$	$e_1\perp e_1=e_1,$ $e_2\perp e_2=e_2,$	$\alpha(e_1)=e_1.$
$\mathcal{TH}_2^{13}$	$e_1\dashv e_2=e_1,$ $e_2\dashv e_2=e_2,$	$e_2\vdash e_1=e_1,$ $e_2\vdash e_2=e_2,$	$e_1\perp e_1=e_1,$ $e_1\perp e_2=e_1,$ $e_2\perp e_1=e_1,$ $e_2\perp e_2=e_1+e_2,$	$\alpha(e_1)=e_1,$ $\alpha(e_2)=e_2.$

Proof Let $\mathcal{A}$ be a two-dimensional vector space. To determine a Hom-associative trialgebra structure on $\mathcal{A}$, we consider $\mathcal{A}$ with respect to one Hom-associative trialgebra operation. Let $\mathcal{TH}_2^m = (\mathcal{A}, \vdash, \alpha)$ be the Hom-algebra

$e_1 \vdash e_1 = -e_1, \quad e_1 \vdash e_2 = e_2,$

$e_2 \vdash e_1 = e_2, \quad \alpha(e_1) = e_1, \quad \alpha(e_2) = -e_2.$

The multiplication operations $\dashv, \perp \; in \; \mathcal{A}$, we define as follows:

$$\begin{array}{ll} e_1 \dashv e_1 = a_1e_1 + a_2e_2, & e_1 \perp e_1 = b_1e_1 + b_2e_2, \\ e_1 \dashv e_2 = a_3e_1 + a_4e_2, & e_1 \perp e_2 = b_3e_1 + b_4e_2, \\ e_2 \dashv e_1 = a_5e_1 + a_6e_2, & e_2 \perp e_1 = b_5e_1 + b_6e_2, \\ e_2 \dashv e_2 = a_7e_1 + a_8e_2, & e_2 \perp e_2 = b_7e_1 + b_8e_2. \end{array}$$

Now verifying Hom-associative trialgebra axioms, we get several constraints for the coefficients $a_i, bi \in \mathbb{R}$ where $1 \leq i \leq 8$.

Applying $(e_1 \dashv e_1) \vdash \alpha(e_1) = \alpha(e_1) \vdash (e_1 \vdash e_1)$, we get $(a_1e_1 + a_2e_2) \vdash e_1 = e_1 \vdash (e_1 \vdash e_1)$ and then $e_1 \vdash e_1 = 1$.

Therefore $a_1 = -1$.

The verification of $(e_1 \vdash e_1) \dashv \alpha(e_1) = \alpha(e_1) \vdash (e_1 \dashv e_1)$ leads to $e_1 \dashv e_1 = e_1 \vdash (e_1 + a_2e_2)$ and from this we get $e_1 + a_2e_2 = e_1$. Hence we obtain $a_2 = 0$.

Consider $(e_1 \dashv e_1) \dashv \alpha(e_1) = \alpha(e_1) \dashv (e_1 \vdash e_1)$.

It implies that $e_1 \dashv e_2 = 1$, therefore $a_3 = 1$ and $a_4 = 0$.

The next relation to consider is $(e_1 \dashv e_2) \dashv \alpha(e_1) = \alpha(e_1) \dashv (e_2 \dashv e_1)$. It implies that $1 = e_1 \dashv (a_5e_1 + a_6e_2)$ and we get $a_5 = 0$ and $a_6 = 1$. To find a_7 and a_8, we note that $(e_2 \dashv e_2) \dashv \alpha(e_1) = \alpha(e_2) \dashv (e_2 \vdash e_1) \Rightarrow (a_7e_1 + a_8e_2) \dashv e_1 = 0 \Rightarrow a_7e_1 + a_6a_8e_2 = 0$. Hence we have $a_7 = 0$, $a_6a_8 = 0$. Finally, we apply $(e_2 \dashv e_2) \dashv \alpha(e_2) = \alpha(e_2) \dashv (e_2 \vdash e_2) \Rightarrow a_8(e_2 \dashv e_2) = 0$, and get $a_8 = 0$.

Applying $(e_1 \perp e_1) \vdash \beta(e_1) = \alpha(e_1) \vdash (e_1 \vdash e_1)$, we get $(y_1e_1 + y_2e_2) \vdash e_1 = e_1 \dashv (e_1 \vdash e_1)$ and then $e_1 \vdash e_1 = 0$. Therefore $y_1 = 0 \; and \; y_2 = 0$. Consider $(e_1 \perp e_2) \dashv \beta(e_1) = \alpha(e_1) \perp (e_2 \dashv e_1)$. It implies that $e_1 \perp e_2 = 0$, therefore $y_3 = 0$ and $y_4 = 0$.

The next relation to consider is $(e_1 \perp e_2) \dashv \alpha(e_1) = \alpha(e_2) \perp (e_2 \dashv e_1)$.

It implies that $0 = e_2 \perp (b_5e_1 + b_6e_2)$ and we get $b_5 = 0$ and $b_6 = 0$. To find b_7 and , b_8 we note that $(e_2 \perp e_2) \dashv \alpha(e_1) = \alpha(e_2) \dashv (e_2 \vdash e_1) \Rightarrow (b_7e_1 + b_8e_2) \dashv e_1 = 1 \Rightarrow b_7e_1 + b_6b_8e_2 = 1$. Hence we have $b_7 = 0$, $b_6b_8 = 0$. Finally, we apply $(e_2 \perp e_2) \dashv \alpha(e_2) = \alpha(e_2) \dashv (e_2 \vdash e_2) \Rightarrow b_8(e_2 \vdash e_2) = 1$, and get $b_8 = 1$. The verification of all other cases leads to the obtained constraints. Thus, in this case we come to the Hom-associative trialgebra with the multiplication table:

$$\begin{array}{lll} e_1 \dashv e_1 = -e_1, & e_1 \vdash e_1 = -e_1, & e_1 \perp e_1 = -e_1, \\ e_1 \dashv e_2 = e_2, & e_1 \vdash e_2 = e_2, & e_1 \perp e_2 = e_1, \\ e_2 \dashv e_1 = e_2, & e_2 \vdash e_1 = e_2, & e_2 \perp e_2 = e_2, \end{array}$$

$$\alpha(e_1) = e_1, \quad \alpha(e_2) = -e_2.$$

Then $\mathcal{TH}_2^m = (\mathcal{A}, \dashv, \alpha)$ it is isomorphic to $\mathcal{TH}_2^9$. The other Hom-associative trialgebras of the list of Theorem 2.1 can be obtained by minor modification of the observation above.

Theorem 2.2. Any 3-dimensional complex Hom-associative trialgebra is either associative or isomorphic to one of the following pairwise non-isomorphic Hom-associative trialgebras:

Table 2. *3-dimensional complex Hom-associative trialgebras.*

$\mathcal{TH}_3^m$	$\dashv$	$\vdash$	$\perp$	α
$\mathcal{TH}_3^1$	$e_2 \dashv e_2 = e_2 + e_3,$ $e_2 \dashv e_3 = e_2 + e_3,$ $e_3 \dashv e_2 = e_2 + e_3,$	$e_2 \vdash e_2 = e_2 + e_3,$ $e_3 \vdash e_2 = e_2 + e_3,$ $e_3 \vdash e_3 = e_2 + e_3,$	$e_2 \perp e_2 = e_2 + e_3,$ $e_2 \perp e_3 = e_2 + e_3,$ $e_3 \perp e_2 = e_2 + e_3,$ $e_3 \perp e_3 = e_2 + e_3,$	$\alpha(e_1) = e_1.$
$\mathcal{TH}_3^2$	$e_2 \dashv e_2 = e_2 + e_3,$ $e_3 \dashv e_2 = e_2 + e_3,$ $e_3 \dashv e_3 = e_2 + e_3,$	$e_2 \vdash e_2 = e_2 + e_3,$ $e_2 \vdash e_3 = e_2 + e_3,$ $e_3 \vdash e_3 = e_2 + e_3,$	$e_2 \perp e_3 = e_2 + e_3,$ $e_3 \perp e_2 = e_2 + e_3,$ $e_3 \perp e_3 = e_2 + e_3,$	$\alpha(e_1) = e_1.$
$\mathcal{TH}_3^3$	$e_1 \dashv e_1 = ae_2,$ $e_3 \dashv e_3 = e_3,$	$e_1 \vdash e_1 = e_2,$ $e_3 \vdash e_3 = e_2,$	$e_1 \perp e_1 = e_2,$ $e_3 \perp e_3 = e_2,$	$\alpha(e_1) = e_1,$ $\alpha(e_2) = e_2.$
$\mathcal{TH}_3^4$	$e_1 \dashv e_1 = e_3,$ $e_2 \dashv e_2 = e_2,$	$e_1 \vdash e_1 = e_3,$ $e_2 \vdash e_2 = e_2,$	$e_1 \perp e_1 = e_3,$ $e_2 \perp e_2 = e_2,$	$\alpha(e_1) = e_1,$ $\alpha(e_3) = e_3.$
$\mathcal{TH}_3^5$	$e_1 \dashv e_1 = ae_1,$ $e_2 \dashv e_2 = e_2,$ $e_3 \dashv e_2 = e_3,$	$e_1 \vdash e_1 = e_1,$ $e_2 \vdash e_2 = e_2,$ $e_3 \vdash e_2 = e_3,$	$e_1 \perp e_1 = e_1,$ $e_2 \perp e_2 = e_2,$ $e_3 \perp e_2 = e_3,$	$\alpha(e_2) = e_2,$ $\alpha(e_3) = e_3.$
$\mathcal{TH}_3^6$	$e_1 \dashv e_1 = e_1 + e_3,$ $e_1 \dashv e_3 = e_1 + e_3,$ $e_3 \dashv e_1 = e_1 + e_3,$	$e_1 \vdash e_1 = e_1 + e_3,$ $e_1 \vdash e_3 = e_1 + e_3,$ $e_3 \vdash e_1 = e_1 + e_3,$	$e_1 \perp e_1 = e_1 + e_3,$ $e_1 \perp e_3 = e_1 + e_3,$ $e_3 \perp e_1 = e_1 + e_3,$	$\alpha(e_2) = e_2.$

$\mathcal{TH}_3^m$	$\dashv$	$\vdash$	$\perp$	α
$\mathcal{TH}_3^7$	$e_1 \dashv e_1 = e_1 + e_2,$ $e_1 \dashv e_2 = e_1 + e_2,$ $e_2 \dashv e_2 = e_1 + e_2,$	$e_1 \vdash e_1 = e_1 + e_2,$ $e_1 \vdash e_2 = e_1 + e_2,$ $e_2 \vdash e_1 = e_1 + e_2,$ $e_2 \vdash e_2 = e_1 + e_2,$	$e_2 \perp e_1 = e_1 + e_2,$ $e_2 \perp e_2 = e_1 + e_2,$	$\alpha(e_3) = e_3.$
$\mathcal{TH}_3^8$	$e_1 \dashv e_2 = e_1 + e_2,$ $e_2 \dashv e_1 = e_1 + e_2,$ $e_2 \dashv e_2 = e_1 + e_2,$	$e_2 \vdash e_1 = e_1 + e_2,$ $e_2 \vdash e_2 = e_1 + e_2,$	$e_1 \perp e_1 = e_1 + e_2,$ $e_1 \perp e_2 = e_1 + e_2,$ $e_2 \perp e_2 = e_1 + e_2,$	$\alpha(e_3) = e_3.$
$\mathcal{TH}_3^9$	$e_2 \dashv e_2 = ae_2 - be_3,$	$e_2 \vdash e_2 = e_2 + de_3,$	$e_2 \perp e_2 = be_2 + e_3,$	$\alpha(e_1) = e_1,$ $\alpha(e_2) = e_1 + e_2,$ $\alpha(e_3) = e_2 + e_3.$
$\mathcal{TH}_3^{10}$	$e_1 \dashv e_2 = e_1,$ $e_2 \dashv e_1 = e_1,$ $e_2 \dashv e_2 = e_1,$	$e_2 \vdash e_1 = e_1,$ $e_2 \vdash e_2 = e_1,$ $e_2 \vdash e_3 = e_1,$	$e_2 \perp e_2 = e_1,$ $e_2 \perp e_3 = e_1,$ $e_3 \perp e_2 = e_1,$	$\alpha(e_2) = e_1,$ $\alpha(e_3) = e_3.$
$\mathcal{TH}_3^{11}$	$e_2 \dashv e_1 = e_1,$ $e_2 \dashv e_2 = e_1,$ $e_3 \dashv e_2 = e_1,$	$e_1 \vdash e_2 = e_1,$ $e_2 \vdash e_1 = e_1,$ $e_3 \vdash e_2 = e_1,$	$e_2 \perp e_2 = e_1,$ $e_3 \perp e_2 = e_1,$	$\alpha(e_2) = e_1,$ $\alpha(e_3) = e_3.$
$\mathcal{TH}_3^{12}$	$e_2 \dashv e_1 = e_1 + e_3,$ $e_2 \dashv e_2 = e_1 + e_3,$ $e_3 \dashv e_3 = e_1 + e_3,$ $e_1 \dashv e_2 = e_1 + e_3,$	$e_2 \vdash e_1 = e_1 + e_3,$ $e_2 \vdash e_3 = e_1 + e_3,$	$e_2 \perp e_2 = e_1 + e_3,$ $e_2 \perp e_3 = e_1 + e_3,$ $e_3 \perp e_3 = e_1 + e_3,$	$\alpha(e_2) = e_1.$
$\mathcal{TH}_3^{13}$	$e_3 \dashv e_2 = e_1 + e_3,$ $e_3 \dashv e_3 = e_1 + e_3,$	$e_1 \vdash e_2 = e_1 + e_3,$ $e_2 \vdash e_3 = e_1 + e_3,$ $e_3 \vdash e_2 = e_1 + e_3,$ $e_3 \vdash e_3 = e_1 + e_3,$	$e_2 \perp e_2 = e_1 + e_3,$ $e_3 \perp e_2 = e_1 + e_3,$ $e_3 \perp e_3 = e_1 + e_3,$	$\alpha(e_2) = e_1.$
$\mathcal{TH}_3^{14}$	$e_2 \dashv e_3 = e_1 + e_3,$ $e_3 \dashv e_2 = e_1 + e_3,$ $e_3 \dashv e_3 = e_1 + e_3,$	$e_2 \vdash e_2 = e_1 + e_3,$ $e_3 \vdash e_2 = e_1 + e_3,$ $e_3 \vdash e_3 = e_1 + e_3,$	$e_3 \perp e_2 = e_1 + e_3,$ $e_3 \perp e_3 = e_1 + e_3,$	$\alpha(e_2) = e_1.$
$\mathcal{TH}_3^{15}$	$e_1 \dashv e_2 = e_1 + e_3,$ $e_3 \dashv e_2 = e_1 + e_3,$ $e_3 \dashv e_3 = e_1 + e_3,$	$e_2 \vdash e_3 = e_1 + e_3,$ $e_3 \vdash e_2 = e_1 + e_3,$ $e_3 \vdash e_3 = e_1 + e_3,$	$e_2 \perp e_1 = e_3,$ $e_2 \perp e_3 = e_3,$ $e_3 \perp e_3 = e_3,$	$\alpha(e_2) = e_1.$
$\mathcal{TH}_3^{16}$	$e_2 \dashv e_1 = e_3,$ $e_2 \dashv e_2 = e_3,$ $e_3 \dashv e_3 = e_1 + e_3,$	$e_1 \vdash e_2 = e_1,$ $e_3 \vdash e_2 = e_1 + e_3,$ $e_3 \vdash e_3 = e_1,$	$e_3 \perp e_2 = e_3,$ $e_3 \perp e_3 = e_3,$	$\alpha(e_2) = e_1.$
$\mathcal{TH}_3^{17}$	$e_1 \dashv e_2 = e_1 + e_3,$ $e_2 \dashv e_1 = e_1 + e_3,$	$e_1 \vdash e_2 = e_1,$ $e_3 \vdash e_3 = e_1,$	$e_1 \perp e_2 = e_3,$ $e_2 \perp e_1 = e_3,$ $e_3 \perp e_2 = e_1,$ $e_3 \perp e_3 = e_3,$	$\alpha(e_2) = e_1.$
$\mathcal{TH}_3^{18}$	$e_2 \dashv e_3 = e_1 + e_3,$ $e_3 \dashv e_2 = e_1 + e_3,$ $e_3 \dashv e_3 = e_1 + e_3,$	$e_2 \vdash e_1 = e_3,$ $e_2 \vdash e_2 = e_1 + e_3,$ $e_3 \vdash e_3 = e_1 + e_3,$	$e_1 \perp e_2 = e_1 + e_3,$ $e_2 \perp e_3 = e_1,$ $e_3 \perp e_2 = e_1 + e_3,$	$\alpha(e_2) = e_1.$
$\mathcal{TH}_3^{19}$	$e_2 \dashv e_3 = e_2,$ $e_3 \dashv e_2 = e_2,$ $e_3 \dashv e_3 = e_3,$	$e_2 \vdash e_2 = e_2 + e_3,$ $e_3 \vdash e_3 = e_2 + e_3,$	$e_2 \perp e_3 = e_3,$ $e_3 \perp e_2 = e_3,$ $e_3 \perp e_3 = e_3,$	$\alpha(e_1) = e_1.$
$\mathcal{TH}_3^{20}$	$e_1 \dashv e_3 = e_1,$ $e_2 \dashv e_3 = e_1,$ $e_3 \dashv e_3 = e_1,$	$e_3 \vdash e_1 = e_1,$ $e_3 \vdash e_2 = e_1,$ $e_3 \vdash e_3 = e_1,$	$e_1 \perp e_3 = e_1,$ $e_2 \perp e_3 = e_1,$ $e_3 \perp e_3 = e_1,$	$\alpha(e_2) = e_1,$ $\alpha(e_3) = e_2.$
$\mathcal{TH}_3^{21}$	$e_2 \dashv e_3 = e_1,$ $e_3 \dashv e_2 = e_1,$ $e_3 \dashv e_3 = e_1,$	$e_3 \vdash e_1 = e_1,$ $e_3 \vdash e_3 = e_1,$	$e_1 \perp e_3 = e_1,$ $e_2 \perp e_3 = e_1,$ $e_3 \perp e_1 = e_1,$ $e_3 \perp e_2 = e_1,$	$\alpha(e_2) = e_1,$ $\alpha(e_3) = e_2.$

Proof Let $\mathcal{A}$ be a three-dimensional vector space. To determine a Hom-associative trialgebra structure on $\mathcal{A}$, we consider $\mathcal{A}$ with respect to one Hom-associative trialgebra operation. Let $\mathcal{TH}_3 = (\mathcal{A}, \dashv, \alpha)$ be the Hom-algebra

$$e_3 \vdash e_1 = e_1, \quad e_3 \vdash e_2 = e_1, \quad e_3 \vdash e_3 = e_1, \ , \alpha(e_1) = e_1, \quad \alpha(e_3) = e_2.$$

The multiplication operations $\vdash, \perp in\, \mathcal{A}$. We use the same method of the Proof of the Theorem 2.2. Then $\mathcal{TH}_3^m = (\mathcal{A}, \dashv, \alpha)$ it is isomorphic to $\mathcal{TH}_3^{20}$.The other Hom-associative trialgebra of the list of Theorem 2.2 can be obtained by minor modification of the observation.

3. α-Inner-derivations of n-dimensional Hom-associative Trialgebras

This section sets forward a detailed description of α-*Inner-derivations* of Hom-associative trialgebras in dimensions two and three over the field $\mathbb{C}$.

3.1. *Properties of α-Inner-derivations of Hom-associative Trialgebras*

Definition 3.1. An α-*Inner-derivation* of the Hom-associative trialgebra $\mathcal{A}$ is a linear transformation $\mathcal{I} : \mathcal{A} \to \mathcal{A}$ satisfying

$$\begin{aligned} \alpha \circ \mathcal{I} &= \mathcal{I} \circ \alpha, && (19)\\ ad_z(x) &= \mathcal{I}(x) \dashv \alpha(z) - \alpha(x) \dashv \mathcal{I}(z), && (20)\\ ad_z(x) &= \mathcal{I}(x) \vdash \alpha(z) - \alpha(x) \vdash \mathcal{I}(z), && (21)\\ ad_z(x) &= \mathcal{I}(x) \perp \alpha(z) - \alpha(x) \perp \mathcal{I}(z), && (22)\end{aligned}$$

for all $x, y \in \mathcal{A}$.

Let $(\mathcal{A}, \dashv, \vdash, \alpha)$ be a Hom-associative trialgebra over $\mathbb{K}$. For $z \in \mathcal{A}$, we have

$$ad_z(X) = X \dashv z - z \vdash X, \quad (\forall, X \in \mathcal{A}).$$

We can,therefore, prove that ad_z is a α-derivation of $(\mathcal{A}, \dashv, \vdash, \alpha)$. For any $X, Y \in \mathcal{A}$, we get

$$ad_z(X \dashv Y) = (X \dashv Y) \dashv \alpha(z) - \alpha(z) \vdash (X \dashv Y)$$

and

$$\begin{aligned} ad_z(X) \dashv \alpha(Y) \quad &+ \quad \alpha(X) \dashv ad_z(Y)\\ &= (X \dashv z - z \vdash X) \dashv \alpha(Y) + \alpha(X) \dashv (Y \dashv z - z \vdash Y)\\ &= \alpha(X) \dashv (z \dashv Y) - (z \vdash X) \dashv \alpha(Y) + \alpha(X) \dashv (Y \dashv z) - \alpha(X) \dashv (z \dashv Y)\\ &= \alpha(X) \dashv (Y \dashv z) - (z \vdash X) \dashv \alpha(Y)\\ &= \alpha(X) \dashv (Y \perp z) - (z \vdash X) \dashv \alpha(Y).\end{aligned}$$

Hence,

$$ad_z(X \dashv Y) = ad_z(X) \dashv \alpha(Y) + \alpha(X) \dashv ad_z(Y).$$

On the other side,

$$\begin{aligned} ad_z(X \vdash Y) &= (X \vdash Y) \dashv \alpha(z) + \alpha(z) \vdash (X \vdash Y)\\ &= (X \vdash Y) \dashv (z) - \alpha(z) \dashv (X \perp Y)\end{aligned}$$

and

$$\begin{aligned} ad_z(X) \vdash \alpha(Y) \quad &+ \quad \alpha(X) \vdash ad_z(Y)\\ &= (X \dashv z - z \vdash X) \vdash \alpha(Y) + \alpha(X) \vdash (Y \dashv z - z \vdash Y)\\ &= (X \vdash z) \vdash \alpha(Y) - (z \vdash X) \vdash \alpha(Y) + \alpha(X) \vdash (Y \dashv z) - (X \vdash z) \vdash \alpha(Y)\\ &= \alpha(X) \vdash (Y \dashv z) - (z \vdash X) \vdash \alpha(Y)\end{aligned}$$

Thus, it follows that

$$ad_z(X \vdash Y) = ad_z(X) \vdash \alpha(Y) + \alpha(X) \vdash ad_z(Y).$$

3.2. Classification of α-Inner-derivations of Hom-associative Trialgebras

Let $\{e_1, e_2, e_3, \cdots, e_n\}$ be a basis of an n-dimensional Hom-associative trialgebra $\mathcal{A}$. The product of basis is denoted by

$$\mathcal{I}(e_p) = \sum_{q=1}^{n} \mathcal{I}_{qp} e_q.$$

We have

$$\sum_{p=1}^{n} \mathcal{I}_{pk} a_{qp} = \sum_{p=1}^{n} a_{pk} \mathcal{I}_{qp} \quad , \tag{23}$$

$$\sum_{k=1}^{n} \gamma_{ij}^{p} \mathcal{I}_{rp} = \sum_{p=1}^{n} \sum_{q=1} \mathcal{I}_{pi} a_{qj} \gamma_{pq}^{r} \quad - \quad \sum_{p=1}^{n} \sum_{q=1} a_{pi} \mathcal{I}_{qj} \gamma_{pq}^{r}, \tag{24}$$

$$\sum_{k=1}^{n} \delta_{ij}^{p} \mathcal{I}_{rp} = \sum_{p=1}^{n} \sum_{q=1} \mathcal{I}_{pi} a_{qj} \delta_{pq}^{r} \quad - \quad \sum_{p=1}^{n} \sum_{q=1} a_{pi} \mathcal{I}_{qj} \delta_{pq}^{r}, \tag{25}$$

$$\sum_{k=1}^{n} \xi_{ij}^{p} \mathcal{I}_{rp} = \sum_{p=1}^{n} \sum_{q=1} \mathcal{I}_{pi} a_{qj} \xi_{pq}^{r} \quad - \quad \sum_{p=1}^{n} \sum_{q=1} a_{pi} \mathcal{I}_{qj} \xi_{pq}^{r}\,. \tag{26}$$

Theorem 3.1. The *α-Inner-derivations* of 2-dimensional Hom-associative trialgebras have the following form :

Table 3. *α-Inner-derivations of 2-dimensional Hom-associative trialgebras.*

$\mathcal{TH}_2^m$	Der(α-Inner-derivations)	Dim(α-Inner-derivations)
$\mathcal{TH}_2^1$	$\begin{pmatrix} 0 & 0 \\ \mathcal{I}_{21} & 0 \end{pmatrix}$	1
$\mathcal{TH}_2^2$	$\begin{pmatrix} 0 & 0 \\ \mathcal{I}_{21} & 0 \end{pmatrix}$	1
$\mathcal{TH}_2^3$	$\begin{pmatrix} 0 & 0 \\ \mathcal{I}_{21} & 0 \end{pmatrix}$	1
$\mathcal{TH}_2^4$	$\begin{pmatrix} 0 & 0 \\ \mathcal{I}_{21} & \mathcal{I}_{22} \end{pmatrix}$	2
$\mathcal{TH}_2^6$	$\begin{pmatrix} 0 & 0 \\ \mathcal{I}_{21} & 0 \end{pmatrix}$	1

Proof Resting upon Theorem 3.1, we provide the proof only for one case to illustrate the used approach. The other cases can handled similarly with or without modification(s). Let us consider $\mathcal{TH}_2^4$. Applying the systems of equations (23), (24), (25) and (26), we get $\mathcal{I}_{11} = \mathcal{I}_{12} = 0$. Thus, the α-Inner-derivations of $\mathcal{TH}_2^4$ are expressed as follows:

$\mathcal{I}_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $\mathcal{I}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ is the basis of Der(α-*Inner-derivations*) and Dim(Der(α-*Inner-derivations*))=2. The α-Inner-derivations of the remaining parts of the two-dimension Hom-associative trialgebras can be tackled in a similar manner as depicted above.

Theorem 3.2. The *α-Inner-derivations* of 3-dimensional Hom-associative trialgebras have the following form :

Table 4. *α-Inner-derivations of 3-dimensional Hom-associative trialgebras.*

$\mathcal{TH}_3^m$	Der(α-Inner-derivations)	Dim(α-Inner-derivations)	$\mathcal{TH}_3^m$	Der(α-Inner-derivations)	Dim(α-Inner-derivations)
$\mathcal{TH}_3^1$	$\begin{pmatrix} \mathcal{I}_{11} & 0 & 0 \\ 0 & \mathcal{I}_{22} & \mathcal{I}_{23} \\ 0 & -\mathcal{I}_{22} & -\mathcal{I}_{23} \end{pmatrix}$	3	$\mathcal{TH}_3^2$	$\begin{pmatrix} \mathcal{I}_{11} & 0 & 0 \\ 0 & \mathcal{I}_{22} & \mathcal{I}_{23} \\ 0 & -\mathcal{I}_{22} & -\mathcal{I}_{23} \end{pmatrix}$	3
$\mathcal{TH}_3^3$	$\begin{pmatrix} \mathcal{I}_{11} & \mathcal{I}_{21} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathcal{I}_{33} \end{pmatrix}$	3	$\mathcal{TH}_3^4$	$\begin{pmatrix} \mathcal{I}_{11} & 0 & \mathcal{I}_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	2

$\mathcal{TH}_3^m$	Der(α-Inner-derivations)	Dim(α-Inner-derivations)	$\mathcal{TH}_3^m$	Der(α-Inner-derivations)	Dim(α-Inner-derivations)
$\mathcal{TH}_3^5$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \mathcal{I}_{23} \\ 0 & 0 & \mathcal{I}_{33} \end{pmatrix}$	2	$\mathcal{TH}_3^6$	$\begin{pmatrix} \mathcal{I}_{11} & 0 & \mathcal{I}_{13} \\ 0 & \mathcal{I}_{22} & 0 \\ -\mathcal{I}_{11} & 0 & -\mathcal{I}_{13} \end{pmatrix}$	3
$\mathcal{TH}_3^7$	$\begin{pmatrix} \mathcal{I}_{11} & \mathcal{I}_{12} & 0 \\ -\mathcal{I}_{11} & -\mathcal{I}_{12} & 0 \\ 0 & 0 & \mathcal{I}_{33} \end{pmatrix}$	3	$\mathcal{TH}_3^8$	$\begin{pmatrix} \mathcal{I}_{11} & \mathcal{I}_{12} & 0 \\ -\mathcal{I}_{11} & -\mathcal{I}_{12} & 0 \\ 0 & 0 & \mathcal{I}_{33} \end{pmatrix}$	3
$\mathcal{TH}_3^{10}$	$\begin{pmatrix} 0 & 0 & 0 \\ \mathcal{I}_{21} & 0 & 0 \\ 0 & 0 & \mathcal{I}_{33} \end{pmatrix}$	2	$\mathcal{TH}_3^{11}$	$\begin{pmatrix} 0 & 0 & 0 \\ \mathcal{I}_{21} & 0 & 0 \\ 0 & 0 & \mathcal{I}_{33} \end{pmatrix}$	2
$\mathcal{TH}_3^{12}$	$\begin{pmatrix} 0 & 0 & 0 \\ \mathcal{I}_{21} & \mathcal{I}_{23} & 0 \\ 0 & 0 & 0 \end{pmatrix}$	2	$\mathcal{TH}_3^{13}$	$\begin{pmatrix} 0 & 0 & 0 \\ \mathcal{I}_{21} & \mathcal{I}_{23} & 0 \\ 0 & 0 & 0 \end{pmatrix}$	2
$\mathcal{TH}_3^{14}$	$\begin{pmatrix} 0 & 0 & 0 \\ \mathcal{I}_{21} & \mathcal{I}_{23} & 0 \\ 0 & 0 & 0 \end{pmatrix}$	2	$\mathcal{TH}_3^{15}$	$\begin{pmatrix} 0 & 0 & 0 \\ \mathcal{I}_{21} & \mathcal{I}_{23} & 0 \\ 0 & 0 & 0 \end{pmatrix}$	2
$\mathcal{TH}_3^{16}$	$\begin{pmatrix} 0 & 0 & 0 \\ \mathcal{I}_{21} & \mathcal{I}_{23} & 0 \\ 0 & 0 & 0 \end{pmatrix}$	2	$\mathcal{TH}_3^{17}$	$\begin{pmatrix} 0 & 0 & 0 \\ \mathcal{I}_{21} & \mathcal{I}_{23} & 0 \\ 0 & 0 & 0 \end{pmatrix}$	2
$\mathcal{TH}_3^{18}$	$\begin{pmatrix} 0 & 0 & 0 \\ \mathcal{I}_{21} & \mathcal{I}_{23} & 0 \\ 0 & 0 & 0 \end{pmatrix}$	2	$\mathcal{TH}_3^{19}$	$\begin{pmatrix} \mathcal{I}_{21} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	1

Proof Departing from Theorem 3.2, we provide the proof only for one case to illustrate the used approach, the other cases can be addressed similarly with or without modification(s). Let's consider $\mathcal{TH}_3^1$. Applying the systems of equations (23), (24), (25) and (26), we get $\mathcal{I}_{21} = \mathcal{I}_{31} = \mathcal{I}_{21} = \mathcal{I}_{31} = 0$. Hence, the derivations of $\mathcal{TH}_3^1$ are indicated as follows

$$\mathcal{I}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{I}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{I}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

is the basis of Der(α-*Inner-derivations*) and DimDer(α-*Inner-derivations*)=3. The α-*Inner-derivations* of the remaining parts of dimension three associative trialgebras can be handled in a similar manner as illustrated above.

Corollary 3.1.

1. The dimensions of the α-*Inner-derivations* of 2-dimensional Hom-associative trialgebras range between zero and one.
2. The dimensions of the α-*Inner-derivations* of 3-dimensional Hom-associative trialgebras range between zero and three.

4. α-Centroids of Low-dimensional Hom-associative Trialgebras

4.1. *Properties of α-centroids of Hom-associative Trialgebras*

In this section, we draw the following results on properties of α-centroids of Hom-associative trialgebras $\mathcal{A}$.

Definition 4.1. Let $(\mathcal{A}, \dashv, \vdash, \alpha)$ be a Hom-associative trialgebra. A linear map $\psi : \mathcal{A} \to \mathcal{A}$ is called an element of α-*centroids* on $\mathcal{A}$ if, for all $x, y \in \mathcal{A}$,

$$\begin{aligned} \alpha \circ \psi &= \psi \circ \alpha, && (27) \\ \psi(x) \dashv \alpha(y) &= \psi(x) \dashv \psi(y) = \alpha(x) \dashv \psi(y), && (28) \\ \psi(x) \vdash \alpha(y) &= \psi(x) \vdash \psi(y) = \alpha(x) \vdash \psi(y), && (29) \\ \psi(x) \perp \alpha(y) &= \psi(x) \perp \psi(y) = \alpha(x) \perp \psi(y). && (30) \end{aligned}$$

The set of all elements of α-*centroid* of $\mathcal{A}$ is denoted $Cent_{(\alpha)}(\mathcal{A})$. The *centroid* of $\mathcal{A}$ is denoted $Cent(\mathcal{A})$.

Definition 4.2. Let $\mathcal{H}$ be a nonempty subset of $\mathcal{A}$. The subset

$$Z_{\mathcal{A}}(\mathcal{H}) = \{x \in \mathcal{H} | \alpha(x) \bullet \mathcal{H} = \mathcal{H} \bullet \alpha(x) = 0\}, \qquad (31)$$

is said to be centralizer of $\mathcal{H}$ in $\mathcal{A}$, where the $\bullet$ is $\dashv$ and $\vdash$, respectively.

Definition 4.3. Let $\psi \in End(\mathcal{A})$. If $\psi(\mathcal{A}) \subseteq Z(\mathcal{A})$ and $\psi(\mathcal{A}^2) = 0$, then ψ is called a central derivation. The set of all central derivations of $\mathcal{A}$ is denoted by $\mathcal{C}(\mathcal{A})$.

Proposition 4.1. Consider $(\mathcal{A}, \dashv, \vdash, \alpha)$ a Hom-associative trialgebra. Then,

i) $\Gamma(\mathcal{A})Der(\mathcal{A}) \subseteq Der(\mathcal{A})$.

ii) $[\Gamma(\mathcal{A}), Dr(\mathcal{A})] \subseteq \Gamma(\mathcal{A})$.

iii) $[\Gamma(\mathcal{A}),\Gamma(\mathcal{A})](\mathcal{A}) \subseteq \Gamma(\mathcal{A})$ and $[\Gamma(\mathcal{A}),\Gamma(\mathcal{A})](\mathcal{A}^2)=0$.

Proof The proof of parts $i)-iii)$ is straightforward with reference to definitions of derivation and centroid.

Proposition 4.2. Let $(\mathcal{A},\dashv,\vdash,\perp,\alpha)$ be a Hom-associative trialgebra and $\varphi \in Cent(\mathcal{A})$, $d \in Der(\mathcal{A})$. Then, $\varphi \circ d$ is an α-derivation of $\mathcal{A}$.

Proof Indeed, if $x,y \in \mathcal{A}$, then

$$\begin{aligned}(\varphi \circ d)(x \bullet y) &= \varphi(d(x)\bullet \alpha(y)+\alpha(x)\bullet d(y))\\ &= \varphi(d(x)\bullet y)+\varphi(x\bullet d(y))\\ &= (\varphi\circ d)(x)\bullet\alpha(y)+\alpha(x)\bullet(\varphi\circ d)(y),\end{aligned}$$

where $\bullet$ is $\dashv,\vdash$ and $\perp$, respectively.

Proposition 4.3. Let $(\mathcal{A},\dashv,\vdash,\perp,\alpha)$ be a Hom-associative trialgebra over a field $\mathbb{F}$. Hence, $\mathcal{C}(\mathcal{A})=Cent(\mathcal{A})\cap Der(\mathcal{A})$.

Proof If $\psi \in Cent(\mathcal{A})\cap Der(\mathcal{A})$, then by definition of $Cent(\mathcal{A}$ and $Der(\mathcal{A})$, we get $\psi(x\bullet y)=\psi(x)\bullet\alpha(y)+\alpha(x)\bullet\psi(y)$ and $\psi(x\bullet y)=\psi(x)\circ\alpha(y)=\alpha(x)\circ\psi(y)$ for $x,y\in\mathcal{A}$. The yields $\psi(\mathcal{A}\mathcal{A})=0$ and $\psi(\mathcal{A})\subseteq Z(\mathcal{A})$, i.e, $Cent(\mathcal{A})\cap Der(\mathcal{A})\subseteq Cent(\mathcal{A})$. The opposite is obvious since $\mathcal{C}(\mathcal{A})$ is in both $Cent(\mathcal{A})$ and $Der(\mathcal{A})$, where $\bullet$ is $\dashv,\vdash$ and $\perp$, respectively.

Proposition 4.4. Let $(\mathcal{A},\dashv,\vdash,\perp,\alpha)$ be a Hom-associative trialgebra. Therefore, for any $d\in Der(\mathcal{A})$ and $\varphi\in Cent(\mathcal{A})$, we have

(i) The composition $d\circ\varphi$ is in $Cent(\mathcal{A})$, if and only if $\varphi\circ d$ is a central α-derivation of $\mathcal{A}$.

(ii) The composition $d\circ\varphi$ is a α-derivation of $\mathcal{A}$, if and only if $[d,\varphi]$ is a central α-derivation of $\mathcal{A}$.

Proof

i) For any $\varphi\in Cent(\mathcal{A})$, $d\in Der(\mathcal{A})$, $\forall\, x,y\in\mathcal{A}$, we have

$$\begin{aligned}d\circ\varphi(x\bullet y) &= d\circ\varphi(x)\bullet y\\ &= d\circ\varphi(x)\bullet y+\varphi(x)\bullet d(y)\\ &= d\circ\varphi(x)\bullet y+\varphi\circ d(x\bullet y)-\varphi\circ d(x)\bullet y.\end{aligned}$$

Thus, $(d\circ\varphi-\varphi\circ d)(x\bullet y)=(d\bullet\varphi-\varphi\circ d)(x)\bullet y$.

ii) Let $d\circ\varphi\in Der(\mathcal{A})$. Using $[d,\varphi]\in Cent(\mathcal{A})$, we get

$$[d,\varphi](x\bullet y)=([d,\varphi](x))\bullet\alpha(y)=\alpha(x)\bullet([d,\varphi](y)) \tag{32}$$

On the other side, $[d,\varphi]\, d\circ\varphi-\varphi\circ d$ and $d\circ\varphi,\varphi\circ d\in Der(\mathcal{A})$. Therefore,

$$\begin{aligned}[d,\varphi](x\bullet y) &= (d(\varphi\circ(x))\bullet\alpha(y)+\alpha(x)\bullet(d\circ\varphi(y))\\ &\quad -(\varphi\circ d(x))\bullet\alpha(y)-\alpha(x)\bullet(\varphi\circ d(y))\end{aligned} \tag{33}$$

Relying upon (32) and (33), we get $\alpha(x)\bullet([d,\varphi])(y)=([d,\varphi])(x)\bullet\alpha(y)=0$.

At this stage of analysis,let $[d,\varphi]$ be a central α-derivation of $\mathcal{A}$. Then,

$$\begin{aligned}d\circ\varphi(x\bullet y) &= [d\circ\varphi](x\bullet y)+(\varphi\circ d)(x\bullet y)\\ &= \varphi(\circ d(x)\bullet\alpha(y))+\varphi(\alpha(x)\bullet d(y))\\ &= (\varphi\circ d)(x)\bullet\alpha(y)+\alpha(x)\bullet(\varphi\circ d)(y),\end{aligned}$$

where $\bullet$ indicates the products $\dashv,\vdash$ and $\perp$, respectively.

4.2. α-centroids of 2-3-dimensional Hom-associative Trialgebras

This section provides pertinent details on α-centroids of Hom-associative trialgebras in dimension two and three over the field $\mathbb{K}$. Let $\{e_1,e_2,e_3,\cdots,e_n\}$ be a basis of an n-dimensional Hom-associative trialgebras $\mathcal{A}$. The product of basis

$$\psi(e_p)=\sum_{q=1}^{n}c_{qp}e_q.$$

$$\sum_{p=1}^{n}c_{pi}a_{qp}=\sum_{p=1}^{n}a_{pi}c_{qp}, \tag{34}$$

$$\sum_{k=1}^{n}\gamma_{ij}^{k}c_{qk}=\sum_{k=1}^{n}\sum_{p=1}^{n}c_{ki}a_{pj}\gamma_{kp}^{q} \tag{35}$$

$$\sum_{k=1}^{n}\gamma_{ij}^{k}c_{qk}=\sum_{k=1}^{n}\sum_{p=1}^{n}a_{ki}c_{pj}\gamma_{kp}^{q}, \tag{36}$$

$$\sum_{k=1}^{n}\delta_{ij}^{k}c_{qk}=\sum_{k=1}^{n}\sum_{p=1}^{n}c_{ki}a_{pj}\delta_{kp}^{q} \tag{37}$$

$$\sum_{k=1}^{n}\delta_{ij}^{k}c_{qk}=\sum_{k=1}^{n}\sum_{p=1}^{n}a_{ki}c_{pj}\delta_{kp}^{q}, \tag{38}$$

$$\sum_{k=1}^{n}\xi_{ij}^{k}c_{qk}=\sum_{k=1}^{n}\sum_{p=1}^{n}c_{ki}a_{pj}\xi_{kp}^{q} \tag{39}$$

$$\sum_{k=1}^{n}\xi_{ij}^{k}c_{qk}=\sum_{k=1}^{n}\sum_{p=1}^{n}a_{ki}c_{pj}\xi_{kp}^{q}. \tag{40}$$

Theorem 4.1. The α-*centroids* of 2-dimensional complex Hom-associative trialgebras are depicted as follows :

Table 5. *α-centroids of 2-dimensional Hom-associative trialgebras.*

$\mathcal{TH}_2^m$	$Cent_{(\alpha)}(\mathcal{A})$	$Dim(Cent_{(\alpha)}(\mathcal{A}))$	$\mathcal{TH}_2^m$	$Cent_{(\alpha)}(\mathcal{A})$	$Dim(Cent_{(\alpha)}(\mathcal{A}))$
$\mathcal{TH}_2^1$	$\begin{pmatrix} 0 & 0 \\ c_{21} & 0 \end{pmatrix}$	1	$\mathcal{TH}_2^2$	$\begin{pmatrix} 0 & 0 \\ c_{21} & 0 \end{pmatrix}$	1
$\mathcal{TH}_2^4$	$\begin{pmatrix} c_{11} & 0 \\ c_{21} & d_{11} \end{pmatrix}$	2	$\mathcal{TH}_2^5$	$\begin{pmatrix} c_{11} & 0 \\ c_{21} & c_{11} \end{pmatrix}$	2
$\mathcal{TH}_2^6$	$\begin{pmatrix} c_{11} & 0 \\ c_{21} & c_{11} \end{pmatrix}$	2	$\mathcal{TH}_2^7$	$\begin{pmatrix} c_{11} & 0 \\ 0 & c_{11} \end{pmatrix}$	1
$\mathcal{TH}_2^8$	$\begin{pmatrix} c_{11} & 0 \\ 0 & c_{11} \end{pmatrix}$	1	$\mathcal{TH}_2^{10}$	$\begin{pmatrix} c_{11} & 0 \\ 0 & 0 \end{pmatrix}$	1
$\mathcal{TH}_2^{11}$	$\begin{pmatrix} 0 & 0 \\ 0 & c_{22} \end{pmatrix}$	1	$\mathcal{TH}_2^{12}$	$\begin{pmatrix} c_{11} & 0 \\ 0 & 0 \end{pmatrix}$	1
$\mathcal{TH}_2^{13}$	$\begin{pmatrix} c_{11} & 0 \\ 0 & c_{11} \end{pmatrix}$	1			

Proof Departing from Theorem 4.1, we provide the proof only for one case to illustrate the used approach, the other cases can be addressed similarly with or without modification(s). Let's consider $\mathcal{TH}_2^4$. Applying the systems of equations (34), (36), (38) and (40), we get $c_{12}=0$ and $c_{22}=c_{11}$. Hence, the α-centroids of $\mathcal{TH}_2^4$ are indicated as follows $c_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $c_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ is the basis of $Cent_{(\alpha)}(\mathcal{A})$ and Dim($Cent_{(\alpha)}$)($\mathcal{A}$)=2. The α-centroids of the remaining parts of dimension two Hom-associative trialgebras can be handled in a similar manner as illustrated above.

Theorem 4.2. The α-*centroids* of 3-dimensional complex Hom-associative trialgebras are depicted as follows :

Table 6. *α-centroids of 3-dimensional complex Hom-associative trialgebras.*

$\mathcal{TH}_3^m$	$Cent_{(\alpha)}(\mathcal{A})$	$Dim(Cent_{(\alpha)}(\mathcal{A}))$	$\mathcal{TH}_3^m$	$Cent_{(\alpha)}(\mathcal{A})$	$Dim(Cent_{(\alpha)}(\mathcal{A}))$
$\mathcal{TH}_3^1$	$\begin{pmatrix} c_{11} & 0 & 0 \\ 0 & c_{22} & c_{23} \\ 0 & -c_{22} & -c_{23} \end{pmatrix}$	3	$\mathcal{TH}_3^2$	$\begin{pmatrix} c_{11} & 0 & 0 \\ 0 & c_{22} & c_{23} \\ 0 & -c_{22} & -c_{23} \end{pmatrix}$	3
$\mathcal{TH}_3^3$	$\begin{pmatrix} 0 & c_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	1	$\mathcal{TH}_3^4$	$\begin{pmatrix} c_{11} & 0 & c_{13} \\ 0 & 0 & 0 \\ 0 & 0 & c_{11} \end{pmatrix}$	2
$\mathcal{TH}_3^5$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & c_{22} & \\ 0 & 0 & c_{33} \end{pmatrix}$	2	$\mathcal{TH}_3^6$	$\begin{pmatrix} c_{11} & 0 & c_{13} \\ 0 & c_{22} & 0 \\ -c_{11} & 0 & -c_{13} \end{pmatrix}$	3
$\mathcal{TH}_3^7$	$\begin{pmatrix} c_{11} & c_{12} & 0 \\ -c_{11} & -c_{12} & 0 \\ 0 & 0 & c_{33} \end{pmatrix}$	3	$\mathcal{TH}_3^8$	$\begin{pmatrix} c_{11} & c_{12} & 0 \\ -c_{11} & -c_{12} & 0 \\ 0 & 0 & c_{33} \end{pmatrix}$	3
$\mathcal{TH}_3^{10}$	$\begin{pmatrix} 0 & 0 & 0 \\ c_{21} & 0 & 0 \\ 0 & 0 & c_{33} \end{pmatrix}$	2	$\mathcal{TH}_3^{11}$	$\begin{pmatrix} 0 & 0 & 0 \\ c_{21} & 0 & 0 \\ 0 & 0 & c_{33} \end{pmatrix}$	2
$\mathcal{TH}_3^{12}$	$\begin{pmatrix} 0 & 0 & 0 \\ c_{21} & c_{23} & 0 \\ 0 & 0 & 0 \end{pmatrix}$	2	$\mathcal{TH}_3^{13}$	$\begin{pmatrix} 0 & 0 & 0 \\ c_{21} & c_{23} & 0 \\ 0 & 0 & 0 \end{pmatrix}$	2
$\mathcal{TH}_3^{14}$	$\begin{pmatrix} c_{11} & 0 & 0 \\ c_{21} & c_{11} & c_{23} \\ -c_{11} & 0 & 0 \end{pmatrix}$	3	$\mathcal{TH}_3^{15}$	$\begin{pmatrix} 0 & 0 & 0 \\ c_{21} & 0 & c_{23} \\ 0 & 0 & 0 \end{pmatrix}$	2
$\mathcal{TH}_3^{16}$	$\begin{pmatrix} 0 & 0 & 0 \\ c_{21} & 0 & c_{23} \\ 0 & 0 & 0 \end{pmatrix}$	2	$\mathcal{TH}_3^{17}$	$\begin{pmatrix} 0 & 0 & 0 \\ c_{21} & 0 & c_{23} \\ 0 & 0 & 0 \end{pmatrix}$	2
$\mathcal{TH}_3^{18}$	$\begin{pmatrix} 0 & 0 & 0 \\ c_{21} & 0 & c_{23} \\ 0 & 0 & 0 \end{pmatrix}$	2	$\mathcal{TH}_3^{20}$	$\begin{pmatrix} 0 & 0 & 0 \\ c_{21} & 0 & 0 \\ c_{31} & c_{21} & 0 \end{pmatrix}$	2

$\mathcal{TH}_3^m$	$Cent_{(\alpha)}(\mathcal{A})$	$Dim(Cent_{(\alpha)}(\mathcal{A}))$	$\mathcal{TH}_3^m$	$Cent_{(\alpha)}(\mathcal{A})$	$Dim(Cent_{(\alpha)}(\mathcal{A}))$
$\mathcal{TH}_3^{21}$	$\begin{pmatrix} 0 & 0 & 0 \\ c_{21} & 0 & 0 \\ c_{31} & c_{21} & 0 \end{pmatrix}$	2			

Proof Departing from Theorem 4.2, we provide the proof only for one case to illustrate the used approach, the other cases can be addressed similarly with or without modification(s). Let's consider $\mathcal{TH}_3^6$. Applying the systems of equations (34), (36), (38) and (40), we get $c_{12} = c_{21} = c_{23} = c_{32} = 0$, $c_{11} = -c_{31}$ and $c_{13} = -c_{33}$. Hence, the α-centroids of $\mathcal{TH}_3^6$ are indicated as follows

$$c_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad c_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and}$$

$$c_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

is the basis of $Cent_{(\alpha)}(\mathcal{A})$ and Dim$Cent_{(\alpha)}(\mathcal{A})$=3. The α-centroids of the remaining parts of dimension three Hom-associative trialgebras can be handled in a similar manner as illustrated above.

Corollary 4.1.

1. The dimensions of the α-*centroids* of two-dimensional of Hom-associative trialgebras range between zero and two.
2. The dimensions of the α-*centroids* of three-dimensional of Hom-associative trialgebras range between zero and three.

Conflicts of Interest

The authors declare no conflicts of interest.

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