Mathematics Letters
2022; 8(2): 32-36
http://www.sciencepublishinggroup.com/j/ml
doi: 10.11648/j.ml.20220802.12
ISSN: 2575-503X (Print); ISSN: 2575-5056 (Online)

Comparison of Results for Some Different Methods of Determination of Fundamental Matrix of Linear Control Systems

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To cite this article:
Stephen Ekwueme Aniaku, Emmanuel Chukwudi Mbah, Christopher Chukwuma Asogwa. Comparison of Results for Some Different Methods of Determination of Fundamental Matrix of Linear Control Systems. *Mathematics Letters*. Vol. 8, No. 2, 2022, pp. 32-36.
doi: 10.11648/j.ml.20220802.12

Received: March 11, 2022; **Accepted**: April 1, 2022; **Published**: July 28, 2022

Abstract: In this paper, three different methods of determination of fundamental matrix of Linear control systems, when the coefficient matrix A is not a nilpotent matrix are compared. In the case where A is nilpotent, the calculation is straight forward and easy. It only needs the calculation of e^{At}: The methods compared, in the case of A not being nilpotent are Faddeve Algorithm method, Sylvester Expansion Theorem method and Diagonalization method. Fundamental matrix plays a very big role in the determination of solution to linear control systems. Based on this, we have to look for the best method of determining it. Here, the level of problems and difficulties encountered in determining the fundamental matrix using these three methods were verified. Worked examples on the use of these three methods to determine the fundamental matrix were given and level of problems and difficulties examined. From the worked example, it was discovered that these three methods have different level of problems and difficulties in finding the fundamental matrix. It was then concluded that based on their different level of problems and difficulties, these methods were compared and conclusion derived. These three methods which are effective ways of determining Fundamental matrix of linear control systems will be preferred in this order: Faddeve Algoprithm method, Sylvester Expansion Theorem method and Diagonalization method.

Keywords: Characteristic Polynomial, Eigenvalues, Eigen Vectors, Nilpotent Matrix

1. Introduction

We wish to consider the problem of finding the solution of non-homogeneous linear control systems of the form.

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ x(t_0) = x_0 \end{cases} \tag{1}$$

where $x(t)$ is the state, u(t) is control function which is expected to be admissible. Also, A is an n x n matrix function and B is n x p matrix. By Lee and Markus [1], the solution $x(t)$ of (1) is given by the fundamental formula, called the variation of parameters formula, for historical reasons,

$$x(t) = \Phi(t)x_0 + \Phi(t)\int_{t_0}^{t} \Phi(s)B(s)u(s)ds \tag{2}$$

Here $\Phi(t)$ is called the fundamental matrix solution of the corresponding homogeneous systems.

$$\dot{x}(t) = Ax(t) \tag{3}$$

where $\Phi(t_0) = I$, the identity matrix.

So, the solution of the non-homogeneous linear control systems (1) can only be found if we can get, first of all, the fundamental matrix $\Phi(t)$. The problem of determining the fundamental matrix of such systems is paramount in finding such solution. See [2 - 10] and the references therein to see big research works done there.

A necessary and sufficient condition for matrix solution of linear control system to be a fundamental matrix has also been extensively verified [18]. It is easy to determine the fundamental matrix $\Phi(t)$ when the matrix A is nilpotent. It was known that fundamental matrix.

$$\Phi(t) = e^{tA} = I + tA + \frac{t^2A^2}{2!} + \frac{t^3A^3}{3!} + \cdots \quad (4)$$

Our wish in this paper is to compare different methods of determining fundamental matrix of linear control system (3) when A is not nilpotent. First of all, let us give the notations and definitions which we shall use in this paper.

2. Notations and Definitions

Notations

$\Phi(t) =$ Fundamental matrix

$\varphi(t) =$ characteristic polynomial of matrix A

tr(A) = trace of the square matrix A.

$\mathcal{L}\{f(x)\} =$ Laplace transform of a function f(x)

$$= \int_0^\infty e^{-sx} f(x)dx = F(s) \quad (5)$$

$\mathcal{L}\{F(s)\} =$ inverse Laplace transform of F(s)

3. Definition

Definition 1: Nilpotent Matrix [11]

A matrix A for which $A^p = 0$, where p is a positive integer, is called nilpotent matrix.

Definition 2: Nilpotent Index. [11]

If p is the smallest positive integer for which $A^p = 0$, then A is said to be nilpotent of index p.

Definition 3: Laplace Transform [12]

The Laplace transform of a function $f(t)$ defined on $[0, \infty)$ is given by

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$$

Definition 4: Inverse Laplace Transform [12]

By the inverse Laplace transform of $F(s)$, which is denoted by $\mathcal{L}^{-1}\{F(s)$, we mean the unique function $f(t)$ that is continous on $[0, \infty)$ and satisfies.

$$\mathcal{L}^{-1}\{f(t)\} = F(s).$$

4. Fundamental Matrix Φ(*t*)

As a result of the very important role the fundamental matrix played, in general, in the analysis of linear control systems, we want to discuss and compare some of the well-known methods of determining it in various cases. We have noted that the simplest fundamental matrix is that which is obtained from simple linear differentiation of the form.

$$\dot{x}(t) = Ax(t) \quad (6)$$

As $\Phi(t) = e^{tA}$ [13, 14],
where $A = (aij)$ is a constant $n \times n$ matrix.

According to Boyce and Diprima [16]

$$\Phi(t) = e^{tA} = I + \sum_{k=1}^{n} \frac{(tA)^k}{k!} \quad (7)$$

So, from (7), it is easy to find fundamental matrix $\Phi(t)$ whenever the matrix A is nilpotent. The determination of fundamental matrix in other case, however, is not straight forward. We give the methods of finding the fundamental matrix $\Phi(t)$, when the matrix A in (6) is not nilpotent. Although other new approach in calculating the fundamental matrix has been studied [19]. We considered three different methods viz.

(a) The use of Faddeve algorithm,
(b) The use of Sylvester Expression Theorem (SET) [17] and
(c) The use of diagonalization methods. We look these methods one by one.

4.1. The Faddeve Algorithm Method

Consider the differential equation.

$$\dot{\Phi}(t) = A\underline{\varphi}(t) \quad (8)$$

$$\Phi(0) = I$$

The required fundamental matrix is the solution $\Phi(t)$ of (8) which can be obtained by the Laplace transform method. So, taking the Laplace transform of each side of (8), we get

$$\mathcal{L}\{\dot{\Phi}(t)\} = \mathcal{L}\{A\Phi(t)\}$$

$$= S\mathcal{L}\{\Phi(t)\} - \Phi(0) = A\mathcal{L}\{\Phi(t)\}.$$

$$\Rightarrow \Phi(t) = \mathcal{L}^{-1}\{(SI - A)^{-1}\} \quad (9)$$

The problem now lies with the evaluation of the series $(SI - A)^{-1}$. In order for us to evaluate this, we must make use of the algorithm called "Faddeve Algorithm", which we hereby state as follows without proof.

Faddeve Algorithm

Suppose the characteristic polynomial of the $n \times n$ matrix A is denoted by det $(SI - A) = \varphi(s)$ i.e.

$$\varphi(s) = S^n + a_{n-1}S^{n-1} + a_{n-2}S^{n-2} + \cdots + a_0 \quad (10)$$

Then

$$(SI - A)^{-1} = \frac{1}{\varphi(s)}(\Gamma_{n-1}S^{n-1} + \Gamma_{n-2}S^{n-2} + \cdots + \Gamma_0) \quad (11)$$

provided $a_{n-1}, a_{n-2}, \ldots a_0$ and $\Gamma_{n-1}, \Gamma_{n-2}, \ldots, \Gamma_0$ satisfy the following recursive formulae from which the Γ_s and a_s can be calculated.

(Note that tr(A) denotes the trace of A)

$$\left.\begin{array}{c} \Gamma_{n-1} = I;\ a_{n-1} = -tr(A\Gamma_{n-1}) \\ \Gamma_{n-2} = A\Gamma_{n-1} + a_{n-1}I;\ a_{n-2} = \frac{-tr(A\Gamma_{n-2})}{2} \\ \cdots\cdot \\ \cdots\cdot \\ \cdots\cdot \\ \Gamma_i = A\Gamma_{i+1} + a_{i+1}I;\ a_i = \frac{-tr(A\Gamma_i)}{i} \\ \cdots\cdot \\ \vdots\vdots\vdots\vdots \\ \Gamma_0 = A\Gamma_1 + a_1I;\ a_0 = \frac{-tr(A\Gamma_0)}{n} \end{array}\right\} \quad (12)$$

From (11), it is clear that in using (12) to evaluate $(SI-A)^{-1}$, we at the same time utilize the inverse Laplace transform method since $\Phi(t)=\mathcal{L}^{-1}\{(SI-A)^{-1}\}$ by (9).

Let us use a concrete example to illustrate the use clearer.

Example 1

A linear control system is governed by the following state equation.

$$\dot{x}(t)=Ax(t)+Bu(t)$$

Where $A=\begin{bmatrix}-3 & 1\\ 1 & -3\end{bmatrix}, B=\begin{bmatrix}0\\1\end{bmatrix}$.

We want to find the fundamental matrix of A for this system.

So, the state equation is of the form.

$$\dot{x}(t)=\begin{pmatrix}-3 & 1\\ 1 & -3\end{pmatrix}x(t)+\begin{pmatrix}0\\1\end{pmatrix}u(t)$$

$$\Rightarrow A=\begin{pmatrix}-3 & 1\\ 1 & -3\end{pmatrix}$$

$$\Rightarrow (SI-A)=\begin{pmatrix}S+3 & -1\\ -1 & S+3\end{pmatrix}$$

Then

$$\det(SI-A)=\begin{bmatrix}S+3 & -1\\ -1 & S+3\end{bmatrix}=(x+3)^2-1$$

$\Rightarrow \varphi(s)=S^2+6s+8$ [characteristic polynomial].

As A is a 2×2 matrix, we only need Γ_0 and Γ_1 (as $n=2$). From (12), we have

$$\Gamma_{2-1}=\Gamma_1=1=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \text{ and}$$

$$a_{2-1}=a_1=-tr(A\Gamma_1) \text{ [since } \Gamma_1=I]$$

$$=-tr(A)=6$$

$$\Gamma_{2-2}=\Gamma_0=A\Gamma_1+a_1I=A+6I$$

$$=\begin{bmatrix}-3 & 1\\ 1 & -3\end{bmatrix}+\begin{bmatrix}6 & 0\\ 0 & 6\end{bmatrix}=\begin{bmatrix}3 & 1\\ 1 & 3\end{bmatrix}$$

$$(SI-A)^{-1}=\frac{1}{\varphi(s)}[\Gamma_1S+\Gamma_0] \text{ [from 2.6]}$$

$$=\frac{1}{S^2+6s+8}\left[\begin{pmatrix}S & 0\\ 0 & S\end{pmatrix}+\begin{pmatrix}3 & 1\\ 1 & 3\end{pmatrix}\right]$$

$$=\frac{1}{(S+2)(s+4)}\begin{bmatrix}S+3 & 1\\ 1 & S+3\end{bmatrix}$$

$$\begin{bmatrix}\frac{S+3}{(S+2)(s+4)} & \frac{1}{(S+2)(s+4)}\\ \frac{1}{(S+2)(s+4)} & \frac{3}{(S+2)(s+4)}\end{bmatrix} \text{ (A)}$$

Resolving each of the four entries of (A) into partial fraction, we get

$$\frac{S+3}{(S+2)(s+4)}=\frac{1}{2(S+2)}-\frac{1}{2(S+4)} \, (i)$$

And

$$\frac{1}{(S+2)(s+4)}=\frac{1}{2(S+2)}+\frac{1}{2(S+4)} \, (ii)$$

Substituting (i) and (ii) into (A), we get

$$(SI-A)^{-1}=\begin{bmatrix}\frac{1}{2(S+2)}-\frac{1}{2(S+4)} & \frac{1}{2(S+2)}+\frac{1}{2(S+4)}\\ \frac{1}{2(S+2)}+\frac{1}{2(S+4)} & \frac{1}{2(S+2)}-\frac{1}{2(S+4)}\end{bmatrix}$$

Then

$$\Phi(t)=\mathcal{L}^{-1}(SI-A)^{-1}=\frac{1}{2}\begin{bmatrix}e^{-2t}-e^{-4t} & e^{-2t}+e^{-4t}\\ e^{-2t}+e^{-4t} & e^{-2t}-e^{-4t}\end{bmatrix}$$

We note that the same procedure can be employed also for a 3 x 3 matrix and so for general n x n matrix.

4.2. The Use of Sylvester Expression Theorem (SET)

We now consider a second method of determining the fundamental matrix $\Phi(t)$. We hereby examine the application of the so called "Sylvester Expression Theorem" which we state without the proof.

Sylvester Expression Theorem: SET

If $\lambda_1, \lambda_2, \ldots \lambda_n$ are distinct eigenvalues of the constant matrix A, then the series f(A) defined by

$$f(A)=f(\lambda_i)\frac{(A-\lambda_1I)\ldots(A-\lambda_{i-1}I)(A-\lambda_{i+1}I)\ldots(A-\lambda_nI)}{(\lambda_i-\lambda_1)\ldots(\lambda_i-\lambda_{i-1})(\lambda_i-\lambda_{i+1})\ldots(\lambda_i-\lambda_n)} \quad (13)$$

determines the fundamental matrix of A so that

$$f(A)=e\hat{\ }tA=\Phi(t) \quad (14)$$

We use a concrete example of illustrate the use of the above SET method.

Example 2

Consider a linear control system expressed by the differential equation.

$$\dot{x}(t)=\begin{pmatrix}-1 & 1\\ 0 & -2\end{pmatrix}x(t)+\begin{pmatrix}0\\1\end{pmatrix}u(t)$$

where u has the usual meaning and x is the state. We want to determine, by the application of SET, the fundamental matrix $\Phi(t)$ of the given control system.

We start by finding the eigenvalues λ_1 *and* λ_2 of A. this is found to be $\lambda_1=-1$ and $\lambda_2=-2$, which are distinct. Using SET,

$$f(A)=e^{tA}=\Phi(t)$$

From (13), we have that

$$f(A)=e^{tA}=\Phi(t)$$

$$=\sum_{i=1}^{2}f(\lambda_i)\frac{(A-\lambda_2I)\ldots(A-\lambda_1I)}{(\lambda_i-\lambda_2I)\ldots(\lambda_i-\lambda_1I)}$$

$$\Phi(t)=f(\lambda_1)\frac{(A-\lambda_2I)}{(\lambda_1-\lambda_2)}+f(\lambda_2)\frac{(A-\lambda_1I)}{(\lambda_2-\lambda_1)} \quad (15)$$

We now determine the two components of $\Phi(t)$ from (15) one by one as follows:

For $\lambda_i=-1$

$$f(\lambda_1)\frac{(A-\lambda_2 I)}{(\lambda_1-\lambda_2)} = e^{-t}\frac{A+2I}{-1-(-2)} = e^{-t}(A+2I)$$

$$= e^{-t}\left\{\begin{bmatrix}-1 & 1\\ 0 & -2\end{bmatrix}+\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}\right\} = e^{-t}\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}\ (B)$$

Similarly, for $\lambda_2 = -2$, we have

$$f(\lambda_2)\frac{(A-\lambda_1 I)}{(\lambda_2-\lambda_1)} = e^{-2t}\frac{A+I}{(-2—1)} = -e^{-2t}(A+I)$$

$$= -e^{-2t}\left\{\begin{bmatrix}-1 & 1\\ 0 & -2\end{bmatrix}+\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right\}$$

$$= e^{-2t}\begin{bmatrix}0 & 1\\ 0 & -1\end{bmatrix}\ (C)$$

Consequently, adding (B) and (C), we get

$$\Phi(t) = (B) + (C) = \begin{bmatrix}e^{-t} & e^{-t}\\ 0 & 0\end{bmatrix}+\begin{bmatrix}0 & e^{-2t}\\ 0 & e^{-2t}\end{bmatrix}$$

$$\Rightarrow \Phi(t) = \begin{pmatrix}e^{-t} & e^{-t}-e^{-2t}\\ 0 & e^{-2t}\end{pmatrix}\ ■$$

For 3×3 matrix, the process is exactly the same as in this 2×2 matrix. In general, for $n \times n$ matrix A, the SET method is done in the same way.

Let us now examine the diagonalization method of determining the fundamental matrix of a linear control systems. The procedures are as follows:

4.3. The Diagonalization Method [15]

Suppose $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ are the distinct eigenvalues of an $n \times n$ matrix A, with $V_1, V_2, V_3, \ldots, V_n$ as the corresponding eigenvectors. Then, it can be shown that the operator $T(V_1, V_2, V_3, \ldots, V_n)$ with the eigenvectors $V_1, V_2, V_3, \ldots, V_n$ as columns can reduce A to a diagonal Λ so that if $T = T(V_1, V_2, V_3, \ldots, V_n)$, then

$$\Lambda = T^{-1}A\ (A = T\Lambda T^{-1}) \tag{16}$$

Furthermore, if A is associated with the state equation of the form.

$$\dot{x}(t) = Ax(t) \tag{17}$$

It is easy to show that in terms of the diagonalization matrix Λ of A given in equation (16), the required fundamental matrix $\Phi(t)$ is given by

$$\Phi(t) = e^{tA} = e^{t(T\Lambda T^{-1})} = e^{T\Lambda T^{-1}t} \Rightarrow \Phi(t) = e^{tA} = Te^{\Lambda}T^{-1} \tag{18}$$

We now give a concrete example to illustrate the use of this method.

Example 3

We consider the diagonalization method of determining the fundamental matrix $\Phi(t) = e^{tA}$ of a linear control system where

$$A = \begin{bmatrix}-1 & 1\\ 0 & -2\end{bmatrix}$$

We first of all find the eigenvalues λ_1 and λ_2 of the given matrix A. The eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = -2$, which are distinct.

Then, the eigen vectors V_1, V_2 corresponding to $\lambda_1 = -1$ and $\lambda_2 = -2$ are respectively given by $V_1 = (1\quad 0)^T$ and $V_2 = (1\quad -1)^T$

So that the operator T = T(V_1, V_2)

$$T = \begin{bmatrix}1 & 1\\ 0 & -1\end{bmatrix}$$

Then by row reduction,

$$T^{-1} = \begin{bmatrix}1 & 1\\ 0 & -1\end{bmatrix}$$

$$\Phi(t) = Te^{\Lambda t}T^{-1}$$

$$= \begin{pmatrix}1 & 1\\ 0 & -1\end{pmatrix}\begin{pmatrix}e^{-t} & 0\\ 0 & e^{-2t}\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & -1\end{pmatrix}$$

$$= \begin{pmatrix}1 & 1\\ 0 & -1\end{pmatrix}\begin{pmatrix}e^{-t} & 0\\ 0 & e^{-2t}\end{pmatrix}$$

Then,

$$\Phi(t) = \begin{pmatrix}e^{-t} & e^{-t}-e^{-2t}\\ 0 & e^{-2t}\end{pmatrix}\ ■$$

So, we note that the determination of the fundamental matrix for 3×3 matrix follows exactly the similar procedures. In general, the procedure is exactly the same for $n \times n$ constant matrix A.

5. Comparison of Results

We have seen three different methods of determining the fundamental matrix of a linear control systems. Any one of these methods: that is the use of Feddeve algorithm, Sylvester Expression Theorem and Diagonalization, where the constant matrix A is not nilpotent, can successfully be used to find $\Phi(t)$ unless one is constrained to use any particular method. What we are to check is the difficulty that is involved in using any of the three methods we note that in the three methods considered, one needs to be patience, careful and determined in order to find correctly the fundamental matrix $\Phi(t)$. The determination of the eigenvalues of the constant matrix A is required using these three methods except in Faddeve algorithm, where is only the characteristic polynomial that is required.

In the case of Faddeve algorithm, what is highly needed, apart from the characteristic polynomial are the calculation of $a_{n-1}, a_{n-2}, \ldots a_n\ and\ \Gamma_{n-1}, \Gamma_{n-2},\ \Gamma_0$ which satisfy the given recursive formula from which these can be calculated.

There is no big problem in using this method since the three requirements, viz characteristics polynomial, a and Γ_s can be obtained with ease. The only problem one is expected to get using this method is in the calculation of the inverse Laplace transform of the expressions that may be involved.

In both Sylvester Expression Theorem and diagonalization methods, one is required to find the eigenvalues of the constant matrix A, which are expected to be distinct. Then, here we encounter a big problem. The problem is that if some of these eigenvalues are repeated, we cannot use these methods.

We note the use of these three methods in determining the fundamental matrix for a linear control system, where they will be applied, are all accurate methods and boiled down to the same answer. So, one of the restrictions on which method to use, depends on the nature of the eigenvalues.

In the cases where the eigenvalues are not distinct, it is advisable to use the Faddeve algorithm. But suppose that these eigenvalues are distinct, one must ensure that the matrix T, whose columns are the eigenvectors of A is nonsingular, otherwise, the use of diagonalization method will not work. From these, we have the following order of preference of these methods. (1) Faddeve algorithm method, (2) Sylvester Expression Theorem, and Finally (3) The diagonalization method.

6. Recommendation and Conclusion

6.1. Recommendation

Fundamental matrix plays a big role in the determination of solution of linear control systems. We are looking forward to getting a much better and less difficult methods for determination of fundamental matrix than the three methods discussed here.

6.2. Conclusion

The methods of determining fundamental matrix for linear control system $\dot{x} = A(x) + Bu(t)$ can easily be found using the fact that

$$\Phi(t) = e^{tA} = I + \sum_{k=1}^{n} \frac{(tA)^k}{k!}$$

when the constant matrix A is nilpotent.

In the case where the constant matrix A is not nilpotent, these three methods of fundamental matrix determination i.e Faddeve Algorithm, Sylvester Expression Theorem and then Diagonalization methods can be used. Based on the level of problems and difficulties encountered while using these methods, we use any of these three methods preferably in this order: Faddeve algorithm, Sylvester Expression Theorem and then Diagonalization methods.

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