Mathematics Letters
2021; 7(2): 25-29
http://www.sciencepublishinggroup.com/j/jpmt
doi: 10.11648/j.ml.20210702.12
ISSN: 2469-8423 (Print); ISSN: 2469-8431 (Online)

Some New Properties of W_d-fuzzy Implication Algebras

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To cite this article:
Fang-an Deng. Some New Properties of W_d-fuzzy Implication Algebras. *Mathematics Letters*. Vol. 7, No. 2, 2021, pp. 25-29.
doi: 10.11648/j.ml.20210702.12

Received: April 23, 2021; **Accepted:** May 4, 2021; **Published:** May 27, 2021

Abstract: Implication is an important logical connective in practically every propositional logic. In 1987, the so-called Fuzzy implication algebras were introduced by Wu Wangming, then various interesting properties of FI-algebras and some subalgebra of Fuzzy implication algebra, such as regular FI-algebras, commutative FI-algebras, W_d- FI-algebras, and other kinds of FI-algebras were reported. The main aim of this article is to study W_d-fuzzy implication algebras which are subalgebra of fuzzy implication algebras. We showed that W_d -fuzzy implication algebras are regular fuzzy implication algebras, but the inverse is not true. The relations between W_d -fuzzy implication algebras and other fuzzy algebras are discussed. Properties and axiomatic systems for W_d -fuzzy implication algebras are investigated. Furthermore, a few new results on W_d -fuzzy implication algebras has been added.

Keywords: Fuzzy Implication Algebras, W_d -Fuzzy Implication, Regular Fuzzy Implication Algebras, Heyting Type Fuzzy Implication Algebras

1. Introduction

In the past years, fuzzy algebras and their axiomatization have become important topics in theoretical research and in the applications of fuzzy logic. The implication connective plays a crucial role in fuzzy logic and reasoning [2]. Recently, some authors studied fuzzy implications from different perspectives[16]. Naturally, it is meaningful investigating the common properties of some important fuzzy implications used in fuzzy logic. Consequentially, Professor Wu [1] introduced a class of fuzzy implication algebras, FI-algebras for short, in 1990.

In the past two decades, some authors focused on FI-algebras. Various interesting properties of FI-algebras [3-5], regular FI-algebras [1,6,7], commutative FI-algebras [8], W_d- FI -algebras [9], and other kinds of FI-algebras [10] were reported, and some concepts of filter, ideal and fuzzy filter of FI-algebras were proposed [1,11,12]. Relationships between FI-algebra and BCK-algebra [13,14], MV-algebras [15], Rough set algebras [16,17], and were partly investigated, and FI-algebras were axiomatized [18]. In the recent work, the relationship between these FI- algebras and several famous fuzzy algebras were systematically discussed[3,19-24].

The organization of the paper is as follows: preliminary notions and results are introduced in section 2; section 3 relationships between W_d -FI algebras and several classes of important fuzzy algebras are discussed; Main properties of W_d -FI algebras is included in section 4. Lastly, the paper introduces several conclusions and pointers for further research.

2. Preliminaries

In this section,we summarize some definitions and results aboutW_d -FI algebras,which will be used in the following sections of the paper.

First, we recall some definitions and properties about W_d -FI algebras.

Definition 2.1. (see [1]) Let X be a set with a binary operation $\to$, and $0 \in X$. An fuzzy implication algebra, shortly,FI-algebra is an algebra $(X, \to, 0)$ of type(2,0)satisfying

(I1) $x \to (y \to z) = y \to (x \to z)$;
(I2) $(x \to y) \to ((y \to z) \to (x \to z)) = 1$;
(I3) $x \to x = 1$;
(I4) if $x \to y = y \to x = 1$, then $x = y$;
(I5) $0 \to z = 1$, for all $x, y, z \in X$, where $1 = 0 \to 0$.

Definition 2.2. (see [1]). A Hilbert fuzzy implication algebra, shortly,HFI-algebra is an algebra (X,¡ú,0) of type(2,0) which satisfies the following conditions for every $x, y, z \in X$:

(H1) $x \to (y \to x) = 1$;

(H2) $(x \to (y \to z) \to ((x \to y) \to (x \to z)) = 1$;

(H3) if $1 \to x = 1$, then $x = 1$;

(H4) if $x \to y = y \to x = 1$, then $x = y$;

(H5) $0 \to x = 1$,

where $1 = 0 \to 0$.

On an FI-algebra X, one can define a binary relation $\leq$ and operators C, T, S as follows.

$$x \leq y \Longleftrightarrow x \to y = 1, x, y \in X; \quad (1)$$

$$C(x) = x \to 0, x \in X; \quad (2)$$

$$T(x, y) = C(x \to C(y)),$$

$$S(x, y) = C(x) \to y, x, y \in X. \quad (3)$$

Usually, we also say that X is an FI-algebra for convenience.

Obviously, the relation "$\leq$" is a partial ordering on X, i.e., the relation is reflexive, antisymmetric and transitive(see [1]). In fuzzy logic, the property (1) is called the ordering property.

The operator "C" defined in the above definition is a negation on X, i.e., the operator is order-inverting and satisfies $C(0) = 1$, and $C(1) = 0$. "$\leq$" and C are called the partial ordering and the negation induced by the FI- algebra X, respectively.

If an FI-algebra X form a lattice with respect to the partial order "$\leq$", then we called FI-lattice.

Definition 2.3. (see [1,3]). Let X be an FI-algebra.

(i) X is regular FI-algebra, or an RFI-algebra, if $CC(x) = x$, for all $x \in X$.

(ii) X is commutative, or a CFI-algebra, if the binary operation $\odot$ defined by (3) is commutative, or the following condition (I6) holds for all $x, y \in x$:

(I6) $(x \to y) \to y = (y \to x) \to x$.

Then W_d - Fuzzy implication algebra is an algebra of type (2,0). The notion was first formulated in 1996 by Deng and some properties were obtained (see [9]). This notion was originated from the motivation based on fuzzy implication algebra introduced by Wu (see[1]).

Definition 2.4. (see [9]). A (2,0)-type algebra $(X, \to, 0)$ is a W_d-Fuzzy implication algebra, shortly, W_d-FI algebra, if the following conditions hold for all $x, y, z \in X$:

(W1) $x \to (y \to z) = y \to (x \to z)$;

(W2) $(x \to y) \to z = (z \to y) \to x$;

(W3) $x \to x = 1$;

(W4) if $x \to y = y \to x = 1$, then $x = y$;

(W5) $0 \to x = 1$, where $1 = 0 \to 0$.

Example 1 Consider $X = [0, 1]$, for every $x, y \in X$, defined $x \to y = 1$, then $(X, \to, 0)$ is a W_d-FI algebras. Example 2 Let $X = \{0, a, 1\}$ be a finite set of distinct elements. We define

$\to 0$	0	a	1
0	1	1	1
a	0	1	1
1	0	a	1

Then $(X, \to, 0)$ is a FI- algebra, but not W_d-FI algebras. In fact, $(1 \to a) \to 0 = a \to 0 = 0$, but $(0 \to a) \to 1 = 1 \to 1 = 1$, so (W2)does not hold.

3. Relationships Between W_d -FI Algebras and Two Classes of Important Fuzzy Algebras

Lemma 3.1. Let $(X, \to, 0)$ is a W_d-FI algebra, then for any $x, y, z \in X$, the following holds:

(W6) $x \to 1 = 1, 1 \to x = x$, for all $x \in X$;

(W7) if $x \to y = 1, y \to z = 1$, then $x \to z = 1$, for all $x, y, z \in X$;

(W8) $(x \to y) \to ((y \to z) \to (x \to z)) = 1$, for all $x, y, z \in X$;

(W9) $(x \to y) \to (x \to z) = y \to z$, $(x \to z) \to (y \to z) = y \to x$;

(W10) $(x \to (y \to z)) \to ((x \to y) \to (x \to z)) = x$.

Proof. (W6) Indeed $x \to 1 = x \to (0 \to x) = 0 \to (x \to x) = 0 \Rightarrow 1 = 1$. Thus, we have verified that $x \to 1 = 1$.

Besides, $(1 \to x) \to x = (x \to x) \to 1 = 1 \to 1 = 1$, $x \to (1 \to x) = 1 \to (x \to x) = 1 \to 1 = 1$. that is $1 \to x = x$.

(W7) If $x \to y = 1, y \to z = 1$ holds, then $x \to z = 1 \to (x \to z) = (x \to y) \to (x \to z) = ((x \to z) \to y) \to x = ((y \to z) \to x) \to x = (1 \to x) \to x = x \to x = 1$, Thus, $x \to z = 1$.

(W8) Indeed $(x \to y) \to ((y \to z) \to (x \to z)) = (y \to z) \to ((x \to y) \to (x \to z)) = (y \to z) \to (((x \to z) \to y) \to x) = (y \to z) \to (((y \to z) \to x) \to x) = (y \to z) \to ((x \to x) \to (y \to z)) = (y \to z) \to (1 \to (y \to z)) = (y \to z) \to (y \to z) = 1$. Hence, for all $x, y, z \in X$, $(x \to y) \to ((y \to z) \to (x \to z)) = 1$.

The proof of (W9), (W10) is similar to previous ones.

Lemma 3.2. Any W_d -FI algebra must be an FI - algebra. but the inverse is not true.

Proof. From the definition 2.3 and (3) of Lemma 3.1, it is easy to see that any W_d -FI algebra must be an FI - algebra. By example 2, W_d -FI algebra be an proper subalgebra of FI - algebra, but FI algebra must be not W_d -FI algebra.

Theorem 3.1. Any W_d-FI algebra must be an RFI- algebra, but the inverse is not true(see[9]).

We have see that W_d-FI algebra classes are subclasses of RFI-algebras.

Theorem 3.2. W_d -FI algebra must be not CFI- algebra.

Proof. It is easy to proof that if $x \neq y$, then the condition (I6) does not hold, i.e., suppose

$$(x \to y) \to y = (y \to x) \to x,$$

then clearly

$$(y \to y) \to x = (x \to x) \to y.$$

Thus, we have $x = y$, which contradict to assertion.

Theorem 3.3. (see[9]) Relations between W_d -FI algebra and HFI - algebra as following :

(1) If $(X, \to, 0)$ is a W_d -FI algebra such that, for all $x, y, z \in X$,

$$x \to (y \to z) = (x \to y) \to (x \to z)$$

holds, then $(X, \to, 0)$ is a HFI-algebra.

(2) If $(X, \to, 0)$ is a HFI-algebra such that, for all $x, y, z \in X$,

$$(x \to y) \to z = (z \to y) \to x$$

holds, then $(X, \to, 0)$ is a W_d -FI algebra.

4. Main Properties of W_d -FI Algebras

On a W_d -FI algebra X, one define a binary relation $\leq$ as follows, $x \leq y$ if and only if $x \to y = 1, x, y \in X$.

Obviously, the relation "$\leq$" is a partial ordering on X.

Theorem 4.1. Let X be a W_d -FI algebra, then exist a partial ordering in X.

Proof. Indeed, $\forall x, y \in X, x \to y = (1 \to x) \to y = (y \to x) \to 1 = 1$. Thus, we have verified that $x \leq y$. Therefore, for any W_d-FI algebra must be exist a partial ordering in X.

Theorem 4.2. Let X be a W_d -FI algebra, and $x, y, z \in X$. Then

(W11) If $x \leq y$, then $z \to x \leq z \to y, y \to z \leq x \to z$;

(W12) $x \leq CC(x)$;

(W13) $CCC(x) = C(x)$;

(W14) $C(x) \to y = C(y) \to x$;

(W15) (Commutativity) $T(x, y) = T(y, x), S(x, y) = S(y, x)$;

(W16) (Associativity) $T(T(x, y), z) = T(x, T(y, z))$, $S(S(x, y), z) = S(x, S(y, z))$;

(W17) (Monotonicity) If $x \leq y$, then $T(x, z) \leq T(y, z), S(x, Z) \leq S(y, z)$;

(W18) (Identity) $T(x, 1) = x, S(x, 0) = x$;

(W19) (Duality) $S(x, y) = C(T(C(x), C(y))), T(x, y) = C(S(C(x), C(y)))$;

(W20) $S(x, C(x)) = 1, S(x, C(x)) = 0$;

(W21) $x \to (y \to z) = T(x, y) \to z$;

(W22) $T((z \to x), (z \to y)) = z \to T(x, y)$;

(W23) $C(x \to y) = x$.

Proof. (W16) $T(T(x, y), z) = T(C(x \to C(y)), z) = C(C(x \to C(y) \to C(z)) = C(((x \to C(y)) \to 0) \to C(z)) = C((((x \to C(y)) \to 0) \to (z \to 0)) = C(z \to (x \to C(y)) = C(x \to (z \to C(y))) = C(x \to (z \to (y \to 0))) = C(x \to (y \to (z \to 0))) = T(x, T(y, z))$.

Similarly, we have $S(S(x, y), z) = S(x, S(y, z))$.

(W17) Due to $x \leq y \Leftrightarrow x \to y = 1$, then for all $x, y, z \in X$, it is $C(x \to C(z)) \to C(y \to C(z)) = ((x \to C(z)) \to 0) \to ((y \to C(z)) \to 0) = (y \to C(z)) \to (x \to C(z)) = x \to y = 1$. Hence, $T(x, z) \leq T(y, z)$. $S(x, Z) \leq S(y, z)$. Similarly, we have $S(x, z) \leq S(y, z)$.

(W18) $T(x, 1) = C(x \to C(1)) = (x \to C(1)) \to 0 = (0 \to C(1)) \to x = 1 \to x = x$. $S(x, 0) = C(x) \to 0 = (x \to 0) \to 0 = (0 \to 0) \to x = 1 \to x = x$.

(W19) $C(T(C(x), C(y))) = C(C(C(x) \to CC(y))) = C(C(C(x) \to y)) = C(C(x) \to y) \to 0 = (0 \to 0) \to (C(x) \to y) = 1 \to (C(x) \to y) = (C(x) \to y) = S(x, y)$;

$C(S(C(x), C(y))) = C(S(C(x), C(y))) = C(CC(x) \to C(y)) = C(x \to C(y)) = T(x, y)$.

(W20) $S(x, C(x)) = C(x) \to C(x) = 1, T(x, C(x)) = C(x \to CC(x)) = C(x \to x) = C(1) = 1 \to 0 = 0$.

(W21) $T(x, y) \to z = C(x \to C(z)) \to z = ((x \to C(y)) \to 0) \to z = ((x \to (y \to 0)) \to 0) \to z = (z \to 0) \to (x \to (y \to 0)) = x \to ((z \to 0) \to (y \to 0) = x \to (y \to z)$.

(W22) $T((z \to x), (z \to y)) = C((z \to x), C(z \to y)) = ((z \to x) \to ((z \to y) \to 0)) \to 0 = z \to x$, and $z \to T(x, y) = z \to C(x \to C(y)) = z \to ((x \to (y \to 0)) \to 0 = z \to x$.

(W23) $C(x \to y) = (x \to y) \to 0 = (0 \to y) \to x = 1 \to x = x$.

Theorem 4.3. Let X be a W_d -FI algebra, and $x, y, z \in X$. Then $1 \to x = x, (x \to y) \to z = (z \to y) \to x$ imply $x \to (y \to z) = y \to (x \to z)$.

Proof. $x \to (y \to z) = (1 \to x) \to (y \to z) = ((y \to z) \to x) \to 1 = ((x \to z) \to y) \to 1 = (1 \to y) \to (x \to z) = y \to (x \to z)$.

Hence, $x \to (y \to z) = y \to (x \to z)$.

Theorem 4.4. A (2,0)-type algebra $(X, \to, 0)$ is a W_d-fuzzy implication algebra if and only if it satisfies that

$(W1')$ $(x \to y) \to z = (z \to y) \to x$;

$(W2')$ $1 \to x = x$;

$(W3')$ $x \to x = 1$;

$(W4')$ If $x \to y = y \to x = 1$, then $x = y$;

$(W5')$ $0 \to x = 1$, where $1 = 0 \to 0$.

Proof. Immediate from theorem 4.2 and definition 2.3.

Condition (W3) and (W6) states that 1 are a logical unit and the greatest element of W_d -FI algebra. Note that a logical unit is always unique. We say that an W_d -FI algebra X has a negation if X admits a smallest element 0 such that the map: $C : x \mapsto C(x)$ is bijective, where $C(x) = x \to 0$.

For an W_d -FI algebra with negation, we define two binary operation on X as follows: for any $x, y \in X$,

$$x \perp y = C(x) \to y,$$

$$x \top y = C(x \to C(y)).$$

Theorem 4.5. Let X be a W_d -FI algebra.Then for any $x, y, z \in X$ we have:

1)$x \perp y = y \perp x, x \top y = y \top x$;

2)$(x \perp y) \perp z = x \perp (y \perp x), (x \top y) \top z = x \top (y \top x)$;

3)$x \perp 1 = 1, x \top 1 = x, x \perp 0 = x$;

4)$x \top C(x) = 0, x \perp C(x) = 1$;

5)$x \perp y = C(C(x) \top C(y)), x \top y = C(C(x) \perp C(y))$;

6)$C(x) \to C(y) = y \to x, C(x) \to y = C(y) \to x$;

Proof. 1) Applying $(W2)$ and the definition of operator $\perp$,we have $x \perp y = C(x) \to y = (x \to 0) \to y = (y \to 0) \to x = y \perp x$. By $(W1)$ and the definition of operator $\top$,we can obtain that $x \top y = C(x \to C(y)) = (x \to (y \to 0)) \to 0 = (y \to (x \to 0)) \to 0 = C(y \to C(x)) = y \top x$.

2) Using $(W2)$ and the definition of operator $\perp$, we get $(x \perp y) \perp z = C(x \perp y) \to z = ((x \to 0) \to y) \to z = (z \to 0) \to (x \to 0) \to y) = (x \to 0) \to ((x \to 0) \to y)) = C(x) \to (C(z) \to y) = x \perp (z \perp y) = x \perp (y \perp z)$.

Similarly, $(x \top y) \top z = C((x \top y) \to C(z)) = C(C(x \to C(y)) \to (z \to 0)) = c(((x \to (y \to 0)) \to 0) \to (z \to 0) = C(((z \to 0) \to 0) \to (x \to (y \to 0)) = C(z \to (x \to (y \to 0)))$,

$x \top (y \top z) = x \top (C(y \to C(z)) = C(x \to CC(y \to C(z))) = C(x \to (y \to (z \to 0))) = C(x \to (z \to (y \to 0))) = C(z \to (x \to (y \to 0)))$,

Hence, $(x \top y) \top z = x \top (y \top z)$.

3) $x \perp 1 = C(x) \to 1 = (x \to 0) \to 1 = (1 \to 0) \to x = 0 \to x = 1$,

$x \top 1 = C(x \to C(1)) = c(x \to (1 \to 0)) = C(x \to 0) = CC(x) = x$,

$x \perp 0 = C(x) \to 0 = (x \to 0) \to 0 = (0 \to 0) \to x = 1 \to x = x$.

4) $x \top C(x) = C(x \to CC(x)) = C(x \to x) = C(1) = 1 \to 0 = 0$,

$x \perp C(x) = C(x) \to C(x) = 1$;

5) $C(C(x) \top C(y)) = C(C(x) \to CC(y))) = C(C(x) \to y))C(x) \to y = x \perp y, i.e. x \top y = C(C(x) \top C(y))$,

$C(C(x) \perp C(y)) = C(CC(x) \to C(y)) = C(x \to C(y)) = x \top y, i.e. x \top y = C(C(x) \perp C(y))$;

6)By (W2), we have $C(x) \to C(y) = (x \to 0) \to (y \to 0) = y \to ((x \to 0) \to 0) = y \to x$ and

$C(x) \to y = (x \to 0) \to y = (y \to 0) \to x = C(y) \to x.$

5. Conclusion

The main aim of this article is to study W_d-fuzzy implication algebras which are subalgebra of fuzzy implication algebras. We showed that W_d -fuzzy implication algebras are regular fuzzy implication algebras, but the inverse is not true. The relations between W_d -fuzzy implication algebras and other fuzzy algebras are discussed. Properties and axiomatic systems for W_d -fuzzy implication algebras are investigated. Furthermore, a few new results on W_d -fuzzy implication algebras has been added, two new operations were introduced in W_d -fuzzy implication algebras and some further properties were given.

The work of this paper clearly suggests that W_d-fuzzy implication algebras provide a fertile area for future research. In future we will study the following topics:

(1) A HFI algebra form a W_d-fuzzy implication algebra under what conditions?

(2) A RFI algebra form a W_d-fuzzy implication algebra under what conditions?

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