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On the Paper: Numerical Radius Preserving Linear Maps on Banach Algebras

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Abstract: We give an example of a unital commutative complex Banach algebra having a normalized state which is not a spectral state and admitting an extreme normalized state which is not multiplicative. This disproves two results by Golfarshchi and Khalilzadeh.

Keywords: Banach Algebra, Regular Norm, Normalized State, Spectral State

1. Preliminaries

Let $(A,\|.\|)$ be a complex normed algebra with an identity e such that $\|e\|=1$. Let $D(A,e)=\{f\in A': f(e)=\|f\|=1\}$, where A' is the dual space of A. The elements of $D(A,e)$ are called normalized states on A. For $a\in A$, let $V(A,a)=\{f(a)\colon f\in D(A,e)\}$, $V(A,a)$ is called the numerical range of a. Let $sp(a)$ be the spectrum of $a\in A$, and let $co(sp(a))$ be the convex hull of $sp(a)$. We say that a linear functional f on A is a spectral state if $f(a)\in co(sp(a))$ for all $a\in A$. We denote by $M(A)$ the set of all non-zero continuous multiplicative linear functionals on A.

2. Result

2.1. Counterexample

Golfarshchi and Khalilzadeh proved the following results [4]:

[4, Theorem 2]. Let A be a unital complex Banach algebra, and let f be a linear functional on A. Then f is a normalized state on A if and only if $f(a)\in co(sp(a))$ for all $a\in A$.

[4, Theorem 3]. Let A be a unital commutative complex Banach algebra. Then each extreme normalized state on A is multiplicative.

Here we give a counterexample disproving the above results. We also remark that Theorems 5 and 6 [4] are called into question since the authors used Theorem 3 [4] to prove these results.

Let $(A,\|.\|)$ be a non-zero commutative radical complex Banach algebra [6, p.316]. Let $A_e=\{a+\lambda e\colon a\in A,\ \lambda\in\mathbb{C}\}$ be the unitization of A with the identity e, and the norm $\|a+\lambda e\|_1=\|a\|+|\lambda|$ for all $a+\lambda e\in A_e$. $(A_e,\|.\|_1)$ is a unital commutative complex Banach algebra, and $M(A_e)=\{\varphi_\infty\}$, where φ_∞ is the continuous multiplicative linear functional on A_e defined by $\varphi_\infty(a+\lambda e)=\lambda$ for all $a+\lambda e\in A_e$.

(1). let a be a non-zero element of A, $V(A_e,a)=\{z\in\mathbb{C}\colon |z|\le\|a\|\}$ by [2, Remark 3.8], and $sp(a)=\{\varphi_\infty(a)\}=\{0\}$, hence $co(sp(a))=\{0\}$ is strictly included in $V(A_e,a)$ since $\|a\|\neq 0$. Therefore the direct implication of [4, Theorem 2] does not hold.

(2). By [1, Lemma 1.10.3], $D(A_e,e)$ is a non-empty weak* compact convex subset of A_e', then $ext(D(A_e,e))$ is a non-empty set. Assume that each extreme normalized state on A_e is multiplicative, then $ext(D(A_e,e))=\{\varphi_\infty\}$. Let a be a non-zero element of A, by [1, Corollary 1.10.15] there exists $f\in D(A_e,e)$ such that $f(a)\neq 0=\varphi_\infty(a)$. Therefore $\overline{co}(ext(D(A_e,e)))=\{\varphi_\infty\}$ is strictly included in $D(A_e,e)$, which contradicts the Krein-Milman Theorem. This shows that [4, Theorem 3] is not valid.

2.2. Regular Norm and the Operator Seminorm

Let $(A,\|.\|)$ be a non-unital complex Banach algebra, and let $A_e=\{a+\lambda e\colon a\in A,\ \lambda\in\mathbb{C}\}$ be the unitization of A with the identity e. Let $\|a+\lambda e\|_{op}=sup\{\|(a+\lambda e)x\|,\|x(a+\lambda e)\|\colon x\in A,\ \|x\|\le 1\}$ for all $a+\lambda e\in A_e$, $\|.\|_{op}$ is an algebra seminorm on A_e. We say that $\|.\|$ is regular if $\|.\|_{op}=\|.\|$ on A. If $\|.\|$ is regular, it is well known that $(A_e,\|.\|_{op})$ is a complex Banach algebra. The following question was asked [3]: If $(A_e, \|.\|_{op})$ is a complex Banach algebra, is the norm $\|.\|$ regular?

Orenstein tried to give an answer to this question in the commutative case [5], but his proof is not correct since it is essentially based on the direct implication of [4, Theorem 2].

3. Conclusion

In this note, we show that Theorems 2 and 3 [4] are false by giving a counterexample. We also remark that Theorems 5 and 6 [4] and Theorem 1.1 [5] are called into question since the authors used Theorems 2 or 3 [4] to prove these results.

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