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New Approaches to Pythagorean Fuzzy Averaging Aggregation Operators

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Abstract: In this paper, we present two Pythagorean fuzzy averaging aggregation operators such as, Pythagorean fuzzy weighted averaging (PFWA) operator, Pythagorean fuzzy ordered weighted averaging (PFOWA) operator and also introduce some of their basic properties.

Keywords: Pythagorean Fuzzy Sets, PFWA Operator, PFOWA Operator

1. Introduction

Atanassov [1] introduced the concept of IFS characterized by a membership function and a non-membership function. It is more suitable for dealing with fuzziness and uncertainty than the ordinary fuzzy set developed by Zadeh [2] characterized by one membership function. Gau and Buehrer [3] proposed the notion of vague set. Chen and Tan [4] and Hong and Choi [5] presented some techniques for handling multi-criteria fuzzy decision-making problems based on vague sets. Bustine and Burillo [6] showed that the vague set is equivalent to IFS. In 1986, many scholars [7, 8, 9, 10, 11, 12] have done works in the field of AIFS and its applications. Particularly, information aggregation is a very crucial research area in IFS theory that has been receiving more and more focus. Xu [13] developed some basic arithmetic aggregation operators, including IFWA operator, IFOWA operator and IFHA operator and applied them to group decision making. Xu and Yager [14] defined some basic geometric aggregation operators such as, IFWG operator, IFOWG operator and IFHG operator, and applied them to multiple attribute decision making (MADM) based on intuitionistic fuzzy information. Wang and Liu [15] introduced the notion of IFEWG operator and geometric IFEOWG operator and applied them to group decision making. In [16] Wang and Liu also introduced the concept of IFEWA operator and IFEOWA operator. Zhao and Wei [17] introduced the notion of two new types of hybrid aggregation operators such as, IFEHA operator and IFEHG operator. But there are many cases where the decision maker may provide the degree of membership and nonmembership of a particular attribute in such a way that their sum is greater than one. Therefore, Yager [18] introduced the concept of PFS. PFS is more powerful tool to solve uncertain problems. In 2013, Yager and Abbasov [19] introduced the notion of two new Pythagorean fuzzy aggregation operators such as PFWA operator and PFOWA operator. In [20, 21, 22, 23] K. Rahman et al. introduced the concept of PFHA operator, PFWG operator, PFOWG operator and PFHG operator.

Thus keeping the advantage of the above mention aggregation operators, in this paper we introduce two new types Pythagorean fuzzy aggregation operators for Pythagorean fuzzy values such as, PFWA operator, PFOWA operator.

The remainder paper can be constructed as follows. In Section 2, we present some basic definitions and results which will be used in our later sections. In Section 3, we develop some basic operational laws and relations. In Section 4, we develop Pythagorean fuzzy weighted averaging operator and Pythagorean fuzzy ordered weighted averaging operator. In Section 5, we have conclusion.

2. Preliminaries

Definition 1: [18] Let Q be a universal set, then a PFS, E can be defined as:

$$E = \{(q, \Delta_E(q), \nabla_E(q)) \mid q \in Q\}, \tag{1}$$

where $\Delta_E(q)$ and $\nabla_E(q)$ are mappings from E to $[0,1]$, with some conditions such as, $0 \le \Delta_E(q) \le 1, 0 \le \nabla_E(q) \le 1$, $0 \le \Delta_E^2(q) + \nabla_E^2(q) \le 1, \forall q \in Q$.

In this paper, we consider the interval $[\lozenge_E(q), 1-\partial_E(q)]$ is a PFV, and replace equation (1) with

$$E = \{(q, [\lozenge_E(q), 1-\partial_E(q)]) \mid q \in Q\}, \tag{2}$$

correspondingly. Here the PFV $[\lozenge_E(q), 1-\partial_E(q)]$ show that the fixed degree of the membership $\Delta_E(q)$ is not defined. However it can lie between, $\lozenge_E(q) \le \Delta_E(q) \le 1-\partial_E(q)$, where $\lozenge_E^2(q) + \partial_E^2(q) \le 1$.

Definition 2: [13] Let $\wp_j = \left[\Delta_{\wp_j}, 1-\partial_{\wp_j}\right]$ be a collection of IFVs, then IFWA operator can be defined as:

$$IFWA_{\mathrm{I}}(\wp_1, \wp_2, ..., \wp_n) = \mathrm{I}_1\wp_1 \oplus \mathrm{I}_2\wp_2 \oplus ... \oplus \mathrm{I}_n\wp_n, \tag{3}$$

where $\mathrm{I} = (\mathrm{I}_1, \mathrm{I}_2, \mathrm{I}_3, ..., \mathrm{I}_n)^T$ be the weighted vector of $\wp_j$ with $\mathrm{I}_j \in [0,1]$ and $\sum_{j=1}^{n} \mathrm{I}_j = 1$.

Definition 3: [13] Let $\wp_j = \left[\varsigma_{\wp_j}, 1-\partial_{\wp_j}\right]$ be a collection of IFVs, then IFOWA operator can be defined as:

$$IFOWA_{\mathrm{I}}(\wp_1, \wp_2, ..., \wp_n) = \mathrm{I}_1\wp_{\tau(1)} \oplus \mathrm{I}_2\wp_{\tau(2)} \oplus ... \oplus \mathrm{I}_n\wp_{\tau(n)}, \tag{4}$$

where $(\tau(1), \tau(2), \tau(3), ..., \tau(n))$ is any permutation of $(1, 2, ..., n)$ such that $\wp_{\tau(j-1)} \ge \wp_{\tau(j)}$.

3. Operational Laws and Relations

Definition 4. Let $\aleph_1 = \left[\hbar_{\aleph_1}, 1-\partial_{\aleph_1}\right]$, $\aleph_2 = \left[\hbar_{\aleph_2}, 1-\partial_{\aleph_2}\right]$ be the two *PFVs*, then $S(\aleph_1) = \hbar_{\aleph_1}^2 - \partial_{\aleph_1}^2$, $S(\aleph_2) = \hbar_{\aleph_2}^2 - \partial_{\aleph_2}^2$ be the scores *of* $\aleph_1$, $\aleph_2$ *and* $H(\aleph_1) = \hbar_{\aleph_1}^2 + \partial_{\aleph_1}^2$, $H(\aleph_2) = \hbar_{\aleph_2}^2 + \partial_{\aleph_2}^2$ *be the accuracy degrees of* $\aleph_1$, $\aleph_2$ *respectively, then*

(1) If $S(\aleph_1) < S(\aleph_2)$, then $\aleph_1 < \aleph_2$,

(2) If $S(\aleph_1) = S(\aleph_2)$, then

(a) If $H(\aleph_1) = H(\aleph_2)$, then $\aleph_1 = \aleph_2$,

(b) If $H(\aleph_1) < H(\aleph_2)$ then $\aleph_1 < \aleph_2$,

(c) If $H(\aleph_1) > H(\aleph_2)$ then $\aleph_1 \succ \aleph_2$,

Theorem 1: Let $\Re_1 = \left[\hbar_{\Re_1}, 1-\partial_{\Re_1}\right]$, $\Re_2 = \left[\hbar_{\Re_2}, 1-\partial_{\Re_2}\right]$ be the collection of two PFVs, if $\hbar_{\Re_1} \le \hbar_{\Re_2}$ and $\partial_{\Re_1} \ge \partial_{\Re_2}$, then $\Re_1 \le \Re_2$.

Proof: Since $S(\Re_1) = \hbar_{\Re_1}^2 - \partial_{\Re_1}^2$ and $S(\Re_2) = \hbar_{\Re_2}^2 - \partial_{\Re_2}^2$. Now $\hbar_{\Re_1} \le \hbar_{\Re_2}, \partial_{\Re_1} \ge \partial_{\Re_2}$, then

$$\begin{aligned} S(\Re_1) - S(\Re_2) &= \left(\hbar_{\Re_1}^2 - \partial_{\Re_1}^2\right) - \left(\hbar_{\Re_2}^2 - \partial_{\Re_2}^2\right) \\ &= \hbar_{\Re_1}^2 - \partial_{\Re_1}^2 - \hbar_{\Re_2}^2 + \partial_{\Re_2}^2 \\ &= \left(\hbar_{\Re_1}^2 - \hbar_{\Re_2}^2\right) + \left(\partial_{\Re_2}^2 - \partial_{\Re_1}^2\right). \end{aligned}$$

If

$$\hbar_{\Re_1}^2 = \hbar_{\Re_2}^2 \Leftrightarrow \hbar_{\Re_1} = \hbar_{\Re_2}. \tag{5}$$

And

$$\partial_{\Re_2}^2 = \partial_{\Re_1}^2 \Leftrightarrow \partial_{\Re_2} = \partial_{\Re_1}. \tag{6}$$

From equation (5) and equation (6), we have

$$\Re_1 = \Re_2. \tag{7}$$

Otherwise we have $S(\Re_1) - S(\Re_2) < 0$ i.e., $S(\Re_1) < S(\Re_2)$. Thus

$$\Re_1 < \Re_2. \tag{8}$$

From equation (7) and equation (8), we have $\Re_1 \le \Re_2$.

Definition 5: Let $\Re_1 = \left[\hbar_{\Re_1}, 1-\partial_{\Re_1}\right]$, $\Re_2 = \left[\hbar_{\Re_2}, 1-\partial_{\Re_2}\right]$ and $\Re_3 = \left[\hbar_{\Re_3}, 1-\partial_{\Re_3}\right]$ be the collection of three PFVs, and $\lambda > 0$, then

(1) $\Re_1 \oplus \Re_2 = \left[\sqrt{\hbar_{\Re_1}^2 + \hbar_{\Re_2}^2 - \hbar_{\Re_1}^2\hbar_{\Re_2}^2}, 1-\partial_{\Re_1}\partial_{\Re_2}\right]$

(2) $\lambda\Re = \left[\sqrt{1-\left(1-\hbar_{\Re}^2\right)^{\lambda}}, 1-\left(\partial_{\Re}\right)^{\lambda}\right]$

Let $S_P\left(\hbar_{\Re_1}, \hbar_{\Re_2}\right) = \sqrt{\hbar_{\Re_1}^2 + \hbar_{\Re_2}^2 - \hbar_{\Re_1}^2\hbar_{\Re_2}^2}$ and $T_P\left(\partial_{\Re_1}, \partial_{\Re_2}\right) = \partial_{\Re_1}\partial_{\Re_2}$, then the operational law (1) can be rewritten as follows:

$$\Re_1 \oplus \Re_2 = \left[S_P\left(\hbar_{\Re_1}, \hbar_{\Re_2}\right), 1-T_P\left(\partial_{\Re_1}, \partial_{\Re_2}\right)\right], \tag{9}$$

where $S_P\left(\hbar_{\Re_1}, \hbar_{\Re_2}\right) = \sqrt{\hbar_{\Re_1}^2 + \hbar_{\Re_2}^2 - \hbar_{\Re_1}^2\hbar_{\Re_2}^2}$ is called t - conorm.

Especially, if $\hbar_{\Re_1} = 1-\partial_{\Re_1}$ and $\hbar_{\Re_2} = 1-\partial_{\Re_2}$, then both

$\Re_1=\left[\hbar_{\Re_1},1-\partial_{\Re_1}\right]$ and $\Re_2=\left[\hbar_{\Re_2},1-\partial_{\Re_2}\right]$ are converted to $\hbar_{\Re_1}$ and $\hbar_{\Re_2}$, respectively. Here the above mention laws are converted into the following positions:

$$(1)\ \Re_1\oplus\Re_2=S_P\left(\hbar_{\Re_1},\hbar_{\Re_2}\right)$$

$$(2)\qquad \lambda\Re=\sqrt{1-\left(1-\hbar_{\Re}^2\right)^{\lambda}},\lambda>0$$

Let $\delta(\lambda,\Re)=\sqrt{1-\left(1-\hbar_{\Re}^2\right)^{\lambda}}$, $\lambda>0$, then expression (2) is a unit interval monotone increasing function on λ and $\Re$ having the following good properties.

(1) $0\le\delta(\lambda,\Re)\le1$. Especially, $\delta(\lambda,0)=0,\delta(\lambda,1)=1$ and $\delta(1,\Re)=\Re$,

(2) If $\lambda\to0$ and $0<\Re<1$, then $\delta(\lambda,\Re)\to0$,

(3) If $\lambda\to+\infty$ and $0<\tilde{a}<1$, then $\delta(\lambda,\tilde{a})\to1$,

(4) If $\lambda_1>\lambda_2$, then $\delta(\lambda_1,\Re)>\delta(\lambda_2,\Re)$,

(5) If $\Re_1>\Re_2$, then $\delta(\lambda,\Re_1)>\delta(\lambda,\Re_2)$.

These desirable properties provide a theoretic basis for the application of the operational law (2) to the aggregation of PFVs.

Theorem 2: Let $\Re_1=\left[\hbar_{\Re_1},1-\partial_{\Re_1}\right]$, $\Re_2=\left[\hbar_{\Re_2},1-\partial_{\Re_2}\right]$ and $\Re_3=\left[\hbar_{\Re_3},1-\partial_{\Re_3}\right]$ be the collection of three PFVs and let $\beta_1=\Re_1\oplus\Re_2$ and $\beta_2=\lambda\Re(\lambda>0)$, then both β_1 and β_2 are also PFVs.

Proof: Since: Let $\Re_1=\left[\hbar_{\Re_1},1-\partial_{\Re_1}\right]$, $\Re_2=\left[\hbar_{\Re_2},1-\partial_{\Re_2}\right]$ be two PFVs, then $\hbar_{\Re_1}^2+\partial_{\Re_1}^2\le1,\hbar_{\Re_2}^2+\partial_{\Re_2}^2\le1$. Then we have

$$\begin{aligned}&\left(\sqrt{\hbar_{\Re_1}^2+\hbar_{\Re_2}^2-\hbar_{\Re_1}^2\hbar_{\Re_2}^2}\right)^2+1-\left(1-\partial_{\Re_1}^2\partial_{\Re_2}^2\right)\\&\le\left(1-\partial_{\Re_1}^2\partial_{\Re_2}^2\right)+1-\left(1-\partial_{\Re_1}^2\partial_{\Re_2}^2\right)\\&=1.\end{aligned}$$

Thus β_1 is a PFV. Now $\sqrt{1-\left(1-\hbar_{\Re}^2\right)^{\lambda}}\ge0$ and $\left(\partial_{\Re}\right)^{\lambda}\ge0$, then

$$\begin{aligned}&\left(\sqrt{1-\left(1-\hbar_{\Re}^2\right)^{\lambda}}\right)^2+\left(\partial_{\Re}^2\right)^{\lambda}\\&\le\left(\sqrt{1-\left(1-\hbar_{\Re}^2\right)^{\lambda}}\right)^2+\left(1-\hbar_{\Re}^2\right)^{\lambda}\\&=1-\left(1-\hbar_{\Re}^2\right)^{\lambda}+\left(1-\hbar_{\Re}^2\right)^{\lambda}\\&=1.\end{aligned}$$

Thus β_2 is a PFV.

Now we are define some basic cases as follows:

(1) If $\Re=\left[\hbar_{\Re},1-\partial_{\Re}\right]=[1,1]$, i.e., $\hbar_{\Re}=1,1-\partial_{\Re}=1$, then

$$\begin{aligned}\lambda\Re&=\left[\sqrt{1-\left(1-\hbar_{\Re}^2\right)^{\lambda}},1-\left(\partial_{\Re}\right)^{\lambda}\right],\lambda>0\\&=[1,1].\end{aligned}$$

(2) If $\Re=\left[\hbar_{\Re},1-\partial_{\Re}\right]=[0,0]$, i.e., $\hbar_{\Re}=0,1-\partial_{\Re}=0$, then

$$\begin{aligned}\lambda\Re&=\left[\sqrt{1-\left(1-\hbar_{\Re}^2\right)^{\lambda}},1-\left(\partial_{\Re}\right)^{\lambda}\right],\lambda>0\\&=[0,0].\end{aligned}$$

(3) If $\Re=\left[\hbar_{\Re},1-\partial_{\Re}\right]=[0,1]$, i.e., $\hbar_{\Re}=0,1-\partial_{\Re}=1$, then

$$\begin{aligned}\lambda\Re&=\left[\sqrt{1-\left(1-\hbar_{\Re}^2\right)^{\lambda}},1-\left(\partial_{\Re}\right)^{\lambda}\right],\lambda>0\\&=[0,1].\end{aligned}$$

(4) If $\lambda\to0$ and $0<\hbar_{\Re},\partial_{\Re}<1$, then

$$\begin{aligned}\lambda\Re&=\left[\sqrt{1-\left(1-\hbar_{\Re}^2\right)^{\lambda}},1-\left(\partial_{\Re}\right)^{\lambda}\right]\\&=[0,0].\end{aligned}$$

(5) If $\lambda\to+\infty$ and $0<\hbar_{\Re},\partial_{\Re}<1$, then

$$\begin{aligned}\lambda\Re&=\left[\sqrt{1-\left(1-\hbar_{\Re}^2\right)^{\lambda}},1-\left(\partial_{\Re}\right)^{\lambda}\right]\\&=[1,1].\end{aligned}$$

(6) If $\lambda=1$, then

$$\begin{aligned}\lambda\Re&=\left[\sqrt{1-\left(1-\hbar_{\Re}^2\right)^{\lambda}},1-\left(\partial_{\Re}\right)^{\lambda}\right]\\&=\Re.\end{aligned}$$

Theorem 3: Let $\Re_1=\left[\hbar_{\Re_1},1-\partial_{\Re_1}\right]$, $\Re_2=\left[\hbar_{\Re_2},1-\partial_{\Re_2}\right]$ be two PFVs and $\lambda_1,\lambda_2,\lambda_3>0$, then we

(1) $\Re_1\oplus\Re_2=\Re_2\oplus\Re_1$,

(2) $\lambda(\Re_1\oplus\ \Re_2)=\lambda\Re_1\oplus\lambda\Re_2$,

(3) $\lambda_1\Re\oplus\lambda_2\Re=(\lambda_1\oplus\lambda_2)\Re$.

Proof (1) Since

$$\begin{aligned}\Re_1 \oplus \Re_2 &= \left[\sqrt{\hbar^2_{\Re_1} + \hbar^2_{\Re_2} - \hbar^2_{\Re_1}\hbar^2_{\Re_2}}, 1-\partial_{\Re_1}\partial_{\Re_2}\right] \\ &= \left[\sqrt{\hbar^2_{\Re_2} + \hbar^2_{\Re_1} - \hbar^2_{\Re_2}\hbar^2_{\Re_1}}, 1-\partial_{\Re_2}\partial_{\Re_1}\right] \\ &= [\Re_2 \oplus \Re_1].\end{aligned}$$

(2) Since

$$\Re_1 \oplus \Re_2 = \left[\sqrt{\hbar^2_{\Re_1} + \hbar^2_{\Re_2} - \hbar^2_{\Re_1}\hbar^2_{\Re_2}}, 1-\partial_{\Re_1}\partial_{\Re_2}\right]$$

Then

$$\begin{aligned}\lambda(\Re_1 \oplus \Re_2) &= \begin{bmatrix}\sqrt{1-\left(1-\left(\sqrt{\hbar^2_{\Re_1}+\hbar^2_{\Re_2}-\hbar^2_{\Re_1}\hbar^2_{\Re_2}}\right)^2\right)^{\lambda}}, \\ 1-\left(\partial_{\Re_1}\partial_{\Re_2}\right)^{\lambda}\end{bmatrix} \\ &= \begin{bmatrix}\sqrt{1-\left(1-\hbar^2_{\Re_1}\right)^{\lambda}\left(1-\hbar^2_{\Re_2}\right)^{\lambda}}, \\ 1-\left(\partial_{\Re_1}\partial_{\Re_2}\right)^{\lambda}\end{bmatrix}\end{aligned} \tag{10}$$

Now

$$\lambda\Re_1 = \left[\sqrt{1-\left(1-\hbar^2_{\Re_1}\right)^{\lambda}}, 1-\left(\partial_{\Re_1}\right)^{\lambda}\right],$$

$$\lambda\Re_2 = \left[\sqrt{1-\left(1-\hbar^2_{\Re_2}\right)^{\lambda}}, 1-\left(\partial_{\Re_2}\right)^{\lambda}\right].$$

Since

$$\begin{aligned}\lambda\Re_1 \oplus \lambda\Re_2 &= \begin{bmatrix}\sqrt{\begin{array}{c}\left(\sqrt{1-\left(1-\hbar^2_{\Re_1}\right)^{\lambda}}\right)^2+\left(\sqrt{1-\left(1-\hbar^2_{\Re_2}\right)^{\lambda}}\right)^2- \\ \left(\sqrt{1-\left(1-\hbar^2_{\Re_1}\right)^{\lambda}}\right)^2\left(\sqrt{1-\left(1-\hbar^2_{\Re_2}\right)^{\lambda}}\right)^2\end{array}}, \\ 1-\left(\partial_{\Re_1}\right)^{\lambda}\left(\partial_{\Re_1}\right)^{\lambda}\end{bmatrix} \\ &= \left[\sqrt{1-\left(1-\hbar^2_{\Re_1}\right)^{\lambda}\left(1-\hbar^2_{\Re_2}\right)^{\lambda}}, 1-\left(\partial_{\Re_1}\partial_{\Re_2}\right)^{\lambda}\right]\end{aligned} \tag{11}$$

From equation (10) and equation (11) we have

$$\lambda(\Re_1 \oplus \Re_2) = \lambda\Re_1 \oplus \lambda\Re_2.$$

(3) Since

$$\lambda_1\Re = \left[\sqrt{1-\left(1-\hbar^2_{\Re}\right)^{\lambda_1}}, 1-\left(\partial_{\Re}\right)^{\lambda_1}\right],$$

$$\lambda_2\Re = \left[\sqrt{1-\left(1-\hbar^2_{\Re}\right)^{\lambda_2}}, 1-\left(\partial_{\Re}\right)^{\lambda_2}\right].$$

Then

$$\begin{aligned}\lambda_1\Re \oplus \lambda_2\Re &= \begin{bmatrix}\sqrt{\begin{array}{c}\left(\sqrt{1-\left(1-\hbar^2_{\Re}\right)^{\lambda_1}}\right)^2+\left(\sqrt{1-\left(1-\hbar^2_{\Re}\right)^{\lambda_2}}\right)^2 \\ -\left(\sqrt{1-\left(1-\hbar^2_{\Re}\right)^{\lambda_1}}\right)^2\left(\sqrt{1-\left(1-\hbar^2_{\Re}\right)^{\lambda_2}}\right)^2\end{array}}, \\ 1-\left(\partial_{\Re}\right)^{\lambda_1}\left(\partial_{\Re}\right)^{\lambda_2}\end{bmatrix} \\ &= \begin{bmatrix}\sqrt{\begin{array}{c}1-\left(1-\hbar^2_{\Re}\right)^{\lambda_1}+1-\left(1-\hbar^2_{\Re}\right)^{\lambda_2}- \\ \left(1-\left(1-\hbar^2_{\Re}\right)^{\lambda_1}\right)\left(1-\left(1-\hbar^2_{\Re}\right)^{\lambda_2}\right)\end{array}}, \\ 1-\left(\partial_{\Re}\right)^{\lambda_1}\left(\partial_{\Re}\right)^{\lambda_2}\end{bmatrix} \\ &= (\lambda_1 \oplus \lambda_2)\Re.\end{aligned}$$

4. Pythagorean Fuzzy Weighted Averaging Aggregation Operators

In this section we introduced the notion of two new types of aggregation operators such as, Pythagorean fuzzy weighted averaging aggregation operator and Pythagorean fuzzy ordered weighted averaging aggregation operator and also discuss some of their basic properties.

4.1. Pythagorean Fuzzy Weighted Averaging Operator

Definition 6: Let $\aleph_j = \left[b_{\aleph_j}, 1-\partial_{\aleph_j}\right]$ (j=1,2,3,…,n) be a collection of PFVs, and let $PFWA : \mho^n \to \mho,\ if$

$$PFWA_{\chi}(\aleph_1, \aleph_2, \ldots, \aleph_n) = \left[\sqrt{1-\prod_{j=1}^{n}\left(1-b^2_{\aleph_j}\right)^{\chi_j}}, 1-\prod_{j=1}^{n}\left(\partial_{\aleph_j}\right)^{\chi_j}\right], \tag{12}$$

then $PFWA$ is called Pythagorean fuzzy weighted averaging $(PFWA)$ operator of dimension n. Especially, if $\varpi = \left(\frac{1}{n}, \frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)^T$, then the PFWA operator is reduced to a PFA operator of dimension n, which can be defined as follows:

$$PFA(\aleph_1, \aleph_2, \ldots, \aleph_n) = \frac{1}{n}(\aleph_1 \oplus \aleph_2 \oplus \ldots \oplus \aleph_n). \tag{13}$$

Theorem 4. Let $\aleph_j = \left[b_{\aleph_j}, 1-\partial_{\aleph_j}\right]$ (j=1,2,3,…,n) be a collection of PFVs, then their aggregated value by using the PFWA operator is also a PFV.

Proof: Straightforward.

Example 1: Let

$\Re_1 = [0.8, 0,5], \Re_2 = [0.7, 0, 6], \Re_3 = [0.6, 0, 7],$
$\Re_4 = [0.5, 0, 8], \Re_5 = [0.4, 0, 9],$

be the PFVs and let $\varpi = (0.1,\ 0.2,\ 0.2,\ 0.2,\ 0.3)^T$ be the weight vector of $\Re_j\ (j = 1, 2, 3, 4, 5)$, then

$\hbar_{\Re_1}=0.8,\hbar_{\Re_2}=0.7,\hbar_{\Re_3}=0.6,\hbar_{\Re_4}=0.5,\hbar_{\Re_5}=0.4,$
and
$1-\partial_{\Re_1}=0.5\Rightarrow\partial_{\Re_1}=0.5, 1-\partial_{\Re_2}=0.6\Rightarrow\partial_{\Re_2}=0.4$
$1-\partial_{\Re_3}=0.7\Rightarrow\partial_{\Re_3}=0.3, 1-\partial_{\Re_4}=0.8\Rightarrow\partial_{\Re_4}=0.2$
$1-\partial_{\Re_5}=0.9\Rightarrow\partial_{\Re_5}=0.1$

Thus
$$\begin{aligned}&PFWA_{\varpi}\left(\Re_1,\Re_2,\Re_3,\Re_4,\Re_5\right)\\&=\left[\sqrt{1-\prod_{j=1}^{5}\left(1-\hbar_{\Re_j}^2\right)^{\varpi_j}}, 1-\prod_{j=1}^{5}\left(\partial_{\Re_j}\right)^{\varpi_j}\right]\\&=\left[\sqrt{1-0.643}, 1-0.221\right]\\&=(0.597, 0.778)\end{aligned}$$

Theorem 5. Let $\aleph_j=\left[\Im_{\aleph_j}, 1-\wp_{\aleph_j}\right]$ $(j=1,2,3,...,n)$ be a collection of PFVs and $\chi=(\chi_1,\chi_2,\chi_2,...,\chi_n)^T$ is the weighted vector of $\aleph_j$ with $\chi_j\in[0,1]$ and $\sum_{j=1}^{n}\chi_j=1,$ then the following conditions hold.

(1) (Idempotency): If $\aleph_j=\aleph,$ for all j, then

$$PFWA_{\chi}\left(\aleph_1,\aleph_2,\aleph_3,...,\aleph_n\right)=\aleph. \tag{14}$$

(2) (Boundary):

$$\aleph^-\le PFWA_{\chi}\left(\aleph_1,\aleph_2,\aleph_3,...,\aleph_n\right)\le\aleph^+, \text{ for all } \chi, \tag{15}$$

Where

$$\aleph^-=\left[\min\left(\Im_{\aleph_j}\right), 1-\max\left(\wp_{\aleph_j}\right)\right], \tag{16}$$

$$\aleph^+=\left[\max\left(\Im_{\aleph_j}\right), 1-\min\left(\wp_{\aleph_j}\right)\right]. \tag{17}$$

(3) (Monotonicity): If $\hbar_{\Re_j}\le\hbar_{\Re_j^*}$. $\partial_{\Re_j}\ge\partial_{\Re_j^*}$ for all j, then

$$PFWA_{\varpi}\left(\Re_1,\Re_2,...,\Re_n\right)\le PFWA_{\varpi}\left(\Re_1^*,\Re_2^*,...,\Re_n^*\right), \tag{18}$$

Proof: (1) As we know that
$PFWA_{\chi}\left(\aleph_1,\aleph_2,...,\aleph_n\right)=\chi_1\aleph_1\oplus\chi_2\aleph_2\oplus...\oplus\chi_n\aleph_n$
Let $\aleph_j=\aleph.$ Then

$$\begin{aligned}&PFWA_{\chi}\left(\aleph_1,\aleph_2,\aleph_3,...,\aleph_n\right)\\&=\chi_1\aleph\oplus\chi_2\aleph\oplus\chi_3\aleph\oplus...\oplus\chi_n\aleph\\&=\left(\chi_1\oplus\chi_2\oplus\chi_3\oplus...\oplus\chi_n\right)\aleph\\&=\left(\sum_{j=1}^{n}\chi_j\right)\aleph\\&=\aleph.\end{aligned}$$

(2) (Boundary): From equation (16) we have

$$\begin{aligned}&\Leftrightarrow\sqrt{\min\left(\Im^2_{\aleph_j}\right)}\le\sqrt{\Im^2_{\aleph_j}}\le\sqrt{\max\left(\Im^2_{\aleph_j}\right)}\\&\Leftrightarrow\sqrt{\prod_{j=1}^{n}\left[1-\max\left(\Im^2_{\aleph_j}\right)\right]^{\chi_j}}\le\sqrt{\prod_{j=1}^{n}\left[1-\Im^2_{\aleph_j}\right]^{\chi_j}}\le\sqrt{\prod_{j=1}^{n}\left[1-\min\left(\Im^2_{\aleph_j}\right)\right]^{\chi_j}}\\&\Leftrightarrow\sqrt{\left[1-\max\left(\Im^2_{\aleph_j}\right)\right]^{\sum_{j=1}^{n}\chi_j}}\le\sqrt{\prod_{j=1}^{n}\left[1-\Im^2_{\aleph_j}\right]^{\chi_j}}\le\sqrt{\left[1-\min\left(\Im^2_{\aleph_j}\right)\right]^{\sum_{j=1}^{n}\chi_j}}\\&\Leftrightarrow\min\left(\Im_{\aleph_j}\right)\le\sqrt{1-\prod_{j=1}^{n}\left[1-\Im^2_{\aleph_j}\right]^{\chi_j}}\le\max\left(\Im_{\aleph_j}\right).\end{aligned} \tag{19}$$

From equation (17) we have

$$\begin{aligned}&\Leftrightarrow\min\left(\wp_{\aleph_j}\right)^{\chi_j}\le\left(\wp_{\aleph_j}\right)^{\chi_j}\le\max\left(\wp_{\aleph_j}\right)^{\chi_j}\\&\Leftrightarrow\prod_{j=1}^{n}\min\left(\wp_{\aleph_j}\right)^{\chi_j}\le\prod_{j=1}^{n}\left(\wp_{\aleph_j}\right)^{\chi_j}\le\prod_{j=1}^{n}\max\left(\wp_{\aleph_j}\right)^{\chi_j}\\&\Leftrightarrow\min\left(\wp_{\aleph_j}\right)^{\sum_{j=1}^{n}\chi_j}\le\prod_{j=1}^{n}\left(\wp_{\aleph_j}\right)^{\chi_j}\le\max\left(\wp_{\aleph_j}\right)^{\sum_{j=1}^{n}\chi_j}\\&\Leftrightarrow\min\left(\wp_{\aleph_j}\right)\le\prod_{j=1}^{n}\left(\wp_{\aleph_j}\right)^{\chi_j}\le\max\left(\wp_{\aleph_j}\right)\\&\Leftrightarrow 1-\max\left(\wp_{\aleph_j}\right)\le 1-\prod_{j=1}^{n}\left(\wp_{\aleph_j}\right)^{\chi_j}\le 1-\min\left(\wp_{\aleph_j}\right).\end{aligned} \tag{20}$$

Let

$$PFWA_{\chi}\left(\aleph_1,\aleph_2,\aleph_3,...,\aleph_n\right)=\aleph=\left[\Im_{\aleph}, 1-\wp_{\aleph}\right]. \tag{21}$$

Then

$$\begin{aligned}S(\aleph)&=\Im^2_{\aleph}-\wp^2_{\aleph}\\&\le\left[\max\left(\Im_{\aleph_j}\right)\right]^2-\left[\min\left(\wp_{\aleph_j}\right)\right]^2\\&=S\left(\aleph^+\right).\end{aligned} \tag{22}$$

Again

$$S(\aleph) = \Im_{\aleph}^2 - \wp_{\aleph}^2$$
$$\geq \left[\min_j \left(\Im_{\aleph_j}\right)\right]^2 - \left[\max_j \left(\wp_{\aleph_j}\right)\right]^2 \quad (23)$$
$$= S\left(\aleph^-\right).$$

If

$$S(\aleph) \prec S\left(\aleph^+\right). \quad (24)$$

And

$$S(\aleph) \succ S\left(\aleph^-\right). \quad (25)$$

Thus from equations (24) and (52), we have

$$\aleph^- < PFWA_{\chi}\left(\aleph_1, \aleph_2, \aleph_3, ..., \aleph_n\right) < \aleph^+. \quad (26)$$

If

$$S(\aleph) = S\left(\aleph^+\right). \quad (27)$$

Then

$$\Leftrightarrow \Im_{\aleph}^2 - \wp_{\aleph}^2 = \left[\max_j \left(\Im_{\aleph}\right)\right]^2 - \left[\min_j \left(\wp_{\aleph}\right)\right]^2$$
$$\Leftrightarrow \Im_{\aleph}^2 = \left[\max_j \left(\Im_{\aleph}\right)\right]^2, \wp_{\aleph}^2 = \left[\min_j \left(\wp_{\aleph}\right)\right]^2$$
$$\Leftrightarrow \Im_{\aleph} = \max_j \left(\Im_{\aleph}\right), \wp_{\aleph} = \min_j \left(\wp_{\aleph}\right)$$

Since

$$H(\aleph) = \Im_{\aleph}^2 + \wp_{\aleph}^2$$
$$= \left[\max_j \left(\Im_{\aleph_j}\right)\right]^2 + \left[\min_j \left(\wp_{\aleph_j}\right)\right]^2 \quad (28)$$
$$= H\left(\aleph^+\right).$$

Thus from equation (28), we have

$$PFWA_{\chi}\left(\aleph_1, \aleph_2, \aleph_3, ..., \aleph_n\right) = \aleph^+. \quad (29)$$

If

$$S(\aleph) = S\left(\aleph^-\right) \quad (30)$$

Then

$$\Leftrightarrow \Im_{\aleph}^2 - \wp_{\aleph}^2 = \left[\min_j \left(\wp_{\aleph}\right)\right]^2 - \left[\max_j \left(\Im_{\aleph}\right)\right]^2$$
$$\Leftrightarrow \Im_{\aleph}^2 = \left[\min_j \left(\wp_{\aleph}\right)\right]^2, \wp_{\aleph}^2 = \left[\max_j \left(\Im_{\aleph}\right)\right]^2$$
$$\Leftrightarrow \Im_{\aleph} = \min_j \left(\wp_{\aleph}\right), \wp_{\aleph} = \max_j \left(\Im_{\aleph}\right).$$

Since

$$H(\aleph) = \Im_{\aleph}^2 + \wp_{\aleph}^2$$
$$= \left[\min_j \left(\wp_{\aleph_j}\right)\right]^2 + \left[\max_j \left(\Im_{\aleph_j}\right)\right]^2 \quad (31)$$
$$= H\left(\aleph^-\right).$$

Thus from equation (31), we have

$$PFWA_{\chi}\left(\aleph_1, \aleph_2, \aleph_3, ..., \aleph_n\right) = \aleph^-. \quad (32)$$

Thus from equations (26), (29), and (32), we have (15) always holds.

(3) (Monotonicity): Follows the above proof.

4.2. Pythagorean Fuzzy Ordered Weighted Averaging

Definition 7. Let $\wp_j = \left[\Im_{\wp_j}, 1 - \varnothing_{\wp_j}\right]$ $(j = 1, ..., n)$, then a PFOWA can be define as:

$$PFOWA_i\left(\wp_1, \wp_2, ..., \wp_n\right) = i_1\wp_{\tau(1)} + i_2\wp_{\tau(2)} + ... + i_n\wp_{\tau(n)}, \quad (33)$$

where $(\tau(1), \tau(2), \tau(3), ..., \tau(n))$ is a permutation of $(1, 2, ..., n)$ with condition $\wp_{\tau(j-1)} \geq \wp_{\tau(j)}$ for all j, and $i = \left(i_1, i_2, i_3, ..., i_n\right)^T$ be the weighted vector of $\wp_j$.

Theorem 6. Let $\aleph_j = \left[\Im_{\aleph_j}, 1 - \wp_{\aleph_j}\right]$ (j=1, 2,…,*n*) be a collection of PFVs, by applying PFOWA operator the result is also a PFV.

Proof: The proof is similar to the Theorem 4.

Example 2. Let

$$\Re_1 = [0.4, 0, 7], \Re_2 = [0.5, 0, 6], \Re_3 = [0.6, 0, 7],$$
$$\Re_4 = [0.8, 0, 5], \Re_5 = [0.8, 0, 4],$$

and $\chi = \left(0.1,\ 0.1,\ 0.2,\ 0.2,\ 0.4\right)^T$, then

$$\hbar_{\Re_1} = 0.4, \hbar_{\Re_2} = 0.5, \hbar_{\Re_3} = 0.6, \hbar_{\Re_4} = 0.8, \hbar_{\Re_5} = 0.8,$$

and

$$1 - \partial_{\Re_1} = 0.7 \Leftrightarrow \partial_{\Re_1} = 1 - 0.7 = 0.3$$
$$1 - \partial_{\Re_2} = 0.6 \Leftrightarrow \partial_{\Re_2} = 1 - 0.6 = 0.4$$
$$1 - \partial_{\Re_3} = 0.7 \Leftrightarrow \partial_{\Re_3} = 1 - 0.7 = 0.3$$
$$1 - \partial_{\Re_4} = 0.5 \Leftrightarrow \partial_{\Re_4} = 1 - 0.5 = 0.5$$
$$1 - \partial_{\Re_5} = 0.4 \Leftrightarrow \partial_{\Re_5} = 1 - 0.4 = 0.6$$

Now we can find the score function $\Re_j\left(j = 1, 2, 3, 4, 5\right)$.

$$S(\Re_1) = (0.4)^2 - (0.3)^2 = 0.07$$
$$S(\Re_2) = (0.5)^2 - (0.4)^2 = 0.09$$
$$S(\Re_3) = (0.6)^2 - (0.3)^2 = 0.27$$
$$S(\Re_4) = (0.8)^2 - (0.5)^2 = 0.39$$
$$S(\Re_5) = (0.8)^2 - (0.6)^2 = 0.28$$

Hence $S(\Re_4) \succ S(\Re_5) \succ S(\Re_3) \succ S(\Re_2) \succ S(\Re_1)$

Thus

$$PFOWA_\chi(\Re_1, \Re_2, \Re_3, \Re_4, \Re_5)$$
$$= \left[\sqrt{1 - \prod_{j=1}^{5}\left(1 - \hbar^2_{\Re_{\tau(j)}}\right)^{\chi_j}}, 1 - \prod_{j=1}^{5}\left(\partial_{\Re_{\tau(j)}}\right)^{\chi_j}\right]$$
$$= \left[\begin{array}{c}\sqrt{\begin{array}{l}1 - (1-0.64)^{0.1}(1-0.64)^{0.1}(1-0.36)^{0.2} \\ (1-0.25)^{0.2}(1-0.16)^{0.4}\end{array}}, \\ 1 - (0.5)^{0.1}(0.6)^{0.1}(0.3)^{0.2}(0.4)^{0.2}(0.3)^{0.4}\end{array}\right]$$
$$= [0.588, 1 - 0.643]$$

Theorem 7. Let $\Re_j = \left[\hbar_{\Re_j}, 1 - \partial_{\Re_j}\right]$ $(j = 1, 2, 3, ..., n)$ be a collection of PFVs and $\varpi = (\varpi_1, \varpi_2, \varpi_3, ..., \varpi_n)^T$ is the weighted vector of $\Re_j$ with $\varpi_j \in [0,1]$ and $\sum_{j=1}^{n} \varpi_j = 1$. Then we have the following.

(1) (Idempotency): If $\Re_j = \Re$, for all j, then

$$PFOWA_\varpi(\Re_1, \Re_2, \Re_3, ..., \Re_n) = \Re. \qquad (34)$$

(2) (Boundary):

$$\Re^- \le PFOWA_\varpi(\Re_1, \Re_2, \Re_2, ..., \Re_n) \le \Re^+, \qquad (35)$$

$$\Re^- = \left[\min_j\left(\hbar_{\Re_j}\right), 1 - \max_j\left(\partial_{\Re_j}\right)\right]$$

$$\Re^+ = \left[\max_j\left(\hbar_{\Re_j}\right), 1 - \min_j\left(\partial_{\Re_j}\right)\right]$$

(3) (Monotonicity): If $\hbar_{\Re_j} \le \hbar_{\Re_j^*}$ and $\partial_{\Re_j} \ge \partial_{\Re_j^*}$ for all j, then

$$PFOWA_\varpi(\Re_1, \Re_2, ..., \Re_n) \le PFOWA_\varpi(\Re_1^*, \Re_2^*, ..., \Re_n^*), \quad (36)$$

Proof: The proof is similar to the Theorem 5.

Theorem 8. Let $\Re_j = \left[\Im_{\Re_j}, 1 - \wp_{\Re_j}\right]$ $(j = 1, 2, ..., n)$ be a group of *PFVs* and $\ell = (\ell_1, \ell_2, \ell_3, ..., \ell_n)^T$ is the weighted vector of $\Re_j$, such that $\ell_j \in [0,1]$ and $\sum_{j=1}^{n} \ell_j = 1$. then

(1) If $\ell = (1, 0, 0, ..., 0)^T$, then

$$PFOWA_\ell(\Re_1, \Re_2, \Re_3, ..., \Re_n) = \max_j \{\Re_j\},$$

(2) If $\ell = (0, 0, 0, ..., 1)^T$, then

$$PFOWA_\ell(\Re_1, \Re_2, \Re_3, ..., \Re_n) = \min_j \{\Re_j\},$$

(3) If $\ell_j = 1$ and $\ell_i = 0 (i \ne j)$, then

$$PFOWA_\ell(\Re_1, \Re_2, \Re_3, ..., \Re_n) = \Re_{\tau(j)},$$

where $\Re_{\tau(j)}$ is the jth greatest of $\Re_i$.

Proof: Straightforward.

5. Conclusion

In this work we have familiarized the idea of PFWA operator and PFOWA operator and also discussed some of their basic properties.

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