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A Note on Some Equivalences of Operators and Topology of Invariant Subspaces

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Abstract: In this paper we investigate the invariant and hyperinvariant subspace lattices of some operators. We give a lattice-theoretic description of the lattice of hyperinvariant subspaces of an operator in terms of its lattice of invariant subspaces. We also study the structure of these lattices for operators in certain equivalence classes of some equivalence relations.

Keywords: Invariant Subspace, Reducing Subspace, Hyperinvariant, Hyper-Reducing, Commutant, Bicommutant, Reducible, Irreducible Operator

1. Introduction

In this paper H will denote a complex separable Hilbert space and $B(H)$ will denote the Banach algebra of bounded linear operators. If $T \in B(H)$, then T^* denotes the adjoint of T, while $Ker(T), Ran(T), \overline{M}$ and $M^\perp$ stands for the kernel of T, range of T, closure of M and orthogonal complement of a closed subspace M of H, respectively. Recall that an operator $T \in B(H)$ is normal if $T^*T = TT^*$, unitary if $T^*T = TT^* = I$, a projection(or idempotent) if $T^2 = T$, an orthogonal projection if $T^2 = T$ and $T^* = T$. An operator $T \in B(H)$ is said to be scalar if it is a scalar multiple of the identity operator. That is, if $T = \alpha I$, where $\alpha \in \mathbb{C}$ and I is the identity operator on H. An operator $T \in B(H)$ is compact if for each bounded set $M \subseteq H$, then the closure of the image $\overline{T(M)}$ is compact. This is equivalent to saying that $\overline{T(B(0,1))}$ is compact, where $B(0,1) = \{x \in H : \|x\| < 1\}$. An operator $T \in B(H)$ is polynomially compact if there exists a non-zero polynomial p such that $p(T)$ is compact.

Two operators $A \in B(H)$ and $B \in B(K)$ are said to be *similar* if there exists an invertible operator $N \in B(H,K)$ such that $NA = BN$ or equivalently $A = N^{-1}BN$, and are *unitarily equivalent* if there exists a unitary operator $U \in B_+(H,K)$ (Banach algebra of all invertible operators in $B(H)$) such that $UA = BU$ (i.e. $A = U^*BU$ equivalently, $A = U^{-1}BU$). An operator $X \in B(H,K)$ is a *quasiaffinity* or a *quasi-invertible* if it is injective and has dense range. Two operators $A \in B(H)$ and $B \in B(K)$ are said to be quasiaffine transforms of each other if there exists a quasiaffinity $X \in B(H,K)$ such that $XA = BX$.

Two operators $A \in B(H)$ and $B \in B(K)$ are *quasisimilar* if there exist quasiaffinities $X \in B(H,K)$ and $Y \in B(K,H)$ such that $XA = BX$ and $AY = YB$. (see [11], [14]).

For any operators $A, B \in B(H)$, we define $[A,B] = AB - BA$.

The *commutant* of $T \in B(H)$ denoted by $\{T\}'$ is the set of all operators that commute with T. That is, $\{T\}' = \{S \in B(H) : ST = TS\}$. (see [4]).

The *bicommutant* or *double commutant* of $T \in B(H)$ denoted by $\{T\}''$ is defined by

$$\{T\}'' = \left\{A \in B(H) : AS = SA, S \in \{T\}'\right\}.$$

It is clear that

$$\{T\}^{''} = \{p(T) : T \in B(H)\} = \bigcap_{S \in \{T\}^{'}} \{S\}^{'}.$$

A subspace $M \subseteq H$ is said to be *invariant* under $T \in B(H)$ if $TM \subseteq M$. In this case, we say that the subspace M is $T-$invariant.

A subspace $M \subseteq H$ is said to be a *reducing subspace* of $T \in B(H)$ if it is invariant under both T and T^* (equivalently, if both M and $M^{\perp}$ are invariant under T). For more details about invariant subspaces and the Invariant Subspace Problem (see [11], [14] and [17]).

A subspace $M \subseteq H$ is said to be a *hyperinvariant subspace* for $T \in B(H)$ if $SM \subseteq M$ for each $S \in \{T\}^{'}$. That is, it is invariant under every operator commuting with T.

A *lattice* is a partially ordered set in which every pair of elements has a least upper bound and a greatest lower bound.

By a *subspace lattice* on a Hilbert space H we mean a family of subspaces of H which is closed under the formation of arbitrary intersections and arbitrary linear spans and which contains the zero subspace $\{0\}$ and H .

The subspace lattice of all invariant, reducing and hyperinvariant subspaces of T is denoted by $Lat(T)$, $\operatorname{Re}d(T)$ and $HyperLat(T)$, respectively. Note that these lattices are complete, in the sense that intersections and closed linear spans of subspaces are also in these lattices. Since T commutes with itself, we have that

$Hyperlat(T) \subseteq Lat(T)$. (see [4] and [12]).

It is also true that

$$\operatorname{Re}d(T) \subseteq Lat(T).$$

We sometimes call the lattice of hyperinvariant subspaces an *hyperlattice*.

Let $A, B \in B(H)$. We say that an operator $T \in B(H)$ *intertwines* the pair (A, B) if $TA = BT$. If T intertwines both (A, B) and (B, A), then we say that T *doubly intertwines* A and B.

A quasiaffinity X is said to have the *hereditary property* with respect to an operator $T \in B(H)$ if $X \in \{T\}^{'}$ and $\overline{X(M)} = M$ for every $M \in Hyperlat(T)$. If T_1 and T_2 are quasisimilar and there exists an implementing pair (X, Y) of quasiaffinities such that XY has the hereditary property with respect to T_1 and YX has the hereditary property with respect to T_2, then we say that T_1 is hyper-quasisimilar to T_2. This is denoted by $T_1 \overset{h}{\approx} T_2$. The notion of hyper-quasisimilarity was introduced by C. Foias etal[2].

We note that hyper-quasisimilarity is strictly stronger than quasisimilarity (see [2], Proposition 2.7). In fact the following inclusion of operator equivalences is true.

$$Similar \subset Hyper-quasisimilar \subset Quasisimilar.$$

We call $a \subseteq B(H)$ a *subalgebra* of $B(H)$ if a is closed under scalar multiplication, addition and composition. If a is also closed under taking adjoint, we call it a **-subalgebra* of $B(H)$. If the identity operator I belongs to the subalgebra a , we say that a is a unital subalgebra of $B(H)$.

We denote by $W^*(T)$ the (unital) weakly closed von Neumann algebra generated by T. We will use this subalgebra to investigate the structures of invariant and hyperinvariant subspace lattices for some operators.

Two lattices L_1 and L_2 are said to be *isomorphic* (denoted by $L_1 \equiv L_2$) if there exists an isomorphism (bijective map) $\varphi : L_1 \to L_2$, and if $l_1 \le l_2$ if and only if $\varphi(l_1) \le \varphi(l_2)$ for $l_1, l_2 \in L_1$. (see [12]).

Let $T \in B(H)$. If M reduces every operator in the commutant of T, then we call M *hyper-reducing* subspace for T. We denote by $Hyper\operatorname{Re}d(T)$ the collection of all subspaces hyper-reducing for $T \in B(H)$. Clearly

$$Hyper\operatorname{Re}d(T) \subseteq Lat(T).$$

The concept of hyper-reducibility of a subspace of a Hilbert space was introduced by Moore[15]. We will prove in Section 5 that

$$Hyper\operatorname{Re}d(T) = \operatorname{Re}d(\{T\}^{'}) = Lat(\{T\}^{'}) \cap Lat(\{T^*\}^{'}).$$

An operator $T \in B(H)$ is said to be *reducible* if it has a nontrivial reducing subspace (equivalently, if it has a proper nonzero direct summand-that is, if there exists a subspace M of H such that M and $M^{\perp}$ are nonzero and $T-$invariant)(see [14]). This is equivalent to saying that if M is nontrivial and invariant under T and T^*. A subspace that is not reducible is said to be *irreducible*. This means that an operator is irreducible if it has no reducing subspace other than $\{0\}$ and H. It has been shown in [8] that an operator $T \in B(H)$ is reducible if and only if there exists a non-scalar operator L such that $LT = TL$ and $T^*L = LT^*$. That is if and only if there exists a non-scalar operator $L \in \{T\}^{'} \cap \{T^*\}^{'}$. Equivalently, T is reducible if and only if both T and T^* lie in $\{L\}^{'}$ for some non-scalar operator L.

We denote by $\left\langle \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle$ the span of vector $\begin{pmatrix} x \\ y \end{pmatrix}$.

2. Main Results

Recall that the commutant $\{a\}^{'}$ of a *-subalgebra $a \subseteq B(H)$ is the set

$$\{a\}' = \{B \in B(H) : BA = AB, \forall A \in a\}.$$

Clearly $\{a\}'$ is weakly closed. Since the weak operator topology is weaker than the strong operator topology, it is also clear that $\{a\}'$ is always strongly closed.

Theorem 2.1 If $T \in B(H)$ then the following statements hold.

(i). $\{T\}'$ is a subalgebra of $B(H)$.

(ii). $\{T\}''$ is a commutative subalgebra of $B(H)$.

(iii). $\{T\}'' = \{\{T\}'\}'$.

We need the following result which verifies a useful property of unital self-adjoint subalgebras of $B(H)$.

Lemma 2.2 Suppose a is a self-adjoint subalgebra of $B(H)$ and let M be a closed subspace of H. Then following statements are equivalent.

(i). $a(M) \subseteq M$.

(ii). $a(M^{\perp}) \subseteq M^{\perp}$.

(iii). $[a, P_M] = 0$, where P_M denotes the orthogonal projection of H onto M.

Proof. (i) $\Rightarrow$ (ii): Suppose $a(M) \subseteq M$. That is $Ay \in M$ for all $A \in a$ and $y \in M$. Let $x \in M^{\perp}$. Then

$$\langle y, Ax \rangle = \langle A^* y, x \rangle = 0.$$

Since $A^* \in a$ we have that $Ax \in M^{\perp}$, so $a(M^{\perp}) \subseteq M^{\perp}$.

(ii) $\Rightarrow$ (i): $a(M^{\perp}) \subseteq M^{\perp}$ implies that $a(M) \subseteq M$ follows from the fact that $M = M^{\perp\perp}$.

(i) $\Rightarrow$ (iii): Suppose that $a(M) \subseteq M$ and let $A \in a$ and $x \in H$. Then

$$\begin{aligned} A(P_M x - P_M(Ax)) &= A(P_M x - P_M(A(P_M x + P_{M^{\perp}} x))) \\ &= A(P_M x - P_M(A(P_M x)) \\ &= 0, \end{aligned}$$

which shows that $[A, P_M] = 0$ for all $A \in a$. Thus $[a, P_M] = 0$.

(iii) $\Rightarrow$ (ii): Suppose that $[a, P_M] = 0$, and let $x \in M$ and $y \in M^{\perp}$. Then

$$\langle Ay, x \rangle = \langle Ay, P_M x \rangle = \langle P_M Ay, x \rangle = \langle AP_M y, x \rangle = 0,$$

for all $A \in a$ and hence $A(M^{\perp}) \subseteq M^{\perp}$, from where we conclude that $a(M^{\perp}) \subseteq M^{\perp}$.

We note that a subspace M with either of the properties in Lemma 2.2 is called *reducing* (with respect to the subalgebra a).

We use Lemma 2.2 to state the Bicommutant/Double Commutant Theorem.

Theorem 2.3(von Neumann Double Commutant Theorem) Let H be a Hilbert space and $a \subseteq B(H)$ be a unital self-adjoint *-subalgebra of $B(H)$. Then the following conditions are equivalent.

(i). $a = \{a\}''$.

(ii). a is closed with respect to the weak operator topology (WOT) on $B(H)$.

(iii). a is closed with respect to the strong operator topology (SOT) on $B(H)$.

If a unital (self-adjoint) *-subalgebra a of $B(H)$ satisfies either of the three equivalent conditions in Theorem 2.3, we say that it is a *von Neumann algebra*.

The Double Commutant Theorem simply asserts that the double commutant $\{a\}''$ of a unital self-adjoint subalgebra a of $B(H)$ is always strongly closed (and hence weakly closed). That is, a is strongly (and hence weakly) dense in $\{a\}''$. Equivalently, it says that the strongly closed unital self-adjoint subalgebras of $B(H)$ are always their own double commutant.

For convenience, we take a von Neumann algebra as a *-subalgebra a of $B(H)$ satisfying $a = \{a\}''$. A von Neumann algebra is a unital, weakly closed and contains an abundance of projections. If a is a von Neumann algebra, then a is generated by the projections in a.

Theorem 2.4 [12, Corollary 3.2.1] Let $T, S \in B(H)$. If $Lat(T) = Lat(S)$, then $Hyperlat(T) = Hyperlat(S)$.

Proof. This follows easily from the definition.

Question 1. Does the condition that $Lat(T) = Lat(S)$ imply that $\{T\}' = \{S\}'$?

Question 2. Does the condition that $\{T\}' = \{S\}'$ imply that

$$Hyperlat(T) = Hyperlat(S)?$$

We note that the converse of Theorem 2.4 need not hold in general. To see this, let $T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $S = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ on the Hilbert space $H = \mathbb{R}^2$. A simple computation shows that

$$\begin{aligned} Lat(T) &= \left\{ \{0\}, \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle, \mathbb{R}^2 \right\} \\ &\neq \left\{ \{0\}, \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle, \left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle, \mathbb{R}^2 \right\} = Lat(S). \end{aligned}$$

Another computation shows that $\{T\}' = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{11} \end{bmatrix} : a_{11}, a_{12} \in \mathbb{R} \right\}$

and

$$\{S\}^{'} = \left\{ \begin{bmatrix} b_{11} & 0 \\ 0 & b_{22} \end{bmatrix} : b_{11}, b_{22} \in \mathbb{R} \right\}.$$

We note that $\{T\}^{'} \cap \{S\}^{'} = \left\{ \begin{bmatrix} a_{11} & 0 \\ 0 & a_{11} \end{bmatrix} : a_{11} \in \mathbb{R} \right\}$, which is the set of scalar operators. Clearly the commutant of T consists of operators similar to scalar operators. This result is true for isometries and co-isometries. Another computation shows that

$$Hyperlat(T) = \left\{ \{0\}, \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle, \mathbb{R}^2 \right\}$$

$$\neq \left\{ \{0\}, \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle, \left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle, \mathbb{R}^2 \right\} = Hyperlat(S).$$

However, it is clear that the subspace $M = \left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle \in Hyperlat(T)$ if $a_{12} = 0$.

This happens if and only if $\{T\}^{'} = \{\alpha I : \alpha \in \mathbb{C}\}$. This extra condition then implies that

$$Hyperlat(T) = Hyperlat(S).$$

Theorem 2.4 can be relaxed as follows.

Corollary 2.5 [12, Corollary 3.2.2] Let $T, S \in B(H)$. If $Lat(T) \equiv Lat(S)$, then $Hyperlat(T) \equiv Hyperlat(S)$.

Question 3. When is the converse of Corollary 2.5 true?

Let $T \in B(H)$. We define $W^*(T)$ to be the von Neumann algebra generated by $\{I, T\}$. Note that $W^*(T) = \{T\}^{''} \cup \{\alpha I : \alpha \in \mathbb{C}\}$.

From the Double Commutant Theorem, if $T = T^*$, then $\{T\}^{''} = W^*(T)$ and $\{T\}^{'}$ is a von Neumann algebra and is therefore generated by its projections. Since the projections in $\{T\}^{'}$ are also in $\{T^*\}^{'}$, it follows that the Double Commutant Theorem has the following reformulation.

$W^*(T) = \{T : PT = TP$, for every projection $P \in \{T\}^{'}\}$.

Corollary 2.6 Let $T \in B(H)$. Then $Lat(T) = Lat(W^*(T))$.

Proof. Since $T \in W^*(T)$, trivially $Lat(W^*(T)) \subseteq Lat(T)$. On the other hand, $W^*(T)$ consists of polynomials in I and T, and hence $Lat(T) \subseteq Lat(W^*(T))$. Combining these two inclusions, equality follows. This proves the claim.

Corollary 2.7 Let $T \in B(H)$. Then $Hyperlat(T) = Lat(\{T\}^{'})$.

Theorem 2.8 Let $T \in B(H)$ and $M \in Hyperlat(T)$. Then the orthogonal projection P_M of H onto M belongs to $W^*(T)$.

Proof. By Theorem 2.3(Double Commutant Theorem), it suffices to show that if

$$Q = Q^2 = Q^* \in \{W^*(T)\}^{'} = \{T\}^{'} \cap \{T^*\}^{'}$$

$$= \{L \in B(H) : LT = TL\} \cap \{L \in B(H) : T^*L = LT^*\}$$

$$= \{L \in B(H) : LT = TL \text{ and } T^*L = LT^*\},$$

then $[P_M, Q] = 0$, which says that $QM \subseteq M$. Since $Q \in \{T\}^{'}$ and $M \in Hyperlat(T)$, we have that $P_M \in W^*(T)$.

Remark. Let $T \in B(H)$. We call the set $\{W^*(T)\}^{'}$ the *-commutant of T and the set $\{W^*(T)\}^{''} = \left\{\{W^*(T)\}^{'}\right\}^{'}$ the *-bicommutant of T.

Theorem 2.8 helps us prove the following result.

Theorem 2.9 Let $A, B \in B(H)$. If $A \in W^*(B)$, then $Lat(B) \subseteq Lat(A)$.

Proof. We know that $Hyperlat(T) \subseteq Lat(T)$ for any $T \in B(H)$ since T commutes with itself (see [4]), that is, $T \in \{T\}^{'}$. Since $A \in W^*(B)$, we have that $QP_M = P_M Q$, where $Q \in \{W^*(B)\}^{'} = \{B\}^{'} \cap \{B^*\}^{'}$ is an orthogonal projection in $\{B\}^{'}$ and $M \in Hyperlat(B)$, and hence $P_M A P_M = P_M A$, where $P_M \in W^*(A)$ is an orthogonal projection of H onto M. This means that

$$M \in Hyperlat(B) \subseteq Lat(B) \Rightarrow M \in Lat(A).$$

Thus, $M \in Lat(B) \Rightarrow M \in Lat(A)$. This proves the claim.

The converse of Theorem 2.9 is not true in general. However, if in addition $AB = BA$, then the converse is true.

Corollary 2.10 Let $A, B \in B(H)$. If $A \in W^*(B)$, then $Hyperlat(B) \subseteq Hyperlat(A)$.

Proof. This follows from the proof of Theorem 2.9 and the fact that $Hyperlat(T) \subseteq Lat(T)$ for any $T \in B(H)$.

Note that if $A \in \{T\}^{'}$, then $\overline{Ran(A)}$, $Ker(A) \in Lat(T)$.

In addition, if $B \in \{T\}^{''}$, then $\overline{Ran(A)}$, $Ker(A) \in Hyperlat(T)$.

A lattice L of subspaces of a Hilbert space H is said to be trivial if $L = \{\{0\}, H\}$.

Proposition 2.11[12] Let $A, B, T \in B(H)$. If T doubly intertwines A and B and $Lat(A) \cap Lat(B)$ is trivial, then either $T = 0$ or T is a quasiaffinity.

Proof. Suppose T doubly intertwines the pair (A, B). Then

$TA = BT$ and $TB = AT$. Since $TA = BT$, we have $\overline{Ran(T)} \in Lat(B)$ and $Ker(T) \in Lat(A)$. Since $TB = AT$, we deduce that

$$\overline{Ran(T)} \in Lat(A) \bigcap Lat(B)$$

and

$$Ker(T) \in Lat(A) \bigcap Lat(B).$$

If $\overline{Ran(T)} = \{0\}$, then $T = 0$. If $\overline{Ran(T)} = H$, then $Ker(T) = \{0\}$ and hence T is injective with dense range, and therefore a quasiaffinity.

Proposition 2.11 can be strengthened into the following results.

Corollary 2.12 Let $A, B, T \in B(H)$. If T commutes with A and B and $Lat(A) \bigcap Lat(B)$ is trivial, then either $T = 0$ or T is a quasiaffinity.

Proof. If T commutes with A and B, then $TA = AT$ and $TB = BT$. Using Proposition 2.8, we have

$$\overline{Ran(T)} \in Lat(A) \bigcap Lat(B)$$

and

$$Ker(T) \in Lat(A) \bigcap Lat(B).$$

By the same argument then either $T = 0$ or T is a quasiaffinity.

Corollary 2.13 Let $A, T \in B(H)$. If T commutes with A and $Lat(A)$ is trivial, then either $T = 0$ or T is a quasiaffinity.

Proof. If T commutes with A then $TA = AT$. Using Corollary 2.9, we have $\overline{Ran(T)} \in Lat(A)$ and $Ker(T) \in Lat(A)$. Triviality of $Lat(A)$ then implies that $\overline{Ran(T)} = \{0\}$, and thus $T = 0$. If $\overline{Ran(T)} = H$, then $Ker(T) = \{0\}$, which proves that T is a quasiaffnity.

3. Invariant Subspace Lattice Operations

Theorem 3.1 Let $T \in B(H)$. Then $M \in Lat(T)$ if and only if $M^{\perp} \in Lat(T^*)$.

Clearly the map $\varphi : M \to M^{\perp}$ of $Lat(T)$ into $Lat(T^*)$ is a (lattice) isomorphism. This map "inverts" the lattice operations:

$$\vee_{\alpha} M_{\alpha} \mapsto \wedge_{\alpha} M_{\alpha}{}^{\perp}$$

and

$$\wedge_{\alpha} M_{\alpha} \mapsto \vee_{\alpha} M_{\alpha}{}^{\perp}.$$

4. Operator Equivalences and Lattices

Recall that if L_1 and L_2 are lattices of subspaces of a Hilbert space H, an isomorphism $\varphi : L_1 \to L_2$ is a one-to-one and onto map with the property that if $M_1, M_2 \in L_1$ then $M_1 \subseteq M_2$ if and only if $\varphi(M_1) \subseteq \varphi(M_2)$.

In this paper, lattice refers to either $Lat(T), Hyperlat(T)$, or $\mathrm{Re}\,d(T)$.

In this section we investigate lattices of operators in some equivalence classes emanating from certain operator equivalence relations.

Theorem 4.1 Similarity of operators preserves non-trivial invariant and non-trivial hyperinvariant subspaces.

Proof. We prove the case for invariance. The proof for hyperinvariance can be proved similarly. Suppose $A, B \in B(H)$ are such that $A = X^{-1}BX$. That is, $XA = BX$. Suppose M is a non-trivial $A-$ invariant subspace. Then $BXM = XAM \subseteq XM$.

Since M is non-trivial and X is invertible, we conclude that XM is a non-trivial invariant subspace for B. Thus M is $A-$ invariant if and only if M is $B-$ invariant.

It has been proved (see [5], [6]) that if A and B are quasisimilar and one has a nontrivial hyperinvariant subspace, then so does the other. However, similar (quasisimilar) operators need not have isomorphic invariant (hyperinvariant) lattices. An example is given in Herrero[5] of two quasisimilar nilpotent operators of the same order but with non-isomorphic hyperlattices. This shows that structure of the hyperlattice of an operator is not preserved under quasisimilarity.

Example 1. Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. A simple computation shows that A and B are similar. However, another computation shows that $Lat(A) = \left\{ \{0\}, \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle, \mathbb{R}^2 \right\}$ and

$$Lat(B) = \left\{ \{0\}, \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle, \left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle, \mathbb{R}^2 \right\}.$$

Clearly $Lat(A)$ and $Lat(B)$ are not isomorphic.

Theorem 4.2 If $T_1, T_2 \in B(H)$ are quasisimilar with quasiaffinities X and Y, then $XY \in \{T_1\}'$ and $YX \in \{T_2\}'$.

Proof. Suppose $T_1X = XT_2$ and $T_2Y = YT_1$, where X and Y are quasiaffinities. Post-multiplying the first equation by Y and using the second equation, we have

$$T_1XY = XT_2Y = XYT_1,$$

which proves that $XY \in \{T_1\}'$. Post-multiplying the second equation by X and using the first equation we have

$$T_2YX = YT_1X = YXT_2,$$

which proves that $YX \in \{T_2\}'$.

Theorem 4.3 Hyper-quasi-similarity is an equivalence relation.

Proof. Suppose $T_1, T_2 \in B(H)$ are quasisimilar and there exists an implementing pair (X, Y) of quasiaffinities such that XY has the hereditary property with respect to T_1 and YX has the hereditary property with respect to T_2. Let $M_1 \in Hyperlat(T_1)$ and $M_2 \in Hyperlat(T_2)$. Then

$XT_1 = T_2X, T_1Y = YT_2$ and $\overline{YXM_1} = M_1$ and $\overline{XYM_2} = M_2$.

Without loss of generality, if we let $T_1 = T_2$, then we have $XT_1 = T_1X, T_1Y = YT_1$ and $\overline{YXM_1} = M_1$ and $\overline{XYM_2} = M_2$. From Theorem 4.2, we know that both XY and YX are quasiaffinities and $XY, YX \in \{T_1\}'$. Thus, $\overline{YXM_1} = M_1$ and $\overline{XYM_2} = M_2$, where $M_1, M_2 \in Hyperlat(T_1)$. This proves reflexivity of $\overset{h}{\approx}$.

Now suppose $T_1 \overset{h}{\approx} T_2$. Then by re-writing the definition above, we have $YT_2 = T_1Y, T_2X = XT_1$ and $\overline{XYM_2} = M_2$ and $\overline{YXM_1} = M_1$, where $M_1 \in Hyperlat(T_1)$ and $M_2 \in Hyperlat(T_2)$. Thus $T_2 \overset{h}{\approx} T_1$. This proves the symmetry property of $\overset{h}{\approx}$.

Now suppose $T_1 \overset{h}{\approx} T_2$ (with implementations as above) and suppose also that $T_2 \overset{h}{\approx} T_3$, for some $T_3 \in B(H)$. Then there exists an implementing pair of quasiaffinities (Z, S) such that $ZT_2 = T_3Z, T_2S = ST_3$ and $\overline{SZM_2} = M_2$ and $\overline{ZSM_3} = M_3$, where $M_2 \in Hyperlat(T_2)$ and $M_3 \in Hyperlat(T_3)$. A simple computation shows that

$$ZXT_1 = T_3ZX, SYT_1 = T_3SY,$$
$$T_1XZ = XZT_3, T_1YS = YST_3,$$
$$ZSXY \in \{T_2\}', YSZX \in \{T_1\}',$$
$$XYSZ \in \{T_2\}', ZXYS \in \{T_3\}'.$$

This is equivalent to

$$\overline{YSZXM_1} = M_1$$

And

$$\overline{ZXYSM_3} = M_3,\ M_1 \in Hyperlat(T_2)$$

and $M_3 \in Hyperlat(T_3)$. This proves that $T_1 \overset{h}{\approx} T_3$, and hence $\overset{h}{\approx}$ is transitive.

Theorem 4.4 ([8], Corollary 4.8) Hyper-quasi-similarity preserves nontrivial hyperinvariant invariant subspaces.

Proof. This follows easily from the fact that hyper-similarity is stronger than quasisimilarity and the fact that quasisimilarity preserves non-trivial hyperinvariant subspaces.

Kubrusly in [8] has shown that non-scalar normal operators have non-trivial hyperinvariant subspaces. Thus Theorem 4.4 ensures that an operator quasisimilar to a non-scalar normal operator has a non-trivial hyperinvariant subspace.

The following result is a strengthening of ([8], Lemma 4.7) and lends credence to Theorem 4.4.

Theorem 4.5 Let $T \in B(H)$ be quasisimilar to a unitary operator $U \in B(K)$ and let $M \subseteq K$. If $M \in Hyperlat(T)$ then $M \in Hyperlat(U)$.

Proof. Suppose $TX = XU$ and $UY = YT$, where $X \in B(K, H), Y \in B(H, K)$ are quasiaffinities. If $A \in \{T\}'$, then

$$\begin{aligned} U(YAX) &= (UY)(AX) = (YT)(AX) \\ &= Y(TA)X \\ &= (YA)(TX) \\ &= (YA)(XU) \\ &= (YAX)U. \end{aligned}$$

This proves that $YAX \in \{U\}'$ for every $A \in \{T\}'$.

Using the computation, we conclude that M is invariant for YAX, and hence for any operator that commutes with U. This proves the claim.

Corollary 4.6 If $T \in B(H)$ is quasisimilar to a unitary operator $U \in B(K)$ then $Hyperlat(T) \subseteq \mathrm{Re}\,d(U)$.

Proof. Follows from the fact that for a unitary operator U, $Hyperlat(U) = \mathrm{Re}\,d(U)$.

Theorem 4.7 Let $T, A \in B(H)$ such that $TA = AT$. If $M \in Hyperlat(T)$ then $M \in \mathrm{Re}\,d(T)$.

Proof. From the hypothesis and definition, it follows that $M \in Hperlat(A) \subseteq Lat(A)$, and hence M reduces $\{A\}'$. In particular M reduces T.

Theorem 4.8 Let $A \in B(H)$ and $B \in B(K)$ be self-adjoint operators. If there exists a quasiaffinity $X \in B(H, K)$ such that $XA = BX$, then A and B are unitarily equivalent.

Theorem 4.8 has been extended to the class of normal operators as a consequence of ([9], Corollary 6.50).

The following result gives a condition when some subspace lattices for two operators are isomorphic.

Theorem 4.9 Suppose that $T \in B(H)$, where H is a finite dimensional Hilbert space and $\varphi: B(H) \to B(H)$ defined by $T \to \varphi(T)$ is a linear map. Then the following statements are equivalent.

(i). $Lat(T) \equiv Lat(\varphi(T))$.

(ii). $Hyperlat(T) \equiv Hyperlat(\varphi(T))$.

(iii). $\mathrm{Re}d(T) \equiv \mathrm{Re}d(\varphi(T))$.

From Theorem 4.9, we can conclude that

$$Lat(T) \equiv Lat(cT),\ Hyperlat(T) \equiv Hyperlat(cT)$$

and

$$\mathrm{Re}d(T) \equiv \mathrm{Re}d(cT),$$

where $0 \neq c \in \mathbb{C}$.

5. Reducibility and Subspace Lattices

It is clear that reducing subspaces are generally easier to treat that arbitrary invariant subspaces.

Theorem 5.1 A subspace M reduces an operator T if and only if $M \in Lat(T) \cap Lat(T^*)$.

Proof. Follows easily from the definition.

Corollary 5.2 Let $T \in B(H)$ and M be a subspace of H. The following statements are equivalent.

(i). M reduces T.

(ii). $M \in Lat(T) \cap Lat(T^*)$.

(iii). $P_M \in \{T\}'$, where P_M is the orthogonal projection of H onto M.

Theorem 5.3 If $T_1 \in B(H_1)$ and $T_2 \in B(H_2)$ are irreducible, then every operator $A \in B(H_1, H_2)$ that intertwines them is either zero or identity.

Theorem 5.4 If an operator A commutes with an irreducible operator T, then A is similar to a scalar operator.

Theorem 5.5 If $T \in B(H)$ is nilpotent of nil-index n, then $\mathrm{Re}d(T) = \{\{0\}, H\}$.

Corollary 5.6 Let $T \in B(H)$. If $\mathrm{Re}d(T) = \{\{0\}, H\}$, then $T = \alpha I + S$, where S is a nilpotent operator.

Bercovici etal[1] have proved that for a nilpotent operator $T \in B(H)$ such that $T^n = 0$, for some integer $n \geq 1$, $Hyperlat(T)$ is generated by the spaces $Ker(T^m)$ and $Ran(T^m), m = 0, 1, 2, 3, \cdots, n$. They have also shown that $Ran(T^{n-1})$ is the smallest non-trivial hyperinvariant subspace and that $Ker(T^{n-1})$ is the largest non-trivial hyperinvariant subspace.

We show the relationship between $Hyperlat(T)$ and $\mathrm{Re}d(T)$, where T is a unitary operator.

Theorem 5.7 Let $T \in B(H)$ be a unitary operator. A subspace $M \subseteq H$ is hyperinvariant for T if and only if M reduces T.

Proof. Suppose that $M \in Hyperlat(T)$ and let P_M be the orthogonal projection of H onto M. Then $AP_M = P_M A P_M$ for every $A \in \{T\}'$. Since T is unitary and hence normal, by Fuglede's theorem, $A^* \in \{T\}'$. Thus $A^* P_M = P_M A^*$ and hence $AP_M = P_M A P_M = P_M A$. By Corollary 5.2, we have that M reduces T. Conversely, suppose M reduces T. Without loss of generality, suppose $AP_M = P_M A$. Then

$$AM = AP_M H = P_M AH \subseteq P_M H = M.$$

This shows that M is invariant under A.

So, if $AP_M = P_M A$ for all $A \in \{T\}'$, then M is hyperinvariant for T.

Remark. Theorem 5.7 says that for a unitary operator T, $Hyperlat(T) = \mathrm{Re}d(T)$.

Theorem 5.7 can be relaxed as follows.

Theorem 5.8 Let $T \in B(H)$ be an isometry. If $M \subseteq H$ is such that $TM = M$ then M reduces T.

Proof. If $TM = M$ then $T^* M = T^* TM = M$. This proves the claim.

Corollary 5.9 Let $T \in B(H)$ be an isometry. If $M \subseteq H$ is such that $TM = M$ then $\mathrm{Re}d(T) = Lat(T)$.

Proof. This follows from Theorem 5.8 and the fact that $\mathrm{Re}d(T) \subseteq Lat(T)$, for any operator T.

Let $T \in B(H)$. If a subspace $M \subseteq H$ reduces every operator in the commutant of T, then we say that M is a *hyper-reducing* subspace for T.

Example 2. A unilateral shift $S: \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ of multiplicity one is irreducible and so is its two-dimensional analogue. The operator

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

is the two-dimensional analogue of the unilateral shift operator of multiplicity one. We note that $Lat(S) = \left\{\{0\}, \left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle, \mathbb{R}^2 \right\}$. A simple computation shows that $M = \left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle \in Hyperlat(S)$ and that $M \notin \mathrm{Re}d(S)$. So S has no non-trivial reducing subspace and hence it is irreducible. Note also, that $Lat(S) \neq Lat(S^*)$ but $Lat(S) \equiv Lat(S^*)$.

The operator

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} = 1 \oplus \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

is reducible in $H=\mathbb{R}^3$.

Clearly, $T\in B(H)$ is irreducible if its commutant $\{T\}'=\{A\in B(H):TA=AT\}$ contains no projections other 0 and the identity I on H. That is, if there is no non-trivial orthogonal projection commuting with T. This is equivalent to saying that $\mathrm{Re}d(T)=\{\{0\},H\}$. The class of irreducible operators is huge. In fact the class of irreducible operators is dense in $B(H)$ in the norm topology.

An operator $T\in B(H)$ is *reductive* if all its invariant subspaces reduce it.

There are several equivalent ways to characterize a reductive operator.

Note that $\mathrm{Re}d(T)=\mathrm{Re}d(T^*)$, for any operator $T\in B(H)$.

Corollary 5.10 Let $T\in B(H)$. If $Lat(T)\subseteq Lat(T^*)$, then T is reductive.

Corollary 5.11 An operator $T\in B(H)$ is reductive if and only if $Lat(T)=\mathrm{Re}d(T^*)$.

Proof. By definition, if T is reductive, then $Lat(T)\subseteq \mathrm{Re}d(T)=\mathrm{Re}d(T^*)$. But the inclusion $\mathrm{Re}d(T)\subseteq Lat(T)$ is obvious. Combining these statements, we have equality. Conversely, suppose that

$$Lat(T)=\mathrm{Re}d(T^*).$$

Then

$$Lat(T)=\mathrm{Re}d(T^*)$$
$$=Lat(T^*)\bigcap Lat(T)$$
$$\subseteq Lat(T^*).$$

Thus $Lat(T)\subseteq Lat(T^*)$.

The class of reducible operators contains the class of reductive operators. However, an operator may be reducible but fail to be reductive. Thus,

$$\mathrm{Re}ductive\subset \mathrm{Re}ducible.$$

Note that every self-adjoint (and by extension, normal operator on a finite dimensional Hilbert space) is reductive. It is also known that every compact normal operator is reductive. It is a known fact that every operator that commutes with a non-scalar normal operator is reducible.

In fact for a normal operator $T\in B(H)$, we have that

$$Lat(T)\equiv Lat(T^*).$$

Example 3. Let $T=\begin{bmatrix}0&1&0\\0&0&0\\0&0&1\end{bmatrix}$.

Then $T=\begin{bmatrix}0&1\\0&0\end{bmatrix}\oplus 1$

and hence T is reducible. A simple computation shows that

$$Lat(T)=\left\{\{0\},\left\langle\begin{pmatrix}1\\0\\0\end{pmatrix}\right\rangle,\left\langle\begin{pmatrix}0\\0\\1\end{pmatrix}\right\rangle,\left\langle\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right\rangle,\mathbb{R}^3\right\}$$

while

$$Lat(T^*)=\left\{\{0\},\left\langle\begin{pmatrix}1\\0\\0\end{pmatrix}\right\rangle,\left\langle\begin{pmatrix}0\\0\\1\end{pmatrix}\right\rangle,\mathbb{R}^3\right\}.$$

Thus $Lat(T)\neq Lat(T^*)$. Therefore T is not reductive.

A simple calculation shows that T is not a normal operator.

Clearly,

$$Normal\subset \mathrm{Re}ductive\subset \mathrm{Re}ducible.$$

The above inclusion is strict. For instance it has been shown in [12] that not every reductive operator in normal. Moore[16] went further and gave some conditions under which a reductive operator is normal: that such a reductive operator T must commute with an injective compact operator or T is polynomially compact or T is expressible as a sum of a normal operator and a commuting compact operator (see [16], Theorem 1 and Corollary 2).

Example 4. The bilateral shift B on $\ell^2(\mathbb{Z})$ defined by $B(\cdots,x_{-2},x_{-1},[x_0],x_1,x_2,\cdots)=(\cdots,x_{-2},x_{-1},x_0,x_1,x_2,\cdots)$, where $x=(\cdots,x_{-2},x_{-1},[x_0],x_1,x_2,\cdots)\in\ell^2(\mathbb{Z})$ and $[x_0]$ denotes the 0-th coordinate of x, is not reductive.

Indeed,

$$M=\{x\in\ell^2(\mathbb{Z}):x_n=0,\text{ if } if\ n<0\}\in Lat(B)\text{ but}$$

$$M\notin Lat(B^*).$$

Theorem 5.12 A reductive operator is normal if and only if it has a non-trivial invariant subspace.

Theorem 5.13 Let $T\in B(H)$. If a subspace $M\subseteq H$ is hyper-reducing then $M\in Lat(\{T\}')\bigcap Lat(\{T^*\}')$.

Example 5. Let $A=\begin{bmatrix}1&0\\0&1\end{bmatrix}$ and $B=\begin{bmatrix}1&1\\0&1\end{bmatrix}$.

Clearly these two operators are not similar. A simple computation shows

that $Lat(A)=\left\{\{0\},\left\langle\begin{pmatrix}1\\0\end{pmatrix}\right\rangle,\left\langle\begin{pmatrix}0\\1\end{pmatrix}\right\rangle,\mathbb{R}^2\right\}=\mathrm{Re}d(A)$,

and

$$Lat(B)=\left\{\{0\},\left\langle\begin{pmatrix}1\\0\end{pmatrix}\right\rangle,\mathbb{R}^2\right\}\neq\left\{\{0\},\mathbb{R}^2\right\}=\operatorname{Re}d(A).$$

Thus A is reductive while B is not since not every invariant subspace of B reduces B. Another computation shows that

$$\{B\}^{'}=\left\{X:X=\begin{bmatrix}\alpha & \beta\\ 0 & \alpha\end{bmatrix},\alpha,\beta\in\mathbb{R}\right\}$$

and

$$\{A\}^{'}=\left\{Y:Y=\begin{bmatrix}\alpha & \beta\\ \gamma & \lambda\end{bmatrix},\alpha,\beta,\gamma,\lambda\in\mathbb{R}\right\},$$

hence

$$Hyperlat(A)=\left\{\{0\},\mathbb{R}^2\right\}$$

and

$$Hyperlat(B)=Lat(B).$$

Theorem 5.14 ([13], Theorem H) If A is a reductive operator then A can be written as a direct sum $A=A_1\oplus A_2$ where A_1 is normal, A_2 is reductive, $\{A\}^{'}=\{A_1\}^{'}\oplus\{A_2\}^{'}$, and all the invariant subspaces of A_2 are hyperinvariant.

Corollary 5.15 [6] Suppose A is a reductive operator such that $A=A_1\oplus A_2$. Then

$$Hyperlat(A)=Hyperlat(A_1)\oplus Hyperlat(A_2)$$

and

$$Lat(A)=Hyperlat(A).$$

From Theorem 5.14 and Corollary 5.15 we conclude that if A is reductive and completely non-normal (that is, A has no normal direct summand) then $Lat(A)=Lat(\{A\}^{'})$.

Theorem 5.16 ([13], Corollary 1) If A is a reductive operator, then every hyperinvariant subspace of A is hyper-reducing.

Corollary 5.17 If A is a reductive operator, then $Hyperlat(A)\subseteq Hyper\operatorname{Re}d(A)$.

Corollary 5.17 says that if A is reductive then $Lat(\{A\}^{'})=Lat(\{A^*\}^{'})$.

Theorem 5.18 Let $T\in B(H)$. Then

$$Hyper\operatorname{Re}d(T)=Lat(\{T\}^{'})\bigcap Lat(\{T^*\}^{'}).$$

Proof.

$$Hyper\operatorname{Re}d(T)=\left\{M\subseteq H:M\in\operatorname{Re}d(\{T\}^{'})\right\}$$

$$=\left\{M\subseteq H:SM\subseteq M,S^*M\subseteq M,S\in\{T\}^{'}\right\}$$

$$=\left\{M\subseteq H:M\in Lat(S)\bigcap Lat(S^*),S\in\{T\}^{'}\right\}$$

$$=\left\{M\subseteq H:M\in Lat(\{T\}^{'})\bigcap Lat(\{T^*\}^{'})\right\}$$

$$=Lat(\{T\}^{'})\bigcap Lat(\{T^*\}^{'}).$$

Theorem 5.19 Let $T\in B(H)$. Then

$$Hyper\operatorname{Re}d(T)=Hyperlat(T)\bigcap Hyperlat(T^*).$$

Proof. The proof follows from Theorem 5.18 and the fact that $Lat(\{T\}^{'})=Hyperlat(T)$ and $Lat(\{T^*\}^{'})=Hyperlat(T^*)$, for any $T\in B(H)$.

Corollary 5.20 Let $T\in B(H)$ be self-adjoint. Then

$$Hyper\operatorname{Re}d(T)=Hyperlat(T).$$

Proof. The proof follows easily from Theorem 5.19 and the fact that self-adjointness of T. The proof also follows from Theorem 5.18, the self-adjointness of T and the fact that $Lat(\{T\}^{'})=Hyperlat(T)$.

Proposition 5.21([1], Proposition 2.2) Let $T\in B(H)$ be normal. Then $Hyperlat(T)=\left\{M\subseteq H:P_M\in W^*(T)\right\}$.

Corollary 5.22 Let $T\in B(H)$ be normal. Then every hyperinvariant subspace of T is hyperinvariant for T^*.

Corollary 5.22 says $Hyperlat(T)\subseteq Hyperlat(T^*)$, for any normal operator $T\in B(H)$.

The converse of Corollary 5.22 is also true. This leads to the following result.

Corollary 5.23 Let $T\in B(H)$ be normal. Then $Hyperlat(T)=Hyperlat(T^*)$.

Proof. Since T is normal if and only if T^* is normal, the result follows from the fact that $T^*\in\{T\}^{'}$ if and only if $T\in\{T^*\}^{'}$.

Theorem 5.24 If $T\in B(H)$ is an invertible reductive operator, then T^{-1} is also reducible.

Proof. Since $T\in B(H)$ is reducible, by Theorem 5.14 it can be expressed as

$$T=\begin{bmatrix}T_1 & 0\\ 0 & T_2\end{bmatrix}=T_1\oplus T_2,$$

with respect to the direct sum decomposition $H=M\oplus M^{\perp}$, where M is a subspace that reduces T. Invertibility of T

implies that of T_1 and T_2. Thus

$$T^{-1} = \begin{bmatrix} T_1^{-1} & 0 \\ 0 & T_2^{-1} \end{bmatrix} = T_1^{-1} \oplus T_2^{-1},$$

with respect to the direct sum decomposition $H = M \oplus M^{\perp}$.

Corollary 5.25 Let $T \in B(H)$ be invertible. If a subspace $M \subseteq H$ reduces T, then M reduces T^{-1}.

Proof. Let P_M be the orthogonal projection of H onto M. Since M reduces T, we have $TP_M = P_M T$. By the proof of Theorem 5.24, $T^{-1}P_M = P_M T^{-1}$. This proves the claim.

Remark. From Theorem 5.24 and Corollary 5.25, we conclude that if $T \in B(H)$ is invertible, then $\mathrm{Re}\,d(T) = \mathrm{Re}\,d(T^{-1})$.

We also note that Corollary 5.25 is not true if we replace reducibility with invariance. This is because in infinite dimensional Hilbert spaces, the invariance of a subspace M for an invertible operator $T \in B(H)$ does not imply invariance under T^{-1}.

Example 6. Consider the bilateral weighted shift T_ω on $\ell^2(\mathbb{Z})$ defined by $T_\omega e_n = \omega_n e_{n+1}$, where $n \in \mathbb{Z}$ and $\{e_n\}$ the canonical orthonormal basis for $\ell^2(\mathbb{Z})$. A simple calculation shows that $T_\omega^{-1} e_n = \frac{1}{\omega_n} e_n$, where $n \in \mathbb{Z}$. If

$M = span\{e_1, e_2, \cdots\}$, then M is invariant for T_ω but is not invariant for T_ω^{-1}.

The following result shows that taking powers of an operator $T \in B(H)$ preserves invariance and reduction.

Theorem 5.26. Let $T \in B(H)$ and $M \subseteq H$. The following statements are true for any integer $n > 1$.

(i). If $M \in Lat(T)$ then $M \in Lat(T^n)$.

(ii). If $M \in \mathrm{Re}\,d(T)$ then $M \in \mathrm{Re}\,d(T^n)$.

Proof. The proofs of (i) and (ii) follow easily by mathematical induction on $n \in \mathbb{N}$. In the proof of (ii), we use the fact that $M \in \mathrm{Re}\,d(T)$ implies that $TM \subseteq M$ and $T^* M \subseteq M$.

Theorem 5.27. Let $T \in B(H)$ and $M \subseteq H$. If $M \in Hyperlat(T)$ then $M \in Hyperlat(T^n)$, for any integer $n > 1$.

Proof. We need to prove that $M \in Lat(S)$, where $S \in \{T\}'$ implies that $M \in Lat(X)$, where $X \in \{T^n\}'$.

By Theorem 5.26(i), if $M \in Lat(S)$ then $M \in Lat(S^n)$, where $S \in \{T\}'$.

By mathematical induction on $n \in \mathbb{N}$, if $S \in \{T\}'$ then $S^n \in \{T\}'$, $T^n \in \{S\}'$ and $S^n \in \{T^n\}'$.

By letting $X = S^n$, and using Theorem 5.26(i) once more, the result follows.

6. Discussion

The invariant subspaces of an operator, their classification and description play an explicitly central role in operator theory. They are a direct analogue of the eigenvectors of a linear operator. Reducing subspaces are special invariant subspaces which are useful in the direct sum decomposition of an operator. They can also be used to classify an operator. The basic motivations for the study of invariant subspaces come from the interest in the structure of operators and from approximation theory to a wide variety of problems in physics (quantum theory), computer science (data mining), and chemistry (lattice theory of crystal analysis).

In particular, reducing subspaces find applications in wavelet expansion, multiresolution analysis (MRA) in image processing and automorphic graph theory.

If $A \in B(H)$ and $x \in H$, then $\bigvee_{n=0}^{\infty}\{A^n x\}$ is an invariant subspace of A. Therefore knowledge of $Lat(A)$ gives information about the vectors which can be approximated by linear combinations of $\{A^n x\}$.

Knowledge of $Hyperlat(A)$ can give information about the structure of the commutant $\{A\}'$ of A. Commutators of the form $AB - BA$ appear in a mathematical formulation of the Heisenberg's Uncertainty Principle.

7. Conclusion

In this paper, several concepts about subspace lattices have been introduced. It has been shown a unitary operator T, $Hyperlat(T) = \mathrm{Re}\,d(T)$. It has also been proved that for a normal operator T, $Hyperlat(T) \subseteq Hyperlat(T^*)$, and that the following inclusions hold:

$$Normal \subset \mathrm{Re}\,ductive \subset \mathrm{Re}\,ducible.$$

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