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On the Performance of Haar Wavelet Approach for Boundary Value Problems and Systems of Fredholm Integral Equations

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Abstract: The Haar wavelet method applied to different kinds of integral equations (Fredholm integral equation, integro-differential equations and system of linear Fredholm integral equations) and boundary value problems (BVP) representation of integral equations. Three test problems whose exact solutions are known were considered to measure the performance of Haar wavelet. The calculations show that solving the problem as integral equation is more accurate than solving it as differential equation. Also the calculations show the efficiency of Haar wavelet in case of F. I. E. S and integro-differential equations comparing with other methods, especially when we increase the number of collocation points. All calculations are done by the Computer Algebra Facilities included in Mathematica 10.2.

Keywords: Integral Equations, Haar Wavelets, BVP, System of Integral Equations, Collocation Method

1. Introduction

Many applications in scientific fields such as physical and engineering problems can be formulated as integral equations or as differential equations. Differential equations can be reformulated in the form of integral equations while not every integral equations can be differentiated to obtain a corresponding differential equation, equivalence requires restricted regularity conditions on the kernels [1, 2]. The problems which can be formulated as integral and differential equations are introduced. The analytical solution has its difficulties which gives the chance to numerical analysis to appear, especially with the huge development in programming and computer systems. The last step in the numerical treatment is the solution of the algebraic system of the mathematical model [3]. Recently a great deal of interest has been focused on the solution of integral and differential equations by the wavelet methods, the first paper which used Haar wavelet method to solve integral equation has presented in 1991. The basic idea of Haar wavelet method is to convert the differential and integral equations into a system of algebraic equations. In this paper we use the Haar wavelet method to solve BVP and its integral representation. We consider the relation between the numerical treatments of the two point BVP, [3, 4]

$$\left.\begin{matrix} y''(x) + \lambda y(x) = f(x);\ (a \le x \le b), \\ y(a) = \alpha, y(b) = \beta \end{matrix}\right\} \tag{1}$$

And its equivalent second kind Fredholm integral representation

$$y(x) = \alpha + \frac{(\beta-\alpha)}{(b-a)}(x-a) + \frac{\lambda}{(b-a)}\int_a^b k(x,t)y(t)dt - \frac{1}{(b-a)}\int_a^b k(x,t)f(t)dt\,; \tag{2}$$

where the kernel $k(x,t)$ is defined as

$$k(x,t) = \begin{cases} (t-a)(b-x)\,;\,(t \le x), \\ (x-a)(b-t)\,;\,(x \le t). \end{cases} \tag{3}$$

We consider also the integro-differential equation

$$y'(x) + g(x)y(x) = \int_a^b k(x,t)[\rho y(t) + \sigma y'(t)]dt + f(x);\ y(a) = \tau \tag{4}$$

A system of linear Fredholm integral equation which appears in many applications in physics and engineering is considered. The linear system of Fredholm integral equations takes the form, [5, 6]

$$\left.\begin{array}{c} y_r(x) = f_r(x) + \sum_{s=1}^{n} \int_0^1 k_{rs}(x,t) y_s(t) dt \\ r = 1,2,\dots,n \end{array}\right\} \tag{5}$$

Where $f_r \in L^2[0,1], k_{rs} \in L^2([0,1]\times[0,1])$ for $r,s = 1,2,\dots,n$ and $y_r$are unknown functions. The system (5) can be written in matrix form as

$$Y(x) = F(x) + \int_0^1 K(x,t)Y(t)dt \tag{6}$$

Where $Y(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ \vdots \\ y_n(x) \end{bmatrix}, F(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{bmatrix},$

$$K(x,t) = \begin{bmatrix} k_{11}(x,t) & k_{12}(x,t) & \dots & k_{1n}(x,t) \\ k_{21}(x,t) & k_{22}(x,t) & \dots & k_{2n}(x,t) \\ \vdots & \vdots & \dots & \vdots \\ k_{n1}(x,t) & k_{n2}(x,t) & \dots & k_{nn}(x,t) \end{bmatrix}$$

2. Haar Wavelet Functions and Its Integration

The idea of Wavelet analysis is that it represents a function in terms of a set of basic functions, called wavelets, constructed from transformation and dilation of mother wavelet. The wavelet function takes the form

$$\psi_{j,k}(t) = 2^{j/2}\psi(2^j t - k), j = 0,1,\dots; 0 \le k < 2^j, 0 \le t < 1 \tag{7}$$

The wavelet basis satisfy the following properties

a) The orthogonality property

$$\int_0^1 \psi_i(t)\psi_j(t)\,dx = \begin{cases} 1\,; i = j \\ 0\,; i \ne j \end{cases}$$

b) Most of the functions have a small interval of support [7-11].

Definition: Let $f \in L^2(R)$ for $n \in Z, T_n: L^2(R) \to L^2(R)$ be given by$(T_n f)(x) = f(x-n)$ and $D: L^2(R) \to L^2(R)$ be given by $(Df)(x) = \sqrt{2} f(2x)$ operators T_n and D are called translation and dilation operator.

There are many types of wavelet functions; one of them is the Haar functions, which are mathematically the simplest wavelets. The orthogonal set of Haar functions is defined as a group of square waves with magnitude of ± 1 in some intervals and zero elsewhere. They are step functions (piecewise constant functions). Haar transform or Haar wavelet transform has been used as an earliest example for orthonormal wavelet transform with compact support

The Haar wavelet family are given as, [6-9]

$$h_i(t) = \begin{cases} 1\, for\ t \in [\alpha, \beta] \\ -1\, for\ t \in [\beta, \gamma] \\ 0\ elsewhere \end{cases} \tag{8}$$

The notations $\alpha = \frac{k}{m}, \beta = \frac{k+0.5}{m}\ \gamma = \frac{k+1}{m}$ are introduced. The integer $m = 2^j, j = 0,1,2,\dots,J$ indicates the level of wavelet (dilation parameter); $k = 0,1,2,\dots,m-1$ is the translation parameter. The integer J determines the maximal level of resolution, the index i obtained from the relation $i = m + k + 1$ which has minimum value $i = 2\ (m = 1, k = 0)$, which called the mother function, and maximum value $i = 2M$ where $M = 2^J$. the index $i = 1$ corresponds to the scaling function

$$h_1(t) = \begin{cases} 1\, for\ 0 \le t < 1 \\ 0\quad elsewhere \end{cases}$$

The following notations are introduced, [9, 10]

$$p_i(t) = \int_0^t h_i(x)dx\ ;\ q_i(t) = \int_0^t p_i(x)dx \tag{9}$$

For the scaling function we have

$$p_1(t) = \begin{cases} t;\ for\ 0 \le t < 1 \\ 0;\ elsewhere \end{cases} ; q_1(t) = \begin{cases} \frac{t^2}{2}; for\ 0 \le t < 1 \\ 0\,;\ elsewhere \end{cases}$$

Else

$$p_i(t) = \begin{cases} t - \alpha\ for\ t \in [\alpha, \beta] \\ \gamma - t\ for\ t \in [\beta, \gamma] \\ 0\ elsewhere \end{cases}$$

$$q_i(t) = \begin{cases} \frac{(t-\alpha)^2}{2}\ for\ t \in [\alpha, \beta] \\ \frac{(\alpha-\beta)^2 + (\beta-\gamma)^2 - (\gamma-t)^2}{2}\ for\ t \in [\alpha, \beta] \\ \frac{(\alpha-\beta)^2 + (\beta-\gamma)^2}{2}\ for\ t \in [\alpha, \beta] \\ 0\ elsewhere \end{cases} \tag{10}$$

To discretize the functions $h_i(t)$ we divide the interval $t \in [0,1]$ into $2M$ parts of equal length $\Delta t = \frac{1}{2M}$. Let us introduce the collocation points

$$\left.\begin{array}{c} t_l = \frac{(l-0.5)}{2M}, \\ l = 1,2,\dots,2M \end{array}\right\} \tag{11}$$

Any function $y(t),\ t \in [0,1]$ can be written as

$$y(t) = \sum_{i=1}^{2M} a_i h_i(t) = a_1 h_1 + a_2 h_2 + a_3 h_3 + \cdots \tag{12}$$

Where a_i are the Haar coefficients which can be determined by multiplying (12) by $h_m(t)$ and integrating from $0 \to 1$ we get

$$\int_0^1 y(t)h_m(t)dt = a_i \sum_{i=1}^{2M} \int_0^1 h_m(t)h_i(t)dt$$

from the orthogonality condition

$$\left.\begin{aligned} \int_0^1 h_m(t)h_i(t)dt = \begin{cases} 2^{-j}; if\ i = m \\ 0;\ if\ i \neq m \end{cases} \\ \Rightarrow a_i = 2^j \int_0^1 y(t)h_m(t)dt \end{aligned}\right\} \quad (13)$$

The discrete form of eq. (12) at collocation points is

$$y(t_l) = \sum_{i=1}^{2M} a_i h_i(t_l) = \sum_{i=1}^{2M} a_i H_{il} \quad (14)$$

The matrix form of (14) is $Y = aH$ where, Y and a are $2M$ row vectors and $H = H_{il} = h_i(t_l)$ is the coeffeficients matrix with dimension $(2M \times 2M)$ [12-15].

3. Method of Solution

In this section we introduce how the Haar wavelet transform method can solve the BVP and its Fredholm integral form and the system of Fredholm integral equations.

3.1. Haar Solution of Second Order Two-Point BVP

The main advantage of the Haar wavelet method is its efficiency and simple applicability for a variety of boundary conditions [10,11,13]. Since the Haar wavelet is defined for the interval [0,1], then we transform the variable $x \in [a,b]$ in equation (1) into the variable $t \in [0,1]$ using

$$t = \frac{x-a}{b-a} \to \text{if } x = a, t = 0 \text{ and if } x = b, t = 1$$

So equation (1) becomes

$$\left.\begin{aligned} y''(t) + \lambda y(t) = f(t);\ (0 \le t \le 1), \\ y(0) = \alpha, y(1) = \beta. \end{aligned}\right\} \quad (15)$$

In equation (15) let

$$y''(t) = \sum_{i=1}^{2M} a_i h_i(t)$$

Integrating twice from $0 \to t$, using the boundary conditions we get

$$y(t) = \alpha + ty'(0) + \sum_{i=1}^{2M} a_i q_i(t)$$

And

$$y'(0) = \beta - \alpha - \sum_{i=1}^{2M} a_i c_i$$

Where $c_i = \int_0^1 p_i(t)dt$
then $y(t)$ can be written as

$$y(t) = \alpha + (\beta - \alpha)t + \sum_{i=1}^{2M} a_i (q_i(t) - tc_i) \quad (16)$$

Using (16) in (15) we get

$$\sum_{i=1}^{2M} a_i[h_i(t) + \lambda(q_i(t) - tc_i)] = f(t) + \lambda(\alpha - \beta)t - \lambda\alpha$$

Satisfying the previous equation at the collocation points (11) we obtain

$$\left.\begin{aligned} \sum_{i=1}^{2M} a_i[h_i(t_l) + \lambda[q_i(t_l) - t_l c_i]] = f(t_l) + \alpha + \lambda(\beta - \alpha)t_l - \lambda\alpha, \\ l = 1, \dots, 2M \end{aligned}\right\} \quad (17)$$

Solving the linear system of algebraic (17) we obtain the unknowns a_i, substituting in (16) we get the solution of the BVP [16,17].

3.2. Haar Solution of Fredholm Integral Equations

In equation (2) let

$$y(x) = \sum_{i=1}^{2M} a_i h_i(x) \quad (18)$$

From (18) equation (2) becomes

$$\sum_{i=1}^{2M} a_i h_i(x) - \sum_{i=1}^{2M} a_i G_i(x) = g(x) \quad (19)$$

Where

$$G_i(x) = \int_a^b k(x,t)h_i(t)dt \quad (20)$$

and

$$g(x) = \alpha + \frac{(\beta-\alpha)}{(b-a)}(x-a) - \frac{\lambda}{(b-a)}\int_a^b k(x,t)f(t)dt \quad (21)$$

Satisfying equation (19) at the collocation points (11)

$$\sum_{i=1}^{2M} a_i[h_i(x_l) - G_i(x_l)] = g(x_l), l = 1,2,\dots,2M \quad (22)$$

Equation (22) is a linear system of algebraic equations in the unknowns a_i, solving this system we obtain a_i, substituting in (18) we solve the Fredholm integral equation (2) [9-11,18].

3.3. Haar Solution of Integro-Differential Equations

To solve the integro-differential equation (4), let

$$y'(x) = \sum_{i=1}^{2M} a_i h_i(x) \Rightarrow y(x) = \sum_{i=1}^{2M} a_i p_i(x) + y(a) \quad (23)$$

Where $p_i(x) = \int_a^x h_i(t)dt$.
Substituting from (23) in (4) we get

$$\sum_{i=1}^{2M} a_i[h_i(x) + g(x)p_i(x) - \rho R_i(x) - \sigma G_i(x)] = -g(x)y(a) + \rho y(a)Q(x) + f(x) \quad (24)$$

Where $G_i(x)$given from (20) and

$$R_i(x) = \int_a^b k(x,t)p_i(t)dt,\ Q(x) = \int_a^b k(x,t)dt$$

Satisfying equation (24) at the collocation points (11) to get the matrix form as

$$\mathrm{A}(H + V - \rho R - \sigma G) = -\tau g + \rho\tau Q + F \quad (25)$$

Where A is a $2M$ vector of the coefficients a_i ; $(g,\ F,\ Q)$ are $2M$ vectors and $H = H_{il} = H_i(x_l)$, $R = R_{il} = R_i(x_l)$, $V = V_{il} = g(x_l)p_i(x_l)$ [6,9,12].

3.4. Haar Solution of Linear System of Fredholm Integral Equations

Using equation (12) in equation (5) we obtain

$$\sum_{i=1}^{2M} a_{ri}h_i(x) - \sum_{i=1}^{2M} a_{ri}G_i(x) = F(x); \quad (26)$$

Where

$$F(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{bmatrix}$$

$$G_i(x) = \begin{bmatrix} G_{11}(x) = \int_0^1 k_{11}(x,t)\,h_i(t)dt & \dots & G_{1n}(x) = \int_0^1 k_{1n}(x,t)\,h_i(t)dt \\ G_{21}(x) = \int_0^1 k_{21}(x,t)\,h_i(t)dt & \dots & G_{2n}(x) = \int_0^1 k_{2n}(x,t)\,h_i(t)dt \\ \vdots & \ddots & \vdots \\ G_{n1}(x) = \int_0^1 k_{n1}(x,t)\,h_i(t)dt & \dots & G_{nn}(x) = \int_0^1 k_{nn}(x,t)\,h_i(t)dt \end{bmatrix}$$

Satisfying equation (26) only at the collocation points (11) we get a linear system of algebraic equations

$$\left.\begin{array}{c} \sum_{i=1}^{2M} a_{ri}h_i(x_l) - \sum_{i=1}^{2M} a_{ri}G_i(x_l) = F(x_l), \\ l = 1,2,\dots,2M; r = 1,2,\dots,2M \end{array}\right\} \quad (27)$$

The matrix form of equation (27) is $AB = F$ where

$$A = \left[\sum_{i=1}^{2M} a_{1i} \quad \sum_{i=1}^{2M} a_{2i} \quad \dots \quad \sum_{i=1}^{2M} a_{ni}\right],$$

$$F = \begin{bmatrix} f_1(x_l) \\ f_2(x_l) \\ \vdots \\ f_n(x_l) \end{bmatrix}, B = \begin{bmatrix} H - G_{11} & -G_{21} & \dots & -G_{n1} \\ -G_{12} & H - G_{22} & \dots & -G_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ -G_{1n} & -G_{2n} & \dots & H - G_{nn} \end{bmatrix}$$

Solving the linear system (27) we obtain a_{ri} [6,9,12].

3.5. Haar Algorithm for Linear System of F. I. E.

a) Divide the interval [0, 1] into $2M$ part of equal length $\Delta t = \frac{1}{2M}$.
b) Compute the G matrix from the equation (26).
c) Solve the system (27) at the collocation points to determine a_{ri} .
d) Substitute in (12), we get the solution of the system (5).

4. Numerical Examples

In this section some numerical examples are considered. The exact solution introduced to show the accuracy and efficiency of the used method. In the first example we compare between the second order two point BVP and its Fredholm integral representation. In the second example we solve a linear system of F. I. E. and compare our method with a domain decomposition method. In the third example, the solution of the integro-differential equation and comparing our solution with another method is introduced.

Example 1:

Consider the second order two point BVP, [3, 19]:

$$\left.\begin{array}{c} -y''(x) + \pi^2 y(x) = 2\,\pi^2\,Sin(\pi\,x), \\ 0 \le x \le 1;\ y(0) = y(1) = 0. \end{array}\right\} \quad (28)$$

whose exact solution was given as

$$y(x) = \sin(\pi\,x). \quad (29)$$

The Fredholm integral form of equation (28) is

$$y(x) = 2\,sin\,\pi x - \pi^2 \int_0^x t(1-x)\,y(t)dt - \pi^2 \int_x^1 x(1-t)\,y(t)dt. \quad (30)$$

It is an easy task to see that this integral equation (30) satisfies the boundary conditions in (28). Moreover the closed form solution (29) satisfies both the differential and the integral equation.

In equation (28) let

$$y''(x) = \sum_{i=1}^{2M} a_i h_i(x)$$

Integrating twice from $0 \to x$ with boundary conditions we get on

$$\boldsymbol{y(x) = xy'(0)} + \sum_{i=1}^{2M} a_i q_i(x)$$

Putting $x = 1$ in the previous equation to find $y'(0)$ we have

$$y'(0) = -\sum_{i=1}^{2M} a_i c_i$$

Where c_i defined before, then

$$y(x) = \sum_{i=1}^{2M} a_i q_i(x) - x\sum_{i=1}^{2M} a_i c_i$$

Substituting in (28) we get

$$\sum_{i=1}^{2M} a_i[\pi^2 q_i(x) - \pi^2 x c_i - h_i(x)] = 2\pi^2 \sin \pi x \quad (31)$$

Satisfying equation (31) at the collocation points (11)

$$\left.\begin{array}{c}\sum_{i=1}^{2M} a_i[\pi^2 q_i(x_l) - \pi^2 x_l c_i - h_i(x_l)] = 2\pi^2 \sin \pi x_l\,, \\ l = 1, \dots, 2M\end{array}\right\} \quad (32)$$

Solving the linear system (32) we determine a_i. The solution of BVP (28), the exact solution and the error at different numbers of collocation points with different values of M given in table 1.

Satisfying equation (30) at the collocation points (11)

$$\left.\begin{array}{c}y(x_l) + \pi^2 \int_0^{x_l} t(1-x_l)\, y(t)dt + \pi^2 \int_{x_l}^1 x_l(1-t)\, y(t)dt = 2\, sin\, \pi x_l\,. \\ l = 1,2, \dots, 2M\end{array}\right\} \quad (33)$$

putting $y(x) = \sum_{i=1}^{2M} a_i h_i(x)$, in equation (33) we have

$$\left.\begin{array}{c}\sum_{i=1}^{2M} a_i[h_i(x_l) + \pi^2(1-x_l) \int_0^{x_l} t\, h_i(t)dt + \pi^2 x_l \int_{x_l}^1 (1-t)\, h_i(t)dt] = 2\, sin\, \pi x_l\,. \\ l = 1,2, \dots, 2M\end{array}\right\} \quad (34)$$

For $J = 3$ the solution of (30), the exact solution and the error between the exact solution and Haar solution are given in table 1.

Table 1. *Comparison between Haar solution of BVP Ex. (1) and Haar solution of its Fredholm form with (J=3).*

X / 32	Exact solution	Haar (BVP)	$\lvert E_{BVP}\rvert$	Haar (F. I. E)	$\lvert E_{F.I.E.}\rvert$
1	0.098017	0.106234	0.008216	0.098098	0.0000811
3	0.290285	0.315259	0.024974	0.290514	0.0002295
5	0.471397	0.514114	0.042717	0.471775	0.0003781
7	0.634393	0.696537	0.062144	0.6349	0.0005067
9	0.77301	0.857028	0.084018	0.77363	0.0006196
11	0.881921	0.991116	0.109194	0.882627	0.0007062
13	0.95694	1.0956	0.138656	0.957705	0.0007647
15	0.995185	1.16875	0.173548	0.995982	0.0007973
17	0.995185	1.21133	0.216149	0.995982	0.0007976
19	0.95694	1.23014	0.2732	0.957706	0.0007658
21	0.881921	1.2256	0.343674	0.882629	0.000708
23	0.77301	1.19498	0.421967	0.773633	0.0006221
25	0.634393	1.14665	0.512255	0.634903	0.0005098
27	0.471397	1.0885	0.617103	0.47178	0.0003834
29	0.290285	1.032	0.74172	0.290519	0.0002339
31	0.098017	0.997273	0.899256	0.098190	0.0000861

The comparison between the BVP in example (1) and its Fredholm integral form at different values of J are given in table 2.

Table 2. *Shows the error for BVP Ex. (1) and its Fredholm from at different number of collocation points.*

J	2 M	e_{BVP}	$e_{F.I.E}$
1	4	0.295356	0.0079622
2	8	0.292566	0.00203375
3	16	0.291799	0.000511918
4	32	0.290915	0.000128373

Example (2):

Consider the following linear system of Fredholm integral equations, [5]

$$\left.\begin{array}{c}y_1(x) = \frac{x}{18} + \frac{17}{36} + \int_0^1 \frac{t+x}{3} y_1(t)dt + \int_0^1 \frac{t+x}{3} y_2(t)dt\,, \\ y_2(x) = x^2 - \frac{19}{12}x + 1 + \int_0^1 xty_1(t)dt + \int_0^1 xty_2(t)dt\end{array}\right\} \quad (35)$$

With exact solution $y_1(x) = x + 1$ and $y_2(x) = x^2 + 1$.

Putting

$$y_1(x) = \sum_{i=1}^{2M} a_{1i} h_i(x)\,;\; y_2(x) = \sum_{i=1}^{2M} a_{2i} h_i(x) \tag{36}$$

Using (36) in (35)

$$\left.\begin{array}{r}\sum_{i=1}^{2M} a_{1i}[h_i(x) - G_{1i}(x)] - \sum_{i=1}^{2M} a_{2i} G_{1i}(x) = \frac{x}{18} + \frac{17}{36}\\ \sum_{i=1}^{2M} a_{1i}[-G_{2i}(x)] + \sum_{i=1}^{2M} a_{2i}[h_i(x) - G_{2i}(x)] = x^2 - \frac{19}{12}x + 1\end{array}\right\} \tag{37}$$

Where

$$G_{1i}(x) = \int_0^1 \frac{t+x}{3} h_i(t)dt = \begin{cases} \frac{x}{3} + \frac{1}{6} & if\ i = 1 \\ \frac{-1}{12m^2} & if\ i > 1 \end{cases};$$

$$G_{2i}(x) = \int_0^1 tx\, h_i(t)dt = \begin{cases} \frac{x}{2} & if\ i = 1 \\ \frac{-x}{4m^2} & if\ i > 1 \end{cases}$$

Satisfying equation (37) at the collocation points (11)

$$\left.\begin{array}{r}\sum_{i=1}^{2M} a_{1i}[h_i(x_l) - G_{1i}(x_l)] - \sum_{i=1}^{2M} a_{2i} G_{1i}(x_l) = \frac{x_l}{18} + \frac{17}{36}\\ \sum_{i=1}^{2M} a_{1i}[-G_{2i}(x_l)] + \sum_{i=1}^{2M} a_{2i}[h_i(x_l) - G_{2i}(x_l)] = {x_l}^2 - \frac{19}{12}x_l + 1\end{array}\right\} \tag{38}$$

Solving the linear system (38) with (16) collocation points ($J = 3 \to M = 8 \to 2M = 1$) the coefficients a_{1i} & a_{2i} are obtained. The Haar solution of the system (35), the exact solution, the error between Haar solution and exact solution and the error between Haar solution and decomposition solution of the same system are given in table 3 and table 4.

Example 3, [20]:

Consider the integro-differential equation

$$y'(x) = xe^x + e^x - x + \int_0^1 xy(t)dt\,, y(0) = 0 \tag{39}$$

Whose exact solution is

$$y(x) = xe^x \tag{40}$$

To solve (39) let

$$y'(x) = \sum_{i=1}^{2M} a_i h_i(x) \Rightarrow y(x) = \sum_{i=1}^{2M} a_i p_i(x) \tag{41}$$

Where $p_i(x) = \int_0^x h_i(t)dt$. Substituting from (41) in (39) gives

$$\sum_{i=1}^{2M} a_i \left[h_i(x) - x\int_0^1 p_i(t)dt\right] = e^x[x+1] - x$$

Satisfying the previous equation at the collocation points (11)

$$\sum_{i=1}^{2M} a_i \left[h_i(x_l) - x_l \int_0^1 p_i(t)dt\right] = e^{x_l}[x_l + 1] - x_l, l = 1, \dots, 2M. \tag{42}$$

Solving the system (42) the coefficients $a_i, i = 1(1)16$ are obtained. The absolute error between the exact solution and Haar solution comparing with the absolute error between the exact solution and the solution using differential transform method, given in [20], are given in table 5.

Table 3. *Comparison between the error of Haar solution and the error by decomposition method for y1 of Ex. (2) with (J=3).*

x/32	Exact y_1	Haar S. y_1	e_{Haar,y_1}	Deco. S. y_1	e_{Decom,y_1}
1	1.03125	1.03097	0.0002788	1.01917	0.012084
3	1.09375	1.09346	0.0002888	1.0805	0.013253
5	1.15625	1.15595	0.0003029	1.14183	0.014421
7	1.21875	1.21843	0.0003167	1.20316	0.01559
9	1.28125	1.28092	0.0003274	1.26449	0.016759
11	1.34375	1.34341	0.0003412	1.32582	0.017928
13	1.40625	1.4059	0.0003543	1.38715	0.019096
15	1.46875	1.46838	0.0003653	1.44848	0.020265
17	1.53125	1.53087	0.0003788	1.50982	0.021434
19	1.59375	1.59336	0.0003926	1.57115	0.022603
21	1.65625	1.65585	0.0004028	1.63248	0.023771
23	1.71875	1.71833	0.0004167	1.69381	0.02494
25	1.78125	1.78082	0.0004301	1.75514	0.026109
27	1.84375	1.84331	0.0004411	1.81647	0.027278
29	1.90625	1.90579	0.0004552	1.8778	0.028446
31	1.96875	1.97615	0.0074043	1.93913	0.029615

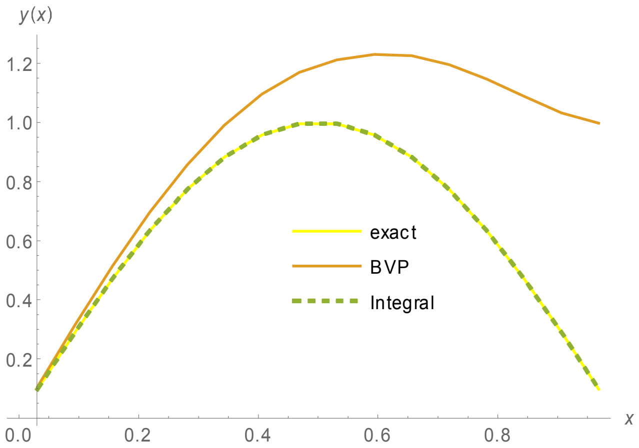

Figure 1. *The comparison between the Haar solution of BVP and Haar solution of its Integral form in Ex. (1) with J=3.*

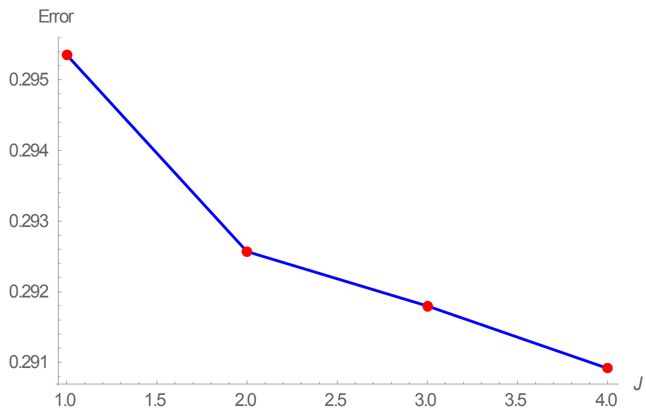

Figure 2. *The effect of increasing the collocation points for Eq. (28).*

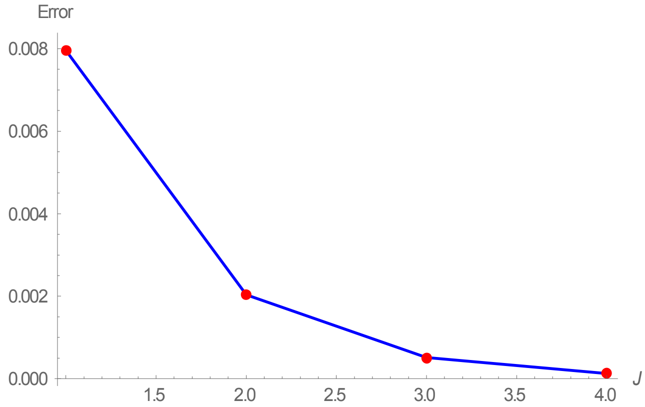

Figure 3. *The effect of the increasing the collocation points for Eq. (30).*

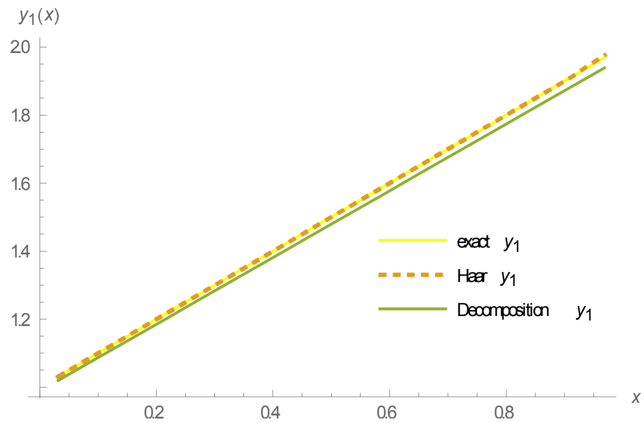

Figure 4. *The comparison between Haar solution and Decomposition solution for y_1 for Ex. (2) with J=3.*

Table 4. *Comparison between the error of Haar solution and the error by decomposition method for y2 of Ex. (2) with (J=3).*

x/32	Exact y_2	Haar S. y_2	e_{Haar,y_2}	Deco. S. y_2	e_{Decom,y_2}
1	1.00098	1.00095	0.0000265	0.99989	0.0010781
3	1.00879	1.00871	0.0000749	1.00555	0.0032347
5	1.02441	1.02429	0.0001271	1.01902	0.0053906
7	1.04785	1.04767	0.000177	1.0403	0.0075468
9	1.0791	1.07887	0.0002287	1.0694	0.0097031
11	1.11816	1.11789	0.000278	1.1063	0.0118594
13	1.16504	1.16471	0.0003293	1.15102	0.0140156
15	1.21973	1.21935	0.0003789	1.20355	0.0161719
17	1.28223	1.2818	0.0004305	1.2639	0.0183281
19	1.35254	1.35206	0.0004804	1.33205	0.0204844
21	1.43066	1.43013	0.0005304	1.40802	0.0226406
23	1.5166	1.51602	0.0005822	1.4918	0.0247969
25	1.61035	1.60972	0.0006329	1.5834	0.0269531
27	1.71191	1.71123	0.0006834	1.6828	0.0291094
29	1.82129	1.82055	0.0007342	1.79002	0.0312656
31	1.93848	1.93769	0.000785	1.90505	0.0334219

Table 5. *Comparison between Haar solution and differential transform method for Ex. (3).*

Method	Haar Wavelet	Differential Transform [18]
Absolute error	$1.9 * 10^{-3}$	$7.3 * 10^{-2}$

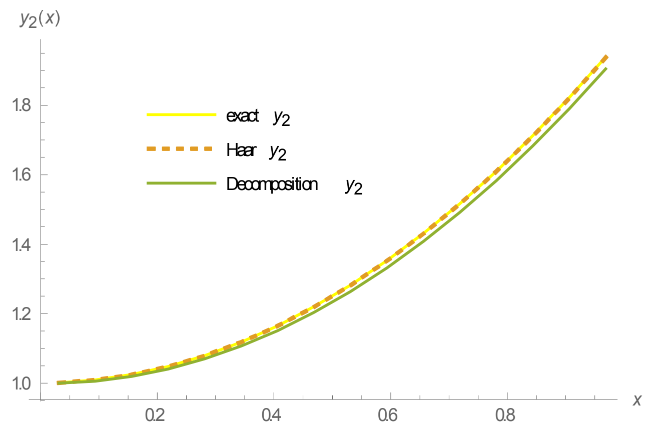

Figure 1. *The comparison between Haar solution and Decomposition solution for y_2 for Ex. (2) with J=3.*

5. Conclusion

Applying the Haar wavelet transform with collocation points method is introduced and we found that:

a) Solving the problem as integral equation is more accurate than solving it as BVP, see table 1 and figure 1.
b) It is observed that if the level of resolution is increased i.e. if the collocation points are increased, then we can get a better solution with less error, see table 2 and figures 2,3.
c) Using Haar wavelet technique to solve the system of integral equations is more efficient than the use of decomposition method, see tables 3, 4 and figures 4, 5.
d) The Haar solution of integro-differential equations gives accuracy more than the differential transform method given in [20], see table 5.
e) It can be concluded that this method is quite suitable, accurate, and efficient in comparison to other classical methods.

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