



The New Proof of Ptolemy's Theorem & Nine Point Circle Theorem

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Abstract: The main purpose of the paper is to present a new proof of the two celebrated theorems: one is "Ptolemy's Theorem" which explains the relation between the sides and diagonals of a cyclic quadrilateral and another is "Nine Point Circle Theorem" which states that in any arbitrary triangle the three midpoints of the sides, the three feet of altitudes, the three midpoints of line segments formed by joining the vertices and Orthocenter, total nine points are concyclic. Our new proof is based on a metric relation of circumcenter.

Keywords: Ptolemy's Theorem, Circumcenter, Cyclic Quadrilateral, Nine Point Circle Theorem, Pedals Triangle, Medial Triangle

1. Introduction

The Ptolemy's Theorem states that "The multiple of the lengths of the diagonals of a Cyclic Quadrilateral is equal to the addition of separate multiples of the opposite side lengths of the Cyclic Quadrilateral" (refer [6]).

In this short paper, we deal with an elementary proof for the Ptolemy's theorem as well as Nine Point Circle Theorem ([24], [25]). Many other different simple approaches for proving the two theorems and their further generalizations are well known in the literature of Euclidean geometry. (some of them can be found in [1], [5], [6], [12], [13], [14], [17], [18], [21], [22], [23] and [26]). In this article we present a new proof for these two theorems based on a metric relation of circumcenter. Our proof actually follows from a lemma related to a circumcircle which gives a necessary condition for the four points to be concyclic and in the conclusion of the article we will try to prove some remarks and related inequalities based on these lemmas.

2. Some Back Ground Material

The standard notation is used throughout.

For any arbitrary triangle ABC, we denote a, b, c for the

The following formulas are well known

sides of the lengths of BC, CA, AB, its semi perimeter by

$s = \frac{1}{2}(a + b + c)$ and its area by Δ . Its classical centers are the

circumcenter (S), the incenter (I), the centroid (G), the orthocenter (O) and nine-point center (N) respectively. Let D, E, F be the feet of the medians and K, L, M are the feet of the altitudes of the triangle ABC which lies on the sides BC, CA, AB and T, U, V are the mid points of AO, BO and CO. The median and the altitude through A (and their lengths) are denoted by m_a and h_a respectively. The classical radii are the circumradius R ($=SA=SB=SC$), the inradius r and the exradii r_1, r_2, r_3 .

The Medial Triangle:

The triangle formed by the feet of the medians is called as Medial triangle. Its sides are parallel to the sides of given triangle ABC and by Thales Theorem the sides and angles of

medial triangle are $\frac{a}{2}, \frac{b}{2}, \frac{c}{2}$, A, B and C respectively. Its

area is $\frac{\Delta}{4}$, circumradius $\frac{R}{2}$, inradius $\frac{r}{2}$.

- (a) $\Delta = \frac{abc}{4R} = rs = r_1(s-a) = r_2(s-b) = r_3(s-c) = \sqrt{s(s-a)(s-b)(s-c)}$
- (b) $(a+b)(b+c)(c+a) = (a+b+c)(ab+bc+ca) - abc$
- (c) $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C = \frac{abc}{2R^3} = \frac{2\Delta}{R^2}$
- (d) For any triangle ABC, $\sum_{a,b,c} a^3 \cos(B-C) = 3abc$
- (e) For any triangle ABC, $a^2 \sin 2A + b^2 \sin 2B + c^2 \sin 2C = \frac{2\Delta}{R^2} (a^2 + b^2 + c^2 - 6R^2)$
- (f) $\cos A \cos B \cos C = \frac{a^2 + b^2 + c^2 - 8R^2}{8R^2}$

3. Basic Lemma's

Lemma-1

Let M be any point in the plane of the triangle ABC, if S its circumcenter then

$$SM^2 = \frac{R^2}{2A} (\sin 2A \cdot AM^2 + \sin 2B \cdot BM^2 + \sin 2C \cdot CM^2 - 2A) \quad (1)$$

$$SM^2 = \frac{1}{16A^2} [a^2 (b^2 + c^2 - a^2) AM^2 + b^2 (a^2 + c^2 - b^2) BM^2 + c^2 (b^2 + a^2 - c^2) CM^2 - 16R^2 A] \quad (2)$$

Proof:

The proof of (1) is available in [3].

Now for (2), we proceed as follows

We have

$$\sin 2A = \frac{4a^2 (b^2 + c^2 - a^2) \Delta}{2a^2 b^2 c^2} = \frac{a^2 (b^2 + c^2 - a^2)}{8R^2 \Delta}$$

By replacing $\sin 2A$, $\sin 2B$, $\sin 2C$ in (1) and by little computation, we get the conclusion (2).

Lemma-2

If X is any point on the circumcircle of the triangle ABC then

$$\sin 2A \cdot AX^2 + \sin 2B \cdot BX^2 + \sin 2C \cdot CX^2 = 4A \quad (3)$$

$$a^2 (b^2 + c^2 - a^2) AX^2 + b^2 (a^2 + c^2 - b^2) BX^2 + c^2 (b^2 + a^2 - c^2) CX^2 = 2a^2 b^2 c^2 \quad (4)$$

Proof:

The proof of (3) is available in [3].

By replacing $\sin 2A$, $\sin 2B$, $\sin 2C$ in (1) and with a little algebra, we get the conclusion (4).

Lemma-3

If D, E, F are the feet of medians of $\triangle ABC$ drawn from the vertices A, B, C on the sides BC, CA, AB and M is any point in the plane of the triangle then

$$4DM^2 = 2CM^2 + 2BM^2 - a^2 \quad (5)$$

$$4EM^2 = 2CM^2 + 2AM^2 - b^2 \quad (6)$$

$$4FM^2 = 2AM^2 + 2BM^2 - c^2 \quad (7)$$

Proof:

Clearly for the triangle BMC, DM is a cevian. So by using Stewart's theorem,

$$\text{We have } DM^2 = \frac{BD}{BC} CM^2 + \frac{CD}{BC} BM^2 - BD \cdot CD$$

By replacing $BD=CD=\frac{a}{2}$, we get the conclusion (5).

In the similar manner we can prove the conclusions (6) and (7).

Lemma-4

If ABCD is a cyclic quadrilateral and P is the point of intersection of diagonals AC and BD such that $AB=a$, $BC=b$, $CD=c$, $DA=d$ and $AC=p$, $BD=q$ then

$$AP \cdot PC = BP \cdot PD \text{ (chords property)} \quad (8)$$

$$\frac{p}{q} = \frac{ad+bc}{ab+cd} \quad (9)$$

Proof:

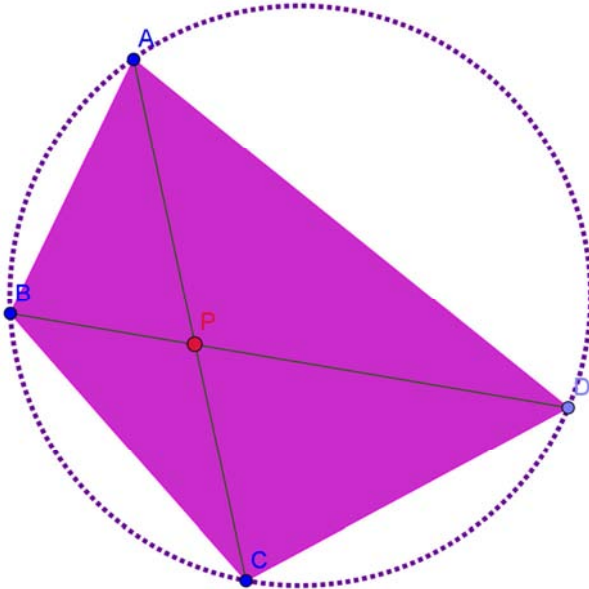


Figure 1. Cyclic quadrilateral.

Clearly from the figure1, Using the property “In any circle Angles in the same segment are equal” and by little angle chasing in the triangles $\triangle APB$, $\triangle DPC$, $\triangle APD$ and $\triangle BPC$.

We can prove that pair of the triangles $\triangle APB$, $\triangle DPC$ and $\triangle APD$, $\triangle BPC$ are similar

It implies that $AP \cdot PC = BP \cdot PD$

Now for (8), we proceed as follows

Since P is the point of intersection of diagonals and BP, CP are the cevians of the triangles $\triangle ABC$ and $\triangle ADC$,

$$\text{So we have } \frac{AP}{CP} = \frac{[\triangle ABP]}{[\triangle BPC]} = \frac{[\triangle APD]}{[\triangle CPD]}$$

($[\triangle XYZ]$ represents area of the triangle XYZ)

$$\begin{aligned} \text{It implies that } \frac{AP}{CP} &= \frac{[\triangle ABP] + [\triangle APD]}{[\triangle BPC] + [\triangle CPD]} = \frac{[\triangle ABD]}{[\triangle BCD]} \\ &= \frac{\frac{1}{2}ad \sin A}{\frac{1}{2}bc \sin(180 - A)} = \frac{ad}{bc} \end{aligned}$$

In the similar manner we can prove that $\frac{BP}{DP} = \frac{ab}{cd}$

Now let $AP=adx$, $CP=bcx$ and $BP=aby$, $DP=cdy$ for some non zero real x, y

But from (8), $AP \cdot PC = BP \cdot PD$

It implies that $x=y$

$$\text{So } \frac{AC}{BD} = \frac{AP+CP}{BP+DP} = \frac{(ad+bc)x}{(ab+cd)y}$$

It implies the conclusion (9).

Lemma-5

If ABCD is a cyclic quadrilateral such that $AB=a$, $BC=b$, $CD=c$, $DA=d$ and $AC=p$, $BD=q$ where $p \neq q$ then

$$[(ac+bd)^2 - p^2 q^2][p^2 - q^2] = [(ab+cd)^2 p^2 - (ad+bc)^2 q^2] \quad (10)$$

$$p^2 q^2 (a^2 + b^2 + c^2 + d^2 - p^2 - q^2) - (a^2 - b^2 + c^2 - d^2)(a^2 c^2 - b^2 d^2) - p^2 (a^2 - d^2)(b^2 - c^2) + q^2 (a^2 - b^2)(c^2 - d^2) = 0 \quad (11)$$

Proof:

Since the Quadrilateral ABCD is a cyclic Quadrilateral, the 4 points A, B, C and D are concyclic. Hence one point among A, B, C and D will lie on the circumcircle of the triangle formed by the remaining three points.

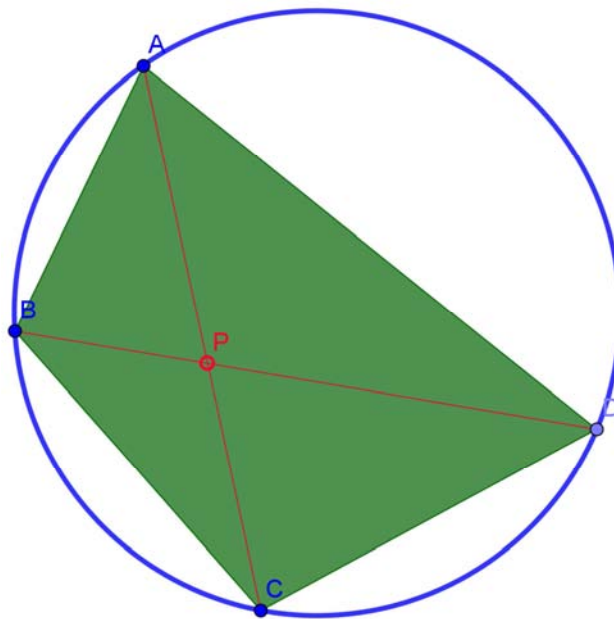


Figure 2. Cyclic quadrilateral.

For triangle ABC, the point D lies on the circumcircle of triangle ABC.

So using lemma-2, by fixing X as D in the conclusion (4),

We get

$$b^2(a^2 + p^2 - b^2)AD^2 + p^2(a^2 + b^2 - p^2)BD^2 + a^2(b^2 + p^2 - a^2)CD^2 = 2a^2b^2p^2 \quad (12)$$

Now for triangle ADC, the point B lies on the circumcircle of triangle ADC.

And using lemma-2, by fixing X as B in the conclusion (4),

We get

$$c^2(d^2 + p^2 - c^2)AB^2 + p^2(c^2 + d^2 - p^2)DB^2 + d^2(c^2 + p^2 - d^2)CB^2 = 2c^2d^2p^2 \quad (13)$$

In the similar manner for triangles $\triangle ADB$, $\triangle ABD$ we get

$$q^2(d^2 + a^2 - q^2)AC^2 + d^2(a^2 + q^2 - d^2)BC^2 + a^2(d^2 + q^2 - a^2)DC^2 = 2a^2d^2q^2 \quad (14)$$

$$c^2(b^2 + q^2 - c^2)BA^2 + q^2(b^2 + c^2 - q^2)CA^2 + b^2(c^2 + q^2 - b^2)DA^2 = 2b^2c^2q^2 \quad (15)$$

Now by replacing $AB=a$, $BC=b$, $CD=c$, $DA=d$, $AC=p$ and $BD=q$ in (12), (13), (14) and (15) we get

$$b^2d^2(a^2 + p^2 - b^2) + p^2q^2(a^2 + b^2 - p^2) + a^2c^2(b^2 + p^2 - a^2) = 2a^2b^2p^2 \quad (16)$$

$$c^2a^2(d^2 + p^2 - c^2) + p^2q^2(c^2 + d^2 - p^2) + b^2d^2(c^2 + p^2 - d^2) = 2c^2d^2p^2 \quad (17)$$

$$p^2q^2(d^2 + a^2 - q^2) + b^2d^2(a^2 + q^2 - d^2) + a^2c^2(d^2 + q^2 - a^2) = 2a^2d^2q^2 \quad (18)$$

$$a^2c^2(b^2 + q^2 - c^2) + p^2q^2(b^2 + c^2 - q^2) + b^2d^2(c^2 + q^2 - b^2) = 2b^2c^2q^2 \quad (19)$$

Now by the computation of [(16)+(17)]-[(18)+(19)], we get

$$[a^2c^2 + b^2d^2 - p^2q^2][p^2 - q^2] = (a^2b^2 + c^2d^2)p^2 - (a^2d^2 + b^2c^2)q^2$$

$$\text{It implies } [(ac + bd)^2 - p^2q^2][p^2 - q^2] = [(ab + cd)^2 p^2 - (ad + bc)^2 q^2]$$

Which completes the proof of (10).

Now for the conclusion (11),

We compute [(16)+(17)+(18)+(19)].

Remark:

The conclusion (11) usually called as **Euler's four point relation** (In this article we proved this relation for a cyclic quadrilateral, actually it is true for any convex quadrilateral). [12]

Lemma-6

If X is any point on the circumcircle of the medial triangle DEF whose angles are A, B, C and area is $\frac{\Delta}{4}$ then

$$\sin 2A \cdot DX^2 + \sin 2B \cdot EX^2 + \sin 2C \cdot FX^2 = \Delta \quad (20)$$

$$(\sin 2B + \sin 2C)AX^2 + (\sin 2A + \sin 2C)BX^2 + (\sin 2A + \sin 2B)CX^2 = \Delta(a^2 + b^2 + c^2 - 5R^2) \quad (21)$$

Proof:

Using the lemma-2(3),

We have, If X is any point on the circumcircle of triangle ABC then

$$\sin 2A \cdot AX^2 + \sin 2B \cdot BX^2 + \sin 2C \cdot CX^2 = 4\Delta$$

Now for medial triangle DEF,

If X is any point on the Circumcircle then

$$\sin 2A \cdot DX^2 + \sin 2B \cdot EX^2 + \sin 2C \cdot FX^2 = 4 \frac{\Delta}{4}$$

It implies the conclusion (20).

Now for the conclusion (21),

We proceed as follows

By using lemma-3(5), (6), (7) Replace DX, EX, FX in (20)

We get

$$\sum 2(\sin 2B + \sin 2C)AX^2 - \sum a^2 \sin 2A = 4\Delta \quad (22)$$

Using (g), we have

$$a^2 \sin 2A + b^2 \sin 2B + c^2 \sin 2C = \frac{2\Delta}{R^2} (a^2 + b^2 + c^2 - 6R^2)$$

By replacing $\sum a^2 \sin 2A$ using (g) in (22) and by little algebra gives the conclusion (21).

Lemma-7

If triangle DEF is a medial triangle and the points K, L and M are the foot of perpendiculars drawn from the Vertices A, B and C of triangle ABC to the sides BC, CA and AB respectively then the 6 points D, E, F, K, L and M are concyclic.

Proof:

By using lemma-6(20),

If X is any point on the circumcircle of medial triangle DEF then

$$\sin 2A \cdot DX^2 + \sin 2B \cdot EX^2 + \sin 2C \cdot FX^2 = \Delta \quad (23)$$

$$\begin{aligned} DK &= |BK - BD| = \left| c \cos B - \frac{a}{2} \right| \sin 2A \cdot DK^2 + \sin 2B \cdot EK^2 + \sin 2C \cdot FK^2 = \sin 2A \left(c \cos B - \frac{a}{2} \right)^2 + \sin 2B \frac{b^2}{4} + \sin 2C \frac{c^2}{4} \\ &= \frac{1}{4} \left[\sum (a^2 \sin 2A) \right] + c \cos B \sin 2A (c \cos B - a) \\ &= \frac{1}{4} \left[\sum (a^2 \sin 2A) \right] - 4\Delta \cos A \cos B \cos C \end{aligned} \quad (24)$$

Now using (e) and (f) we can prove (24) is equal to Δ .

$$\Rightarrow \sin 2A \cdot DK^2 + \sin 2B \cdot EK^2 + \sin 2C \cdot FK^2 = \Delta$$

Hence using (23) we can conclude that the point K lies on the circumcircle of the triangle DEF, that is D, E, F and K are concyclic.

In the similar manner using (23) we can prove L and M also lie on the circumcircle of the triangle DEF.

That is the 6 points D, E, F, K, L and M are concyclic. (see figure 3).

Lemma- 8

If the triangle DEF is a medial triangle and the points T, U and V are the mid points of AO, BO and CO respectively where O is the orthocenter of the triangle ABC then the 6 points D, E, F, T, U and V are concyclic.

Proof:

By using lemma-6 (20),

If X is any point on the circumcircle of medial triangle DEF then

$$\sin 2A \cdot DX^2 + \sin 2B \cdot EX^2 + \sin 2C \cdot FX^2 = \Delta$$

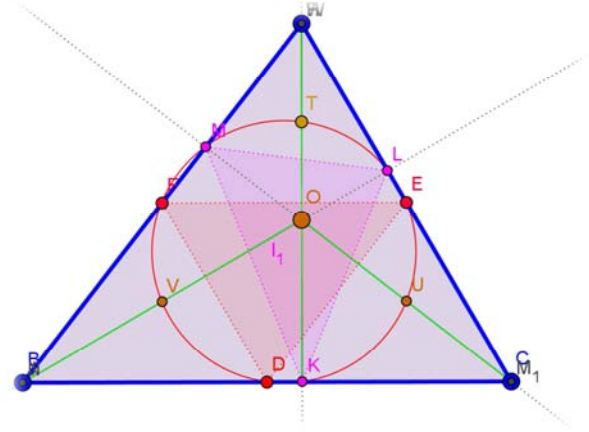


Figure 3. D, E, F, K, L and M are concyclic.

Clearly to prove D, E, F, K, L and M are concyclic, it is enough to prove that the X present in (23) is satisfied by K, L and M simultaneously.

Consider, $\sin 2A \cdot DK^2 + \sin 2B \cdot EK^2 + \sin 2C \cdot FK^2$

It is clear that from the triangles $\triangle BFK$ and $\triangle CEK$, $EK = CE = AE = \frac{b}{2}$, $FK = AF = BF = \frac{c}{2}$ (Mid Point of hypotenuse of a right triangle acts as circumcenter, so it is equidistant from the three vertices)

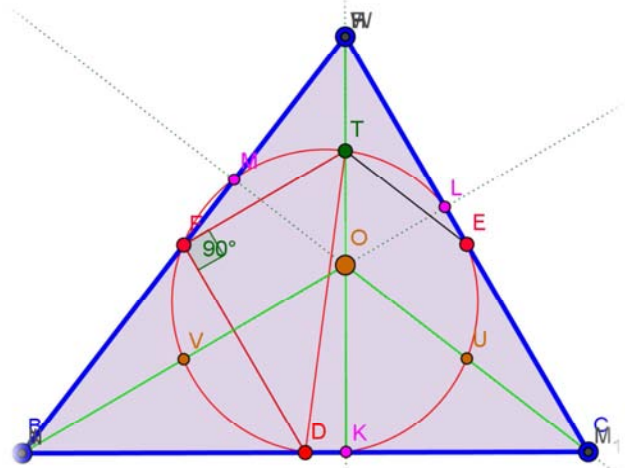


Figure 4. D, E, F, T, U and V are concyclic.

Clearly, to prove D, E, F, T, U and V are concyclic, it is enough to prove that the X present in (23) is satisfied by T, U and V simultaneously.

Consider,
 $\sin 2A \cdot DT^2 + \sin 2B \cdot ET^2 + \sin 2C \cdot FT^2$

Clearly In triangles ΔABO and ΔACO , the points F, E and T are the midpoints of sides AB, AC and AO.

So by the Mid Points Theorem, we can prove that the pairs of the lines (DF, AC), (FT, BO) and (ET, CO) are parallel.

It implies angle AFT = angle ABO = $90^\circ - A$ = angle AET = angle ACO.

And angle BFD = angle BAC = angle A.

$$DT^2 = DF^2 + FT^2 = \frac{b^2}{4} + R^2 \cos^2 B = R^2 \sin^2 B + R^2 \cos^2 B = R^2$$

$$\text{So } \sin 2A \cdot DT^2 + \sin 2B \cdot ET^2 + \sin 2C \cdot FT^2 = \sin 2A \cdot R^2 + \sin 2B \cdot R^2 \cos^2 C + \sin 2C \cdot R^2 \cos^2 B$$

$$= R^2 \left(\sum \sin 2A \right) - 2R^2 \sin B \sin C (\sin B \cos C + \sin C \cos B) \quad (25)$$

Now using (g) we can prove (25) is equal to Δ .

$$\Rightarrow \sin 2A \cdot DT^2 + \sin 2B \cdot ET^2 + \sin 2C \cdot FT^2 = \Delta$$

Hence using (23) we can conclude that the point T lies on the circumcircle of the triangle DEF, that is D, E, F and T are concyclic.

In the similar manner using (23) we can prove U and V also lie on the circumcircle of the triangle DEF.

That is the 6 points D, E, F, T, U and V are concyclic (see figure-4).

Hence proved.

Now we are in a position to deal with the proof of most celebrated theorems in a very prominent way.

4. Main Theorems

A). PTOELMY'S THEOREM

Let ABCD is any arbitrary cyclic quadrilateral such that AC and BD are its diagonals then $pq = ac + bd$

Proof:

Using lemma-4(9),

$$\text{We have } \frac{p}{q} = \frac{ad + bc}{ab + cd},$$

It implies

$$\left[(ab + cd)^2 p^2 - (ad + bc)^2 q^2 \right] = 0 \quad (26)$$

Now using lemma-5(10) we have

$$\left[(ac + bd)^2 - p^2 q^2 \right] \left[p^2 - q^2 \right] = \left[(ab + cd)^2 p^2 - (ad + bc)^2 q^2 \right]$$

By combining (10) and (26) we get $pq = ac + bd$

It completes the proof of the Ptolemy's Theorem.

B). NINE POINT CIRCLE THEOREM

Let D, E, F are the foot of the medians and K, L, M are the

It implies angle DFT = $180^\circ - \text{angle BFD} - \text{angle TFA} = 90^\circ$

And again by Mid Points Theorem we have

$$FT = \frac{BO}{2} = \frac{2R \cos B}{2} = R \cos B \text{ and } ET = \frac{CO}{2} = \frac{2R \cos C}{2} = R \cos C,$$

$$DF = \frac{AC}{2} = \frac{b}{2}$$

Now from triangle DFT, Since angle DFT = 90° ,

So by Pythagoras Theorem, we have

foot of altitudes of the triangle ABC and T, U, V are the midpoints of AO, BO and CO where O is orthocenter then D, E, F, K, L, M, T, U and V are concyclic.

Proof:

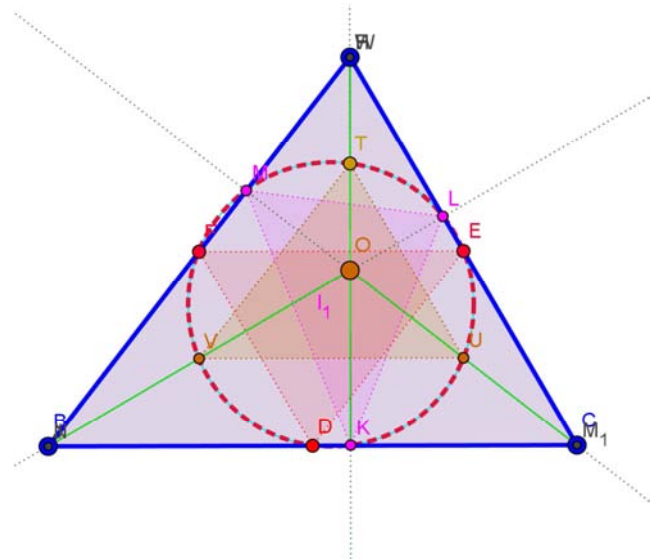


Figure 5. Nine Point Circle.

By combining lemma-7 and lemma-8 we can conclude that the 9 points D, E, F, K, L, M, T, U and V are concyclic (see figure -5).

It proves the Nine Point Circle Theorem.

5. Remarks

$$16\Delta^2 = 2a^2b^2 + 2a^2b^2 + 2a^2b^2 - a^4 - b^4 - c^4 \quad (27)$$

Proof:

We have by lemma-1(2)

$$SM^2 = \frac{1}{16\Delta^2} \left[a^2 (b^2 + c^2 - a^2) AM^2 + b^2 (a^2 + c^2 - b^2) BM^2 + c^2 (b^2 + a^2 - c^2) CM^2 - 16R^2 \Delta \right] \quad (28)$$

Since the conclusion (2) is true for any M, let us fix M as S (circumcenter),
By replacing $AS = BS = CS = R$ and $SM^2 = SS^2 = 0$,
We get

$$0 = \frac{1}{16\Delta^2} \left[a^2(b^2 + c^2 - a^2)R^2 + b^2(a^2 + c^2 - b^2)R^2 + c^2(b^2 + a^2 - c^2)R^2 - 16R^2\Delta \right]$$

Further simplification gives the conclusion (27).

1. If triangle ABC is an Equilateral and X be any point on the circumcircle of the triangle ABC then

$$AX^2 + BX^2 + CX^2 = \frac{8\Delta}{\sqrt{3}} = 6R^2 \quad (29)$$

Proof:

We have from lemma-2 (3),

If X is any point on the circumcircle of triangle ABC then

$$\sin 2A \cdot AX^2 + \sin 2B \cdot BX^2 + \sin 2C \cdot CX^2 = 4\Delta$$

If the triangle ABC is an equilateral triangle then angle A = angle B = angle C = 60°

$$\text{So } \sin 2A = \sin 2B = \sin 2C = \sin 120^\circ = \frac{\sqrt{3}}{2}$$

And we have area of an equilateral triangle $\Delta = \frac{\sqrt{3}}{4}a^2 = \frac{3\sqrt{3}}{4}R^2$, where a and R is the length of the side and circumradius of the triangle respectively.

By replacing these in (3), we get conclusion (29).

2. If X is a point on the circumcircle of triangle ABC which acts as a feet of the internal angular bisector of angle A then

$$\frac{AX}{IX} = \frac{\cos\left(\frac{B-C}{2}\right)}{\cos\left(\frac{B+C}{2}\right)} \quad (30)$$

$$\frac{AX^2 - 2R^2}{IX^2 - 2R^2} = \frac{\cos(B-C)}{\cos(B+C)} \quad (31)$$

$$AX = 2R\cos\left(\frac{B-C}{2}\right), IX = 2R\cos\left(\frac{B+C}{2}\right) \quad (32)$$

Proof:

Since X is the feet of internal angular bisector of angle A,

So by angle chasing we can prove that $BX = CX = IX$

Hence by Ptolemy's Theorem for the cyclic quadrilateral ABXC,

$$\text{We have } AX \cdot BC = AC \cdot BX + AB \cdot CX$$

Further simplification gives

$$\frac{AX}{IX} = \frac{AC + AB}{BC} = \frac{\sin B + \sin C}{\sin A} = \frac{\cos\left(\frac{B-C}{2}\right)}{\cos\left(\frac{B+C}{2}\right)}$$

Which proves (30)

Now for (31) we proceed as follows

We know by lemma-2(3),

If X is any point on the circumcircle of triangle ABC then

$$\sin 2A \cdot AX^2 + \sin 2B \cdot BX^2 + \sin 2C \cdot CX^2 = 4\Delta$$

Clearly $BX = CX = IX$

$$\text{So } \sin 2A \cdot AX^2 + (\sin 2B + \sin 2C) IX^2 = 4\Delta$$

Further simplification (by replacing $\sin 2A$ and 4Δ) using the transformations of the angles gives (31)

Now by combining (30), (31) we get (32).

3. If M is any point in the plane of the triangle ABC which minimizes the sum

$$\sin 2A \cdot AM^2 + \sin 2B \cdot BM^2 + \sin 2C \cdot CM^2 \text{ as } 2\Delta \text{ then M must be circumcenter (S).} \quad (33)$$

Proof:

We have by lemma-2,

$$SM^2 = \frac{R^2}{2\Delta} (\sin 2A \cdot AM^2 + \sin 2B \cdot BM^2 + \sin 2C \cdot CM^2 - 2\Delta)$$

Now Since the square of any real is nonnegative,
We have

$$SM^2 \geq 0$$

And it is clear that the equality holds when M coincides with S (circumcenter).

$$\text{It gives } \sin 2A \cdot AM^2 + \sin 2B \cdot BM^2 + \sin 2C \cdot CM^2 \geq 2\Delta$$

Which proves the required conclusion (33).

$$\sin 2A + \sin 2B + \sin 2C = \frac{2\Delta}{R^2} \quad (34)$$

Proof:

Using transformations of angles we can prove (34) which can be available in any academic trigonometry book.

But here we will prove this identity using lemma-1(1)

We have by lemma-1(1),

If S is the circumcenter of a triangle and M be any point in the plane of triangle then

$$SM^2 = \frac{R^2}{2\Delta} (\sin 2A \cdot AM^2 + \sin 2B \cdot BM^2 + \sin 2C \cdot CM^2 - 2\Delta)$$

It is true for any M, let us fix M as S(circumcenter), then

$$SS^2 = \frac{R^2}{2\Delta} (\sin 2A \cdot AS^2 + \sin 2B \cdot BS^2 + \sin 2C \cdot CS^2 - 2\Delta)$$

By replacing $AS = BS = CS = R$ and $SS=0$

We get conclusion (34).

In the similar manner we can derive some more remarks using the lemma's discussed earlier in section 3.

For few more generalizations and historical details about Ptolemy's theorem and Nine Point Circle Theorem refer [2], [4], [7], [8], [9], [10], [11], [13], [14], [15], [16], [19], [20] and [26].

6. Conclusion

This article has shown New and more elegant approach to prove that the give three points are concyclic, based on this approach we proved two famous theorems related to concyclic points. The proofs presented here only require basic knowledge of trigonometry and no advanced knowledge of synthetic projective geometry, the proofs are certainly an impressive tour –de-force of algebraic-trig manipulation and application. The approach which we dealt in this paper is also useful to study and investigate the further properties of concyclic points. We can extend this approach to prove the necessary condition for the points to be concyclic.

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Biography



Dasari Naga Vijay Krishna is the son of Sri D Adhishesu and Sri D Sulochana, He is renowned faculty in mathematics who has served in all the reputed Institutions in India with more than a decade's of experience in the training of the students for various competitive exams. Currently he is working as a senior lecturer in the Department of Mathematics, Narayana Educational Institutions. He received his B.Sc Honors Degree in Mathematics from Magadha University. He also holds Master Degree in Mathematics from Kalinga University. His primary research is Euclidean Geometry. His several research articles were published in reputed journals.