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Conjugacy Class Lengths of Finite Groups with Prime Graph a Tree

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Abstract: For a finite group G, we write $c\rho(G)$ to denote the prime divisor set of the various conjugacy class lengths of G and $c\sigma(G)$ the maximum number of distinct prime divisors of a single conjugacy class length of G. It is a famous open problem that $|c\rho(G)|$ can be bounded by $c\sigma(G)$. Let G be an almost simple group G such that the graph $\Gamma(G)$ built on element orders is a tree. By using Lucido's classification theorem, we prove $|c\rho(G)| = c\sigma(G)$ except possibly when G is isomorphic to $PSL_2(p^f)\langle\alpha\rangle$, where p is an odd prime and α is a field automorphism of odd prime order f. In the exceptional case, $|c\rho(G)| \le c\sigma(G)+2$. Combining with our known result, we also prove that for a finite group G with $\Gamma(G)$ a forest, the inequality $|c\rho(G)| \le 2c\sigma(G)$ is true.

Keywords: Prime Graph, Conjugacy Class Length, Almost Simple Group

1. Introduction

Throughout the paper, we only consider finite groups. For a finite group G, write $c\rho(G)$ to denote the prime divisor set of the various conjugacy class lengths of G and $c\sigma(G)$ the maximum number of different prime divisors of a single conjugacy class length of G. And $\pi(n)$ stands for prime divisor set of the positive integer n and write $\pi(G)$ for $\pi(|G|)$, here $|G|$ denotes the order of G. Considering some similarities between the influence of character degrees and conjugacy class lengths on groups, in 1989, Huppert once asked [1] whether the inequality $|c\rho(G)| \le 2c\sigma(G)$ holds for every solvable group. It was shown in [2, 4, 6] that this is true when $\Gamma(G)$ is at most 3. Specifically, the case G is proven in [4], the case $\pi(G)$ for the solvable G is in [6], and the cases pq for the nonsolvable $\pi_i, i=1,2,\cdots,$ and $\Gamma(G)$ are finished in [2], respectively. For solvable groups, it is proved $2 \in \pi_1$ in [7], and an improved version G in [20]. Generally, Casolo proved in [1, Corollary 2] that the inequality is true except possibly when $\Gamma(G)$ is $\Gamma(G)$ nilpotent with abelian Sylow G subgroups for at least two prime divisors of $\Gamma(G)$. And yet Casolo and Dolfi in [3, Example 2] show the inequality is invalid by constructing an infinite family of group examples $|c\rho(G)| = c\sigma(G)$. Specifically, the quotient G approaches $PSL_2(p^f)\langle\alpha\rangle$ (from below) as p approaches infinity. The prime number α divides each f. When take $|c\rho(G)| \le c\sigma(G)+2$, we may obtain an infinity family of counterexamples such that each G is metabelian and super solvable and the constant is 3 in that inequality. Note that the subscripts $\Gamma(G)$ in these counterexamples are sufficient large. By observation of these counterexamples, in 1998, Zhang further conjectured [20] the weak version of that inequality should hold by saying that "*Now, it seems true that* $|c\rho(G)| \le 2c\sigma(G)$ *for any finite solvable group and probably also for any finite group*." We attach a prime graph $\Gamma(G)$ to a finite group G: its vertices are $\pi(G)$, and any two vertices are adjacent by an edge just when G has an element of order pq. We use $\pi_i, i=1,2,\cdots,$ to denote the connected components of the graph $\Gamma(G)$, in particular, $2\in\pi_1$ if G is of even order. Furthermore, the graph $\Gamma(G)$ is called a tree when it is a connected graph without any loop; and $\Gamma(G)$ is called a forest when each connected component is a tree. In this note, we prove the following results.

Theorem A. *Suppose that* G *is an almost simple group such that* $\Gamma(G)$ *is a tree. Then* $|c\rho(G)|=c\sigma(G)$ *except possibly when* G *is isomorphic to* $PSL_2(p^f)\langle\alpha\rangle$, *where* p *is an odd prim and* α *is a field automorphism of odd prime order* f. *In the exceptional case,* $|c\rho(G)|\le c\sigma(G)+2$.

Theorem B. *Suppose that* G *is a finite group such that* $\Gamma(G)$ *is a forest. Then* $|c\rho(G)|\le 2c\sigma(G)$.

In the proof of Theorem A, we apply the classification result due to Lucido (Theorem 2.2). In the process, GAP [6] plays a crucial role. In essence, we use finite simpl egroup classification theorem.

Unless otherwise specified, we adopt the standard notation and terminology as presented in [9].

2. Preliminaries

The following fact is useful and basic on conjugacy class lengths, which will be frequently applied without reference.

Lemma 2.1 ((Lemma 33.2 of [9])). Let N be a normal subgroup of G and $\bar{G}=G/N$. Then

1. $|x^N|$ divides $|x^G|$ for any $x\in N$, and so $c\sigma(N)\le c\sigma(G)$.
2. $|\bar{y}^{\bar{G}}|$ divides $|y^G|$ for any $y\in G$, and thus $c\sigma(\bar{G})\le c\sigma(G)$.

Theorem 2.2. If G is an almost simple group with $\Gamma(G)$ a tree, then G is one of the following types of groups.

1. $Aut(A_6)$ for the alternating group A_6 of degree 6.
2. $PSL_2(p^f)\langle\alpha\rangle$ such that $p>3$ is a prime, f is an odd prime and α is a field automorphism of order f.
3. $PGL_4(3)\le G\le Aut(PSL_4(3))$.
4. $Aut(B_2(3))$.
5. $PSU_4(3)\langle\delta\rangle\le G\le Aut(PSU_4(3))$, where δ is a diagonal automorphism of order 2.
6. $PGL_3(4)\le G\le PGL_3(4)\langle\alpha\rangle$ with α is a graph-field automorphism of order 2.
7. $PSL_3(q)\langle\alpha\rangle$, with $q=9,25,49$ and α a field automorphism of order 2.
8. $PGU_3(8)\le G\le Aut(PSU_3(8))$.

$S_Z(2^{2f^2})\langle\alpha\rangle$ with α a field automorphism of order f. Here f is an odd prime.

Proof. This is Lemma 3 of [11].

Lemma 2.3. Assume that G is a finite group with disconnected $\Gamma(G)$. Then the inequality $|c\rho(G)|\le 2c\sigma(G)$ is true.

Proof. This is Theorem A of [8].

Lemma 2.4. Let G be a finite group with $|\pi(G)|\le 8$. Then $|c\rho(G)|\le 2c\sigma(G)$.

Proof. By [2, 3, 5], the inequality is valid when $c\sigma(G)\le 3$. Otherwise, we see $c\sigma(G)\ge 4$ and so $|c\rho(G)|\le|\pi(G)|\le 8\le 2c\sigma(G)$, as wanted.

Theorem 2.5. Let G be a finite group such that $\Gamma(G)$ is a tree, then $|\pi(G)|\le 8$.

Proof. This is Theorem 6 of [11].

Lemma 2.6. Let G be an almost simple group over the nonabelian simple group S, then $\pi(G)=c\rho(G)$.

Proof. It is known $c\rho(G)=\pi(G/Z(G)$ by Theorem 33.4 of [9]. If G has a maximal central Hall subgroup $H>1$, then we can write $G=K\times H$. As S is a simple group, the intersection $S\cap H$ is trivial. It follows that $|S|$ divides $|K|$, and so $S\le K$. We obtain H acts trivially on S, however, this is a contradiction since $G\le Aut(S)$. Therefore H is trivial an $\pi(G)=\pi(G/Z(G))$.

3. Main Results

Theorem 3.1. Suppose that G is an almost simple group such that $\Gamma(G)$ is a tree. Then $|c\rho(G)|=c\sigma(G)$ except possibly when G is isomorphic to $PSL_2(p^f)\langle\alpha\rangle$, where p is an odd prime and α is a field automorphism of odd prime order f. In the exceptional case, $|c\rho(G)|\le c\sigma(G)+2$.

Proof. We apply Theorem 2.2 above to prove the claims.

If G is isomorphic to $Aut(A_6)$, then we know via GAP [6] that $Z(G)$ is trivial and $c\sigma(G)=3=|\pi(G)|$, thus we obtain $|c\rho(G)|=c\sigma(G)$, as desired.

Assume that G is isomorphic to $PSL_2(p^f)\langle\alpha\rangle$. Here $p>3$ is an odd prime and α is a field automorphism of order f which is also an odd prime. By Lemma 4.1of [10], we know $Z(G)$ is trivial and so $c\rho(G)=\pi(G)$. It is known $|PSL_2(p^f)|=(p^f-1)p^f(p^f+1)/2$ when p is an odd prime. As in [4], denote by d an element of order p in

$SL_2(p^f)$ and by z the element of order 2 in the centre of $SL_2(p^f)$. Then it is known that the class size of $d\langle z\rangle$ in $PSL_2(p^f)$ is $(p^f-1)(p^f+1)/2$ (which is the conjugacy class size corresponding to $c\sigma(PSL_2(p^f))$, and so we conclude that $|c\rho(PSL_2(p^f))| = c\sigma(PSL_2(p^f))+1$ for odd p. The field automorphism α indeed leave the class of $d\langle z\rangle$ invariant by [10, Lemma 4.1]. We further get

$$|c\rho(G)| = |\pi(G)| \le |\pi(PSL_2(p^f))|+1$$

$$= c\sigma(PSL_2(p^f))+2 \le c\sigma(G)+2$$

for odd p. Note that f is an odd prime.

Assume now that $PGL_4(3) \le G \le Aut(PSL_4(3))$. Then we see

$$c\sigma(PGL_4(3)) \le c\sigma(G) \le c\sigma(Aut(PSL_4(3)))$$

and

$$c\rho(PGL_4(3)) \subseteq c\rho(G) \subseteq c\rho(Aut(PSL_4(3))),$$

By using GAP [6], we know that $PGL_4(3)$ has 43 conjugacy class lengths,

$c\rho(PGL_4(3)) = \{2,3,5,13\}$ and $c\sigma(PGL_4(3))=4$; and $Aut(PSL_4(3))$ has 56 conjugacy class lengths, $c\rho(Aut(PSL_4(3))) = \{2,3,5,13\}$ and $c\sigma(Aut(PSL_4(3)))=4$. Hence we obtain $|c\rho(G)| = c\sigma(G)$, as required.

If G is isomorphic to $Aut(B_2(3))$, then the application of GAP yields that G has twenty conjugacy class lengths, $c\rho(G)=\{2,3,5\}$ and $c\sigma(G)=3$, and so $|c\rho(G)| = c\sigma(G)$, as wanted.

Next, consider the case that $PSU_4(3)\langle\delta\rangle \le G \le Aut(PSU_4(3))$, where δ is a diagonal automorphism of order 2. Using GAP, we reach that $PSU_4(3)$ has 20conjugacy class lengths, $c\rho(PSU_4(3))=\{2,3,5,7\}$ and $c\sigma(PSU_4(3))=4$. Its automorphism group $Aut(PSU_4(3))$ has 61 conjugacy class lengths, $c\rho(Aut(PSU_4(3))) = \{2,3,5,7\}$ and $c\sigma(Aut(PSU_4(3)))=4$. As G is an almost simple group, we achieve that $c\rho(PSU_4(3)) \subseteq c\rho(G) \subseteq c\rho(Aut(PSU_4(3)))$ and $c\sigma(PSU_4(3)) \le c\sigma(G) \le c\sigma(Aut(PSU_4(3)))$, thus $|c\rho(G)| = 4 = c\sigma(G)$, as desired.

Now, suppose that $PGL_3(4) \le G \le PGL_3(4)\langle\alpha\rangle$ with α is a graph-field automorphism of order 2. As α is of order 2, we attain G is either $PGL_3(4)$ or else $PGL_3(4)\langle\alpha\rangle$. When $G = PGL_3(4)$, by GAP, we know $\pi(PGL_3(4)) = \{2,3,5,7\} = c\rho(PGL_3(4))$ and $c\sigma(G)=4$, and so $|c\rho(G)| = c\sigma(G)$. When $G = PGL_3(4)\langle\alpha\rangle$, we also get that $|c\rho(G)| = |\pi(G)| = c\sigma(G)$ because α is of order 2 which is in $\pi(PGL_3(4))$.

The next case is $PSL_3(q)\langle\alpha\rangle$, with $q=9,25,49$ and α a field automorphism of order 2. For $PSL_3(9)$, we get via GAP that $\pi(PSL_3(9)) = \{2,3,5,7,13\} = c\rho(PSL_3(9))$ and $c\sigma(PSL_3(9))=5$. Also since $c\rho(PSL_3(9)\langle\alpha\rangle) \subseteq \pi(PSL_3(9)\langle\alpha\rangle) = \pi(PSL_3(9))$ this yields $|c\rho(PSL_3(9)\langle\alpha\rangle)| = c\sigma(PSL_3(9)\langle\alpha\rangle)$. For $PSL_3(25)$, by GAP, we attain that $\pi(PSL_3(25)) = \{2,3,5,7,13,31\} = c\rho(PSL_3(25))$ and $c\sigma(PSL_3(25))=6$. Also $c\rho(PSL_3(25)\langle\alpha\rangle) \subseteq \pi(PSL_3(25)\langle\alpha\rangle) = \pi(PSL_3(25))$, this implies $|c\rho(PSL_3(25)\langle\alpha\rangle)| = c\sigma(PSL_3(25)\langle\alpha\rangle)$.

For $PSL_3(49)$, we get by GAP that $\pi(PSL_3(49)) = \{2,3,5,7,19,43\} = c\rho(PSL_3(49))$ and $c\sigma(PSL_3(49))=6$. Also $c\rho(PSL_3(49)\langle\alpha\rangle) \subseteq \pi(PSL_3(49)\langle\alpha\rangle) = \pi(PSL_3(49))$, this shows $|c\rho(PSL_3(49)\langle\alpha\rangle)| = c\sigma(PSL_3(49)\langle\alpha\rangle)$, as desired.

Assume that $PGU_3(8) \le G \le Aut(PSU_3(8))$. By using GAP, it follows that $\pi(PSU_3(8)) = \pi(Aut(PSU_3(8))) = c\rho(PSU_3(8)) = \{2,3,7,19\}$ and $c\sigma(PSU_3(8))=4$. Note that G is an almost simple group with $PSU_3(8) \le G \le Aut(PSU_3(8))$, thus $|c\rho(G)| = c\sigma(G)$.

Assume finally that G is isomorphic to $S_Z(2^{2f^2})\langle\alpha\rangle$, where α is a field automorphism of odd prime order f. Let Q be a Sylow 2-subgroup of $S = S_Z(2^{2f^2})$.

By Proposition 1 of [12], we see $C_S(g) \le Q$ for any nontrivial element $g \in Q$.

Using Lemmas 1 and 2 of [12], we may pick the noncentral element $g=(\alpha,\beta,x) \in S(q;x)$, which is indeed a specific form of Q (by [12, Theorem 7]). Here neither of α or β can equal 0, 1. Then the centralizer $C_S(g)$ is properly contained in Q, and so $\pi(S) = \pi\ |g^S|$. If f belongs to $\pi(S)$, then $|c\rho(G)| = c\sigma(G)$. Otherwise, $\langle\alpha\rangle$ is a Sylow f-subgroup. By [7, Theorem 9], we have $\pi(G) = \pi(|g^G|)$ and so $|c\rho(G)| = c\sigma(G)$. The whole proof is complete.

Some remarks on $c\sigma(PSL_2(p^f))$ are made here. By [4], we see that $c\sigma(PSL_2(p^f)) = \left|\pi(p^{2f}-1)\right|$ for the odd $p^f > 5$ and $c\sigma(PSL_2(p^f)) = \left|\pi(2^{2f}-1)\right|$ for $2^f > 4$.

Lemma 2 of [11] yields $\left|\pi(p^{2f}-1)\right| \geq 3$ unless $p^f = 7,8,9,17$. Some direct calculations by GAP show that $c\sigma(PSL_2(7))$, $c\sigma(PSL_2(9))$ and $c\sigma(PSL_2(17))$ are all equal to 3, but $c\sigma(PSL_2(5))$ and $c\sigma(PSL_2(8))$ are 2. Observe that $PSL_2(4) \cong PSL_2(5) \cong A_5$, $c\rho(PSL_2(5)) = \{2,3,5\}$ and $c\rho(PSL_2(8)) = \{2,3,7\}$, moreover $c\rho(Aut(PSL_2(8))) = \{2,3,7\}$.

Theorem 3.2. Suppose that G is a finite group such that $\Gamma(G)$ is a forest. Then

$$\left|c\rho(G)\right| \leq 2c\sigma(G).$$

Proof. If $\Gamma(G)$ is disconnected, then Lemma 2.3 yields the result. Otherwise, $\Gamma(G)$ is a tree, and so $\left|\pi(G)\right| \leq 8$ (by Lemma 2.5). Applying Lemma 2.4, we get the result.

In this note, we show Huppert's problem has affirmative answer for the finite group whose prime graph a forest, and even has better result when the group is an almost simple group with prime graph a tree.

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