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Thermodynamic Potentials Theory Aspects in External Differential Forms Calculus Representation

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Abstract: Thorough review of external differential forms calculus basic theses presented. Potentialities of this mathematical discipline, which can describe physical properties of dielectric materials, magnets and photonic materials influenced by mechanical, thermal and electromagnetic factors more logically and objectively, then traditional methods, demonstrated. Methodological effectiveness of the differential forms of thermodynamic potentials application in the macroscopic properties of homogeneous mono- and polyvariant systems description has been demonstrated. The simple, fundamental, symmetrical to the thermodynamic variables choice relations demonstrating the calculus of differential forms benefits have been obtained. Using Pfaffian forms thermodynamics, have been demonstrated, that differential forms calculus application to a description of the physical reality allows to operate physical concepts at a deeper level, based on the fundamental physical and mathematical principles.

Keywords: External Differential Forms, Thermodynamic Potentials, External Product, Maxwell's Identity, Photonic Materials

1. Introduction

The calculus of differential forms, which was created at the beginning of the XX century by E. Kartan, is one of the most fundamental and at the same time simple-to-use, mobile and fruitful mathematical method in differential geometry and its applications [1-3]. The universality of concepts and methodological simplicity are factors that confirm the fundamentality of differential forms theory. In the opinion of many mathematicians [1-4], such traditional mathematical tools, as vector, differential and integral calculus, which are the foundation of usual theoretical physics mathematical apparatus, are to a certain extent not full constructive transfer of the more fundamental mathematical constructions - external differential forms (see appendix).

The development of scientific thought has always sought to unify, simplicity and universality of physical concepts, which could be presented using the fundamental nature of the operator symbolism, easily and simply to operate with it. Ultimately, it leaded to the concept of external (or alternated) differential form [1-3].

Many thermal, mechanical, magnetic and electric properties of matter with mono- or polyvariant structure can be satisfactory described by thermodynamic language. A lot of macroscopic matter properties have been accounted by such a way. Thermodynamic approach was found successful from both fundamental and applied points of view. Methodology of thermodynamic potentials (also calling characteristic functions) widely used against standard thermodynamic language background [5-10].

At the same time many fundamental problems haven't due explanation because of traditional mathematic apparatus restrictions. On our opinion, using external differential forms calculus allows to expand thermodynamic language application field, to look to the standard relations from new point of view, consider ones on deeper scientific level.

Authors think, that using in the article mathematic apparatus application will take on more concrete sense and apprehend more adequate after thermodynamic axioms and laws consideration using direct differential calculus [5-10] and, at the same time, external differential forms calculus basic theses [1-4]. Such an approach allows to look deeper to the thermodynamic laws in the view of abstract vector analysis and its geometrical images, which show physical reality nature from one more fundamental side, describing in mathematical physics by external multiplication and external differentiation concepts (see Appendix). Motivation of external differential forms using dictating by this methodology effectiveness, meaning fundamentality and application's simplicity.

Confirmation of initial principles is this article which demonstrated the simplicity of obtaining already known results and provided obtaining new ones conceptual scheme.

2. Thermodynamic Potentials External Differential Forms Calculus

Let's consider a simple, homogeneous, placed in an external constant electric or magnetic field system. Its thermodynamic properties are investigated, using the theory of defined on consistent generalized thermodynamic forces and coordinates manifolds Pfaffian forms potentials.

The variables that characterize the forces denoted by the $X_\nu = \left\{T, P, \vec{E}, \vec{H}, \mu_i\right\}$ (here from left to right: temperature, pressure, external electric and magnetic field intensities, i-th matter component chemical potential); congaed to these forces coordinates denoted by $x_\nu = \left\{S, V, \vec{P}, \vec{M}, n_i\right\}$ (here, respectively: specific entropy, volume, electrical and magnetic polarization, the molar substance component concentrations).

Functionally dependence of the specific thermodynamic potentials (internal energy U, enthalpy W, free energy F, Gibbs potential G) at an appropriate extensive (additive) and intensive consistent variables manifold chosen in the form

$$U = U\left(S, V, \vec{P}, \vec{M}, n_i\right), W = W\left(S, P, \vec{E}, \vec{H}, n_i\right),$$
$$F = F\left(T, V, \vec{P}, \vec{M}, n_i\right), G = G\left(T, P, \vec{E}, \vec{H}, n_i\right)$$

We emphasize that the required functional relationship can be represented in another form for another problem conditions [5, 9, 10].

By themselves, in the differential forms calculus the thermodynamic functions are a 0-form. The action of the external differentiation operator $\tilde{d}$ at the 0-form transforms it into 1-form. This operator is similar to the internal differentiation one, but it has some features (see App.). Each 1-form of potential after formal substitution operator to an ordinary differential operator $\tilde{d}$ will produce a corresponding certain potential Pfaffian form [1-8]. In particular, the internal energy external differential is

$$\tilde{d}U = \frac{\partial U}{\partial S}\tilde{d}S + \frac{\partial U}{\partial V}\tilde{d}V + \frac{\partial U}{\partial x_\nu}\tilde{d}x_\nu + \frac{\partial U}{\partial n_i}\tilde{d}n_i$$

Here $\tilde{d}U$ is a counterpart of usual differential, and partial derivatives are generalized forces. So, let's act to the relevant 0-form by the operator $\tilde{d}$ and obtain 1-form of the thermodynamic potentials

$$\tilde{d}U = T\tilde{d}S - P\tilde{d}V + X_\nu \tilde{d}x_\nu + \mu_i \tilde{d}n_i \quad (1)$$

$$\tilde{d}W = T\tilde{d}S + V\tilde{d}P - x_\nu \tilde{d}X_\nu + \mu_i \tilde{d}n_i \quad (2)$$

$$\tilde{d}F = -S\tilde{d}T - P\tilde{d}V + X_\nu \tilde{d}x_\nu + \mu_i \tilde{d}n_i \quad (3)$$

$$\tilde{d}G = -S\tilde{d}T + V\tilde{d}P - x_\nu \tilde{d}X_\nu + \mu_i \tilde{d}n_i \quad (4)$$

Everywhere (in particular, in (1-4)) meaning summation on doubled indices. Take into account $G = \mu_i N_i$ (or, for specific values, $G = \mu_i n_i$) [5, 6], acting operator $\tilde{d}$ to this equation and obtaining $\tilde{d}G = n_i \tilde{d}\mu_i + \mu_i \tilde{d}n_i$. The basic equation for 1-forms is obtained substituting this expression in (4):

$$S\tilde{d}T - V\tilde{d}P + x_\nu \tilde{d}X_\nu + n_i \tilde{d}\mu_i = 0 \quad (5)$$

relating only intensive variable, and the only extensive quantities are the parameters [5, 6, 8] (extended equation of Duhem-Gibbs in the new terminology).

Note that the thermodynamic potentials are the full differentials in the sense of usual differential calculus, that is why differential calculus corresponding differential form in the sense of external differential calculus is closed (or précising) [1, 2]. Therefore, considerating basic property of double using the operator $\tilde{d}$, we apply the external differential operator to the relations (1-5) again. In consequence occur vanishing 1-form: $\tilde{d}\left(\tilde{d}U\right) = \tilde{d}\left(\tilde{d}W\right) = \tilde{d}\left(\tilde{d}F\right) = \tilde{d}\left(\tilde{d}G\right) = 0$. Take into account anticommutation rules $\tilde{d}S \wedge \tilde{d}T = -\tilde{d}T \wedge \tilde{d}S$ etc., too. In consequence we obtain the basic equation of the external differential calculus thermodynamic potentials theory

$$\tilde{d}T \wedge \tilde{d}S - \tilde{d}P \wedge \tilde{d}V + \tilde{d}X_\nu \wedge \tilde{d}x_\nu + \tilde{d}\mu_i \wedge \tilde{d}n_i = 0 \quad (6)$$

Note that this equation has balanced, "symmetrical" to the differentials, form.

Based on the rules of the external differentiation, from the basic equation (6) we can easily get all the known thermodynamic relations between describing the macroscopic properties of the material characteristic thermodynamic factors. These relations traditionally determines on the Pfaffian forms of characteristic functions (corresponding thermodynamic potentials) basis [5-8]. For example, let's consider only thermal and mechanical variables in (6) (i.e. assuming $\tilde{d}X_\nu \wedge \tilde{d}x_\nu = \tilde{d}\mu_i \wedge \tilde{d}n_i = 0$), we obtain the truncated ratio

$$\tilde{d}T \wedge \tilde{d}S - \tilde{d}P \wedge \tilde{d}V = 0 \quad (7)$$

The 0-forms for the temperature and pressure are defined using the manifold (basis) of variables (S, V):

$$T = T(S,V), P = P(S,V) \tag{8}$$

From the 0-forms (8) we obtain 1-forms

$$\tilde{d}T = \left(\frac{\partial T}{\partial S}\right)_V \tilde{d}S + \left(\frac{\partial T}{\partial V}\right)_S \tilde{d}V \tag{9}$$

$$\tilde{d}P = \left(\frac{\partial P}{\partial S}\right)_V \tilde{d}S + \left(\frac{\partial P}{\partial V}\right)_S \tilde{d}V \tag{10}$$

Substituting (9) and (10) into (7) and taking into account forms properties (particularly, anticommutation: $\tilde{d}S \wedge \tilde{d}S = \tilde{d}V \wedge \tilde{d}V = 0, \tilde{d}S \wedge \tilde{d}V = -\tilde{d}V \wedge \tilde{d}S \neq 0$), (7) leads to the form

$$\left(\frac{\partial T}{\partial V}\right)_S \left(\tilde{d}V \wedge \tilde{d}S\right) + \left(\frac{\partial P}{\partial S}\right)_V \left(\tilde{d}V \wedge \tilde{d}S\right) = 0 \tag{11}$$

From (11) we obtain the well-known Maxwell's relation

$$\left(\frac{\partial T}{\partial V}\right)_S = -\left(\frac{\partial P}{\partial S}\right)_V \tag{12}$$

Using Jacobians technique [5, 6], it is possible to show (12) either as

$$\frac{\partial(T,S)}{\partial(V,S)} = \frac{\partial(P,V)}{\partial(V,S)} \tag{13}$$

or as the calibration ratio [7]

$$\frac{\partial(T,S)}{\partial(P,V)} = 1 \tag{14}$$

Using pair of variables (S, P), (T, V), (T, P) as a basis, after a conversion, similar to demonstrated above, using the external differential forms calculus technique, obtain the corresponding Maxwell's relations, that can be reduced to calibration (14) by Jacobians technique.

To make an analysis of the homogeneous, placed in an external field system, you should consider an appropriate combination of the paired members (6), or the corresponding 2-form. For example, to examine together thermal and mechanical properties in general case (in presence of an electric and magnetic fields), you should consider 2-form (6) (with $\tilde{d}\mu_i \wedge \tilde{d}n_i = 0$)).

The simplest and most accessible to experimental verification are 4-dimensional 2-forms (second degree forms in R^4).

For example we can explore obtained from (6) 2-form, describing the mechanical behavior of dielectric in an electric field

$$\tilde{d}P \wedge \tilde{d}V - \tilde{d}\vec{P} \wedge \tilde{d}\vec{E} = 0 \tag{15}$$

and magnetic material in a magnetic field

$$\tilde{d}P \wedge \tilde{d}V - \tilde{d}\vec{M} \wedge \tilde{d}\vec{H} = 0 \tag{16}$$

Similarly, we can consider only the thermal properties of the dielectric and magnetic material respectively on the base of the corresponding 2-forms

$$\tilde{d}T \wedge \tilde{d}S - \tilde{d}\vec{E} \wedge \tilde{d}\vec{P} = 0 \tag{17}$$

$$\tilde{d}T \wedge \tilde{d}S - \tilde{d}\vec{H} \wedge \tilde{d}\vec{M} = 0 \tag{18}$$

We consider the most known relations for the dielectric (15) and magnet (16) in the isotropic case.

Solving equation (15). Similarly to operations (9)-(14) used to solve (7), select serially bases $\left(P,\left|\vec{P}\right|\right),\left(P,\left|\vec{E}\right|\right),\left(V,\left|\vec{P}\right|\right),\left(V,\left|\vec{E}\right|\right)$. Finally in each case obtain the calibration

$$\frac{\partial(V,P)}{\partial(|\vec{P}|,|\vec{E}|)} = 1 \tag{19}$$

Using the calibration (19) and the Jacobian technique [5, 6], we can obtain any ratio between the characteristic coefficients for given thermodynamic variables and field conditions. For example, multiplying (19) on unit represented as $1 = \frac{\partial(P,|\vec{P}|)}{\partial(P,|\vec{P}|)}$, which can be formally regarded as a fraction, obtain the relation

$$\frac{\partial(V,P)}{\partial(|\vec{P}|,|\vec{E}|)} \cdot \frac{\partial(P,|\vec{P}|)}{\partial(P,|\vec{P}|)} = 1$$

After obvious transformations it leads to the form

$$\frac{\partial(V,P)}{\partial(P,|\vec{P}|)} \cdot \frac{\partial(P,|\vec{P}|)}{\partial(|\vec{P}|,|\vec{E}|)} = \frac{\partial(V,P)/\partial(P,|\vec{P}|)}{\partial(|\vec{P}|,|\vec{E}|)/\partial(P,|\vec{P}|)} = 1$$

Hence we receive connection

$$\frac{\partial(V,P)}{\partial(|\vec{P}|,P)} = \frac{\partial(|\vec{P}|,|\vec{E}|)}{\partial(|\vec{P}|,P)},$$

which can be written in the traditional form

$$\left(\frac{\partial V}{\partial|\vec{P}|}\right)_P = \left(\frac{\partial|\vec{E}|}{\partial P}\right)_{|\vec{P}|} \text{ or } \left(\frac{\partial P}{\partial|\vec{E}|}\right)_{|\vec{P}|} = \left(\frac{\partial|\vec{P}|}{\partial V}\right)_P \tag{20}$$

For the magnetic material mechanical properties study (see (16)) in the adiabatic or temperature constancy case ($\tilde{d}T \wedge \tilde{d}S = 0$) the following calibration is obtained:

$$\frac{\partial(V,P)}{\partial(|\vec{M}|,|\vec{H}|)} = 1 \tag{21}$$

Based on the calibration ratio (21), considering the pair of variables $\left(P,\left|\vec{M}\right|\right),\left(P,\left|\vec{H}\right|\right),\left(V,\left|\vec{M}\right|\right),\left(V,\left|\vec{H}\right|\right)$ and using a Jacobians method, it is possible to obtain all the thermodynamic relations between mechanical-magnetic coefficients characterizing the system. In particular, if in the case (15) choose variables $\left(P,\left|\vec{E}\right|\right)$, and in the case (16) – variables $\left(P,\left|\vec{H}\right|\right)$, we obtain the well-known ratio

$$\left(\frac{\partial V}{\partial\left|\vec{E}\right|}\right)_P=-\left(\frac{\partial\left|\vec{P}\right|}{\partial P}\right)_{\left|\vec{E}\right|};\left(\frac{\partial V}{\partial\left|\vec{H}\right|}\right)_P=-\left(\frac{\partial\left|\vec{M}\right|}{\partial P}\right)_{\left|\vec{H}\right|} \qquad (22)$$

These relations (Maxwell's identity) can be found by the standard thermodynamic approach on the base of differentials fullness condition [5-8].

Obviously, the relations (22) define a volume change caused by electric and magnetic fields respectively: electrostriction $\left(\frac{\partial V}{\partial\left|\vec{E}\right|}\right)_P$ and magnetostriction $\left(\frac{\partial V}{\partial\left|\vec{H}\right|}\right)_P$ The latter are connected with electroelastc $\left(\frac{\partial\left|\vec{P}\right|}{\partial P}\right)_{\vec{E}}$ and magnetoelastic $\left(\frac{\partial\left|\vec{M}\right|}{\partial P}\right)_{\vec{H}}$ effects. In the absence of external fields ($\vec{E}=0$ and $\vec{H}=0$) e) electric and magnetic effects caused by elastic forces is called [5] piezoelectric and piezomagnetic respectively.

3. Remarks

Let's make a remark about the methodology of thermodynamic potentials. Maxwell's relations, obtained in a standard Pfaffian forms calculus as a result of characteristic functions mixed derivatives equality, usually connected some values describing the mechanical, thermal etc. system properties [5, 6, 8]. The establishment of such relations is a content of the thermodynamic potentials method. For example, thermodynamic potential derivatives in the T, S, P, V variables determine the thermal, adiabatic, isochoric, isobaric, caloric system parameters characterizing its thermal and mechanical properties. The relationship between these parameters can be determined basing on different potentials. At the same time, the thermodynamic potential only in their own variables satisfies the differential fullness condition and is a real characteristic function of them.

Calibration relations have certain universal character, and in the majority of cases they are invariant to the variables change. Calibration violation is an indication of matter abnormal properties in relevant thermodynamic variables space points (for instance, water [5, 6]).

Additionally we'll make a brief summary, based on the provisions of [1-3], in order to more fully revealing the meaning operations used in the paper.

Note the differential forms calculus fundamental provisions and the obvious comparisons.

Note the following about comparison. In three-dimensional space R^3 external multiplication operation may be associated with the vector multiplication operation in the standard vector calculus. Accordingly, the external differentiation operator $\tilde{d}$, acting on the 1-form (p=1) in three-dimensional space, associated with the rotor of the vector field.

The $\tilde{d}$ and Λ operations fundamental properties are the following.

Operator $\tilde{d}$ converting form to another form, increasing its degree per unit – this is its main property. So if $\tilde{\varphi}$ is a form, then $\tilde{d}\tilde{\varphi}$ is a form too, and its degree is one unit higher.

Next integral-differential rule, that is correct for the forms of degrees 0 and 1, holds for higher degrees too. In a one-dimensional space (n = 1) operator $\tilde{d}$ turns a 0-form (p=0) to such a 1-form (p=1): $\tilde{d}f(x)=\frac{df}{dx}\tilde{d}x$. For the latter standard integral calculus basic formula is true:

$$\int_a^b\tilde{d}f\equiv\int_a^b df=f(b)-f(a)$$

In the differential forms calculus operator's $\tilde{d}$ action to forms of high degrees is similar to one's action to 0-form. If $\tilde{\varphi}(p)$ is a form of degree p, then $\tilde{d}\tilde{\varphi}(p)=\tilde{\psi}(p+1)$. For forms $\tilde{\varphi}(p)$ and $\tilde{\psi}(q)$ of the p and q degree respectively, the following basic relations are satisfied:

$$\begin{aligned}
&\tilde{\varphi}\Lambda\tilde{\psi}=(-1)^{pq}\tilde{\psi}\Lambda\tilde{\varphi}\\
&\tilde{d}(\tilde{\varphi}+\tilde{\psi})=\tilde{d}\tilde{\varphi}+\tilde{d}\tilde{\psi}\\
&\tilde{d}(\tilde{\varphi}\Lambda\tilde{\psi})=\tilde{d}\tilde{\varphi}\Lambda\tilde{\psi}+(-1)^p\tilde{\varphi}\Lambda\tilde{d}\tilde{\psi}\\
&\tilde{d}(\tilde{d}\tilde{\varphi})=0
\end{aligned}$$

The article listed the differential forms calculus provisions have been applied to the thermodynamic potentials and their differentials, which can be regarded as a 0-form and 1-forms, respectively. At the same time the second external differentials of these functions, according to the operator's $\tilde{d}$ general properties, vanish [1, 2].

4. Conclusions

This paper is given a visual representation of the vector calculus fundamental nature, which is essentially based on the external differential forms calculus. As an example of the required formalism chosen Pfaffian forms methodology, used in thermodynamic potentials theory. The mathematical simplicity of forms using and the efficiency of obtaining physical results is shown.

In the article fundamental relationship between external differential form calculus and abstract vector analysis principles have been demonstrated. This apparatus is a generalization of standard differential and integral calculus.

Article shows potentials of external differential forms using for analyze electromagnetic fields influence on condensed media, composite materials and compound high-molecular systems, in particular photonic materials.

Using mathematical apparatus methodological peculiarities, proving its fundamentality, which causes one's academic necessity and practical expedience, demonstrated.

Authors think [11], that due to the fundamental nature of the external differential forms calculus apparatus and based on geometrical principles (adequate the describing physical reality nature) concepts application using differential forms calculus as a method of mentioned reality study in the near future will be, as projected in the literature cited in an article, the needed fundamental mathematical tool in the making steps towards understanding the laws of nature investigators arsenal.

Appendix

1. Basic Theses

To refining using material, following [3], let's consider ordered set $(\vec{\xi}_1, \vec{\xi}_2, ..., \vec{\xi}_p)$ of p vectors. Let exists function $a(\vec{\xi}_1, \vec{\xi}_2, ..., \vec{\xi}_p)$, which compares real number to such a set of vectors. This function named polylinear form of p degree (or p-form), if it is a linear form for each argument.

If the polylinear form changes its sign when any pair of arguments permute, it named skew-symmetric (anti-symmetric, alternating) form:

$$a(\vec{\xi}_1, ..., \vec{\xi}_i, ...\vec{\xi}_j, ..., \vec{\xi}_p) = -a(\vec{\xi}_1, ..., \vec{\xi}_j, ...\vec{\xi}_i, ..., \vec{\xi}_p)$$

Representation of arbitrary polylinear form $a(\vec{\xi}_1, \vec{\xi}_2, ..., \vec{\xi}_p)$ in arbitrary orthonormalized basis of some n-dimension vector space V defined as

$$a(\vec{\xi}_1, \vec{\xi}_2, ..., \vec{\xi}_p) = \sum_{i_1=1}^{n} ... \sum_{i_p=1}^{n} a_{i_1...i_p} \xi_1^{i_1} ... \xi_1^{i_p} ,$$

where $a_{i_1...i_p} = a_{i_1...i_p}(\vec{e}_{i_1}, ..., \vec{e}_{i_p})$ - some numbers, $\left\{\xi_1^{i_1}, ..., \xi_p^{i_p}\right\}$ - vectors $\vec{\xi}$ components in this basis: $\vec{\xi}_k = \sum_{i=1}^{n} \xi_k^i \vec{e}_i ; k = 1, ..., p$.

The special case is a polylinear skew-symmetric form, which can be represented by expansion on given basis:

$$\omega = \omega(\vec{\xi}_1, \vec{\xi}_2, ..., \vec{\xi}_p) = \sum_{i_1=1}^{n} ... \sum_{i_p=1}^{n} \omega_{i_1...i_p} \xi_1^{i_1} ... \xi_1^{i_p}$$

In this case numbers $\omega_{i_1...i_p} = \omega_{i_1...i_p}(\vec{e}_{i_1}, ..., \vec{e}_{i_p})$ change their signs when indexes pair permutes.

Main operation in alternating form theory is operation of exterior multiplicity. It needs of some comments.

Let's consider polylinear form, which is simply a product of two skew-symmetric p- and q-dimension forms:

$$a(\vec{\xi}_1, ..., \vec{\xi}_{p+q}) = \omega^p(\vec{\xi}_1, ..., \vec{\xi}_p) \cdot \omega^q(\vec{\xi}_{p+1}, ..., \vec{\xi}_{p+q})$$

Generally, this form is not alternating, because when arguments $\vec{\xi}_i$ and $\vec{\xi}_j$ (where 1≤i≤p, p+1≤j≤p+q) permute, resulting form may change not only a sign, but a module too. Just this fact leaded to introducing exterior product concept.

This concept is connected with a permutations theory. Let σ(k) – some permutation of numbers set $k = \{1, ..., m\}$. A set of such permutations denoted as Σ_m. For two different permutations σ and τ from this set exists their superposition $\sigma\tau \in \Sigma_m$. For any permutation σ exists inverse one σ^{-1}, satisfying $\sigma\sigma^{-1}=\sigma\sigma^{-1}=\varepsilon$, where ε – identical permutation. Permutation which exchange only two numbers and keeps all otters fixed, named transposition. For transposition there is $\sigma=\sigma^{-1}$. Any permutation can be expanded into transpositions. Number of expansion members is independent of expansion means. Parity of transpositions number in the permutation expansion named parity of this permutation.

Such a way, exterior product $\omega=\omega^p\Lambda\omega^q$ of forms ω^p и ω^q is a form $\omega^p \Lambda \omega^q = \omega(\vec{\xi}_1, ..., \vec{\xi}_{p+q}) = \sum_{\sigma} sgn(\sigma) \cdot \sigma a$ (summation by all permutations). Here σa is a function of p+q vectors, obtained from defined above function a by argument permutation σ; sgn(σ) is 1, if permutation is even, and -1, if it is odd. As a simple example of exterior product let's view a product of two linear 1-forms gives a bilinear form:

$$f(\vec{\xi}_1)\Lambda g(\vec{\xi}_2) = \sum_{\sigma} sgn(\sigma) \cdot \sigma f(\vec{\xi}_1) g(\vec{\xi}_2) =$$
$$= f(\vec{\xi}_1) g(\vec{\xi}_2) - f(\vec{\xi}_2) g(\vec{\xi}_1)$$

Exterior product of 1-form and q-form (q>1) is a form of q+1 degree:

$$\omega(\vec{\xi}_0, \vec{\xi}_1, ..., \vec{\xi}_q) = f(\vec{\xi}_0)\Lambda g(\vec{\xi}_1, ..., \vec{\xi}_q) =$$
$$= \sum_{\sigma} sgn(\sigma) \cdot \sigma f(\vec{\xi}_0) g(\vec{\xi}_1, ..., \vec{\xi}_q) =$$
$$= \sum_{i=0}^{q} (-1)^i f(\vec{\xi}_i) g(\vec{\xi}_0, ..., \vec{\xi}_{i-1}, \vec{\xi}_{i+1}, ..., \vec{\xi}_q)$$

So, by definition, p-linear form $\widetilde{\varphi}$ is a linear function of p vectors. Number p is a form degree, and φ is polylinear form of p degree. If the form is alternating, then φ is an exterior p-form. Each linear 1-form (p=1) is exterior; exterior 0-forms (p=0) are, by definition, real numbers.

Schematically physics and mathematics usually operate with letter T, denoting some real space of tangent vectors. Symbol T^p denotes p-fold Cartesian product T on itself – set of all ensembles of p vectors from T space $\left(\vec{\xi}_1, ..., \vec{\xi}_p\right)$. By T* denotes a dual space (like direct and inverse space in the solid state physics), or vector space of linear forms in T space [1-3].

For all p exterior p-forms combine into real vector space E^p, named p-fold Grassmann product on T space. At that E^0 is R, and E^1 is T*. Underline, that exterior multiplication reflected Cartesian product $E^p \times E^q$ to E^{p+q} space. I.e., forms exterior product $\varphi(p)\Lambda\psi(q)$, that is exterior (p+q)-form, don't belong to E^p and E^q, when p>0 and q>0.

2. Concrete Applications

From methodological point of view exterior differential form formalism is much simply then vector calculus one [1-3]. By definition, p-degree differential form in n-dimension Euclidean space is infinitely differentiable vector function $\omega(x,dx)$ (where $x=\left(x^1,x^2,...,x^n\right)$ is an element of mentioned space, and differential symbol $dx=\left(dx_1,dx_2,...,dx_n\right)$ is a vector) [3]. This form is skew-symmetric p-form by *dx*, when *x* are fixed [1-3]:

$$\omega(x,dx)=\sum_{i_1<...<i_p}\omega_{i_1...i_p}(x)dx^{i_1}\wedge...\wedge dx^{i_p} \tag{A1}$$

where $i_1,...,i_p$ are ordering indexes.

To underline exterior differentiation distinction from usual differentiation, sometimes forms and their derivatives denotes as φ, ψ or α, β, and exterior differentials sign by differential operator $\tilde{d}$; then $\tilde{d}\varphi,\tilde{d}\alpha$ are the standard notations of φ and α forms exterior differentials.

Differential forms algebra is formalized by exterior differentiation $\tilde{d}$ and exterior (anticommutative) multiplication Λ rules. This algebra is more easily and at the same time more effective and more fundamental, that vector analysis [1-3].

A set of all forms of any degree with exterior multiplication operation between them defines as Grassmann algebra [1-3].

For the forms φ(p) and ψ(q) of p and q degrees it's true a commutation rule

$$\varphi(p)\wedge\psi(q)=(-1)^{pq}\psi(q)\wedge\varphi(p) \tag{A2}$$

Operator's $\tilde{d}$ action to high degree forms is analogous to one's action to 0-forms. Exterior differentiation increases form degree per unit (if φ is a p-form, then $\tilde{d}\varphi$ is a (p+1)-form).

If $\tilde{d}\varphi$ is an exact form [1, 2] ($d\varphi$ is a full differential in a usual differential calculus), then double exterior differentiation operation leads to form vanishing: $\tilde{d}(\tilde{d}\varphi)=0$.

Exterior differentiation rules are the similar to usual differentiations ones, taking into account Λ operation anticommutative properties:

$$\tilde{d}x_i\wedge\tilde{d}x_j=\left(\delta_{ij}-1\right)\tilde{d}x_j\wedge\tilde{d}x_i \tag{A3}$$

Linearity of exterior differential forms resulting from (λ_1,λ_2 are numbers):

$$\begin{aligned}&\tilde{d}\left(\lambda_1\varphi(p)+\lambda_2\psi(q)\right)=\lambda_1\tilde{d}\varphi(p)+\lambda_2\tilde{d}\psi(q)\\&\tilde{d}\left(\lambda_1\varphi(p)\wedge\lambda_2\psi(q)\right)=\\&=\lambda_1\lambda_2\left\{\tilde{d}\varphi(p)\wedge\psi(q)+(-1)^p\varphi(p)\wedge\tilde{d}\psi(q)\right\}\end{aligned} \tag{A4}$$

Thus, systematical consideration of differential forms bases on the next theses. Differential form characterizes by its dimension, or degree (p), and by Euclidean space dimension (n): R^n. Space dimension is equal to number of manifold variables that form is defined on. Obvious relation $n\geq p$ is true.

Form of zero degree (p=0) is an any infinitely differentiable function

$$\omega\left(x,\tilde{d}_0x\right)=\omega(x,0)=\omega(x)=f(x)=f\left(x^1,x^2,...,x^n\right) \tag{A5}$$

Obviously, in one-dimension case $\omega(x)=f(x)$, in two-dimension case $\omega(x)=\omega\left(x^1,x^2\right)=\omega(x,y)=f(x,y)$ Similar representations of 0-form there are for any dimension of space.

Form of degree 1 (p=1), or 1-form, is

$$\omega(x,dx)=\sum_{j=1}^{n}\omega_j(x)\tilde{d}x_j \tag{A6}$$

Particularly, when n=1, we have linear differential form

$$\omega\left(x,\tilde{d}x\right)=f(x)\tilde{d}x \tag{A7}$$

Form of degree 2 (p=2), or 2-form, is

$$\omega\left(x,\tilde{d}x_1,\tilde{d}x_2\right)=\sum_{i<k}\omega_{ik}(x)\tilde{d}x_i\wedge\tilde{d}x_k \tag{A8}$$

Particularly, for the minimal space dimension n=p=2

$$\vec{x}=(x^1,x^2),\tilde{d}x=\tilde{d}x\left(\tilde{d}_1x(\tilde{d}_1x^1,\tilde{d}_1x^2),\tilde{d}_2x(\tilde{d}_2x^1,\tilde{d}_2x^2)\right) \tag{A9}$$

$$\tilde{\omega}(x,\tilde{d}x)=\tilde{\omega}\left(x,\tilde{d}x_1,\tilde{d}x_2\right)=f(x)\begin{vmatrix}\tilde{d}x_1^{\,1}&\tilde{d}x_1^{\,2}\\\tilde{d}x_2^{\,1}&\tilde{d}x_2^{\,2}\end{vmatrix} \tag{A10}$$

Here determinant $\tilde{d}x_1^{\,1}\tilde{d}x_2^{\,2}-\tilde{d}x_1^{\,2}\tilde{d}x_2^{\,1}$ is equal to defined on vectors $\tilde{d}x_1,\tilde{d}x_2$ area element. When n=p=3 and variables are

$$\begin{aligned}&\vec{x}=x(x^1,x^2,x^3),\\&\tilde{d}x=\tilde{d}x\begin{pmatrix}\tilde{d}_1x(\tilde{d}_1x^1,\tilde{d}_1x^2,\tilde{d}_1x^3),\tilde{d}_2x(\tilde{d}_2x^1,\tilde{d}_2x^2,\tilde{d}_2x^3),\\\tilde{d}_3x(\tilde{d}_3x^1,\tilde{d}_3x^2,\tilde{d}_3x^3)\end{pmatrix}\end{aligned}$$

we have corresponding to vectors $\tilde{d}x_1,\tilde{d}x_2,\tilde{d}x_3$ volume element, which is similarly equal to determinant. 3-form is, respectively

$$\tilde{\omega}(x,\tilde{d}x)=\tilde{\omega}\left(x,\tilde{d}x_1,\tilde{d}x_2,\tilde{d}x_3\right)=f(x)\begin{vmatrix}\tilde{d}x_1^{\,1}&\tilde{d}x_1^{\,2}&\tilde{d}x_1^{\,3}\\\tilde{d}x_2^{\,1}&\tilde{d}x_2^{\,2}&\tilde{d}x_2^{\,3}\\\tilde{d}x_3^{\,1}&\tilde{d}x_3^{\,2}&\tilde{d}x_3^{\,3}\end{vmatrix} \tag{A11}$$

Let's specify differential form formalism for a vector field case. In this case remind, that exterior differential $\tilde{d}\omega$ of the linear differential form ω of p degree defines by relation

$$\tilde{d}\omega = \sum_{i_1,\dots,i_p} \tilde{d}\omega_{i_1\dots i_p} \wedge \tilde{d}x_{i_1} \wedge \dots \wedge \tilde{d}x_{i_p} \quad \text{(A12)}$$

where

$$\tilde{d}\omega_{i_1\dots i_p} = \sum_{k=1}^{n} \frac{\partial \omega_{i_1\dots i_p}}{\partial x^k} \tilde{d}x_k \quad \text{(A13)}$$

Operator $\tilde{d}$ action to zero degree form ($\omega(x) = f(x)$) formally agrees with usual differentiation:

$$\tilde{d}\omega(x) = \tilde{d}f(x) = \sum_{k=1}^{n} \frac{\partial f}{\partial x^k} \tilde{d}x_k \quad \text{(A14)}$$

External differential of linear form (p=1) calculation (when n>1) leads to

$$\begin{aligned} \tilde{d}\omega\left(x, \tilde{d}x\right) &= \tilde{d}\left(\sum_{i=1}^{n} \omega_i(x)\tilde{d}x_i\right) = \\ &= \sum_{k<i}\left(\frac{\partial \omega_i}{\partial x^k} - \frac{\partial \omega_k}{\partial x^i}\right)\tilde{d}x_k \wedge \tilde{d}x_i \end{aligned} \quad \text{(A15)}$$

For example, if n=2, for 1-form $\omega\left((x,y), \tilde{d}x, \tilde{d}y\right) = P(x,y)\tilde{d}x + Q(x,y)\tilde{d}y$ obtain

$$\tilde{d}\omega = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\tilde{d}x \wedge \tilde{d}y \quad \text{(A16)}$$

Mark out rules, which define exterior differentiation operator action for fixed form degree p and space dimension n.

Operator $\tilde{d}$ action to defined in 1-dimension space R^1 0-form (p=0) gives 1-form (p=1). I.e., in R^1 operator $\tilde{d}$ increase form degree, and dimension of space, which form defined on, is invariant:

$$\tilde{d}\omega(x) = \tilde{d}f(x) = \frac{df}{dx}\tilde{d}x \quad \text{(A17)}$$

Operator $\tilde{d}$ action to 0-form, defined in n-dimension space R^n, also given 1 form – linear combination of n differential terms:

$$\tilde{d}\omega(x) = \tilde{d}f(x^1,\dots,x^n) = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}\tilde{d}x_i \quad \text{(A18)}$$

Differential forms of higher degrees (p>1) generate either by lower degrees form exterior multiplication, or by exterior differentiation operator action to form of degree, lower per unit.

For example, if 1-form in R^3 is

$$\begin{aligned} \varphi &= \omega\left((a_1,a_2,a_3), \tilde{d}x_1, \tilde{d}x_2, \tilde{d}x_3\right) = \\ &= a_1\tilde{d}x_1 + a_2\tilde{d}x_2 + a_3\tilde{d}x_3 \end{aligned} \quad \text{(A19)}$$

where $a_i = a_i(x^1,x^2,x^3)$ are a components of vector field $\vec{a}$, which are functions of R^3 space variables, then operator $\tilde{d}$ translates this 1-form into a 2-form in R^3:

$$\tilde{d}\varphi = c_1\tilde{d}x_2 \wedge \tilde{d}x_3 + c_2\tilde{d}x_3 \wedge \tilde{d}x_1 + c_3\tilde{d}x_1 \wedge \tilde{d}x_2 \quad \text{(A20)}$$

Here (c_1,c_2,c_3) - components of vector $\vec{c}$, defining as

$$c_1 = \frac{\partial a_3}{\partial x^2} - \frac{\partial a_2}{\partial x^3},\ c_2 = \frac{\partial a_1}{\partial x^3} - \frac{\partial a_3}{\partial x^1},\ c_3 = \frac{\partial a_2}{\partial x^1} - \frac{\partial a_1}{\partial x^2} \quad \text{(A21)}$$

In traditional vector calculus mentioned vector calls rotor: $\vec{c} = rot(\vec{a})$. If we'll consider 2-form in R^3

$$\begin{aligned} \psi &= \omega\left((b_1,b_2,b_3), \tilde{d}x_1, \tilde{d}x_2, \tilde{d}x_3\right) = \\ &= b_1\tilde{d}x_2 \wedge \tilde{d}x_3 + b_2\tilde{d}x_3 \wedge \tilde{d}x_1 + b_3\tilde{d}x_1 \wedge \tilde{d}x_2, \end{aligned} \quad \text{(A22)}$$

where b_i - components of some vector (depend on от x^1, x^2, x^3), then exterior differentiation operator translates it into 3-form in R^3

$$\tilde{d}\psi = c\tilde{d}x_1 \wedge \tilde{d}x_2 \wedge \tilde{d}x_3 \quad \text{(A23)}$$

Obtained 3-form characterizes by value

$$c = \frac{\partial b_1}{\partial x^1} + \frac{\partial b_2}{\partial x^2} + \frac{\partial b_3}{\partial x^3} \quad \text{(A24)}$$

analogous to a divergence $c = div(\vec{b})$ in vector analyze.

3. Formalism of Integration

Forms are connected with an elementary volume in the n-dimension manifold. For the volume form we can change any n-form. This choice follows from problem conditions. Two-dimension square is a singular volume of three-dimension space.

Integration of the function on the manifold essentially is multiplication of a function value on volume element and summation of obtained numbers.

If $\tilde{\omega}$ is n-form, defined in area U of n-dimension manifold M with coordinates $\{x^1, x^2, \dots, x^n\}$, then exists such function $f(x^1, x^2, \dots, x^n)$, that $\tilde{\omega} = f\tilde{d}x^1 \wedge \dots \wedge \tilde{d}x^n$, and integral of $\tilde{\omega}$ on U by definition is

$$\begin{aligned} \int \tilde{\omega} &\equiv \int f(x^1,\dots,x^n)dx^1 \dots dx^n \equiv \\ &\equiv \int f(x^1,\dots,x^n)\tilde{d}x^1 \wedge \dots \wedge \tilde{d}x^n \end{aligned} \quad \text{(A25)}$$

In two-dimension case, i.e. on manifold (λ, μ), in differential forms terms integral will be written as

$$\int \tilde{\omega} \equiv \int f(\lambda,\mu) d\lambda d\mu \equiv \int f(\lambda,\mu) \tilde{d}\lambda \wedge \tilde{d}\mu \quad \text{(A26)}$$

Variables change, or transfer to new coordinates (x, y), makes in according to exterior differentiation rules for the function compositions:

$$\begin{aligned} \tilde{d}\lambda &= \tilde{d}\lambda(x,y) = \frac{\partial\lambda}{\partial x}\tilde{d}x + \frac{\partial\lambda}{\partial y}\tilde{d}y \\ \tilde{d}\mu &= \tilde{d}\mu(x,y) = \frac{\partial\mu}{\partial x}\tilde{d}x + \frac{\partial\mu}{\partial y}\tilde{d}y \end{aligned} \quad \text{(A27)}$$

Taking into account antisymmetry of the exterior multiplication operator Λ:

$\tilde{d}x\Lambda\tilde{d}y = -\tilde{d}y\Lambda\tilde{d}x, \tilde{d}x\Lambda\tilde{d}x = \tilde{d}y\Lambda\tilde{d}y = 0$, we'll receive transfer from «square» $\tilde{d}\lambda\Lambda\tilde{d}\mu$ to $\tilde{d}x\Lambda\tilde{d}y$. Result is

$$\begin{aligned} \tilde{d}\lambda\Lambda\tilde{d}\mu &= \left(\frac{\partial\lambda}{\partial x}\tilde{d}x + \frac{\partial\lambda}{\partial y}\tilde{d}y\right)\Lambda\left(\frac{\partial\mu}{\partial x}\tilde{d}x + \frac{\partial\mu}{\partial y}\tilde{d}y\right) = \\ &= \frac{\partial(\lambda,\mu)}{\partial(x,y)}\tilde{d}x\Lambda\tilde{d}y \end{aligned} \quad \text{(A28)}$$

Thus, in the differential form calculus integrals of function *f* on variables λ,μ and one on variables x, y are connected by traditional way, through Jacobean $\frac{\partial(\lambda,\mu)}{\partial(x,y)}$.

At the same time differential form apparatus using is sensitive to choose of coordinate system. A sign of $\int\tilde{\omega}$, or a sign of Jacobean, causes by starting basis. In other words, integral of form $\tilde{\omega}$ depends on coordinate system orientation only.

According to common point of view, we must choose basis type: « right » or « left ».

Traditional vector integral equations have respective formulas in exterior differential form calculus. For example, well-known equation

$$\int_{\partial F} (\vec{a}\cdot d\vec{l}) = \int_F rot(\vec{a})\cdot d\vec{S} \quad \text{(A29)}$$

corresponds to

$$\int_{\partial F} \varphi = \int_F \tilde{d}\varphi \quad \text{(A30)}$$

Formula

$$\int_{\partial G} (\vec{b}\cdot d\vec{S}) = \int_G div(\vec{b}) dx^1 dx^2 dx^3 \quad \text{(A31)}$$

looks like

$$\int_{\partial G} \psi = \int_G \tilde{d}\psi \quad \text{(A32)}$$

Traditional vectors in (A29) and (A31) replace by form calculus vectors in (A30), (A32). Latter's components are functions of three variables (defined in R^3).

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