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Analyzing the Beam's Deformation Behavior on an Elastoplastic Foundation

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Abstract: The idea of beams on elastic foundation has been widely applied in the design of geotechnical structures such railway tracks, rigid and flexible highway pavement, building and habour structures. Winkler was the first to present the analysis of a beam on an elastic foundation for the analysis of railroad track deflection, based on the premise that the foundation reaction forces are proportionate to the deflection of the beam at that location. An elastic material regains its original shape on unloading whereas plastic material do not; an elastoplastic material undergoes coupled elastic (recoverable) and plastic (unrecoverable) deformations during loading and unloading. Soils are really elastoplastic material. At stresses below the yield stresses soil to responds elastically, whereas at stresses beyond yield stress soil to respond elastoplastically. The conventional analysis of plate on elastic foundation is inadequate which necessitated this study. This study focuses on the analysis of a beam on an elastoplastic foundation. Though the derivation started with winkler's model, elastoplastic condition was considered. It was also assumed that the soil is homogeneous and isotropic; and that the beam on elastoplastic system is symmetrical with law of superposition applying. The derivation was further confirmed using Buckingham Pi theorem for dimensional analysis.

Keywords: Beam on Elastic Foundation, Winkler's Model, Beam on Elastoplatic Foundation, Soil Structure Interaction

1. Introduction

There are various cases in geotechnical engineering where the engineer must size the footings using simple empirical approaches to transmit loads from the superstructure to the soil beneath. While acceptable bearing capacity value approaches are used to design spot footings, most engineers use a similar method to build continuous footings. The computation of the vertical displacements of the footing along the longitudinal direction becomes important when continuous footings carry distributed and concentrated loads to evaluate the probable differential settlements of the footing. If we suppose that the continuous footing behaves like a beam, we can analyze it using the beam-on-elastic-foundation model. The loads on the beam are transferred from the beam and into the soil in a complex manner depending on the longitudinal stiffness of the beam. In other words, the behaviour of the beam under load is determined by both the soil's material properties and its own stiffness characteristics. The beam on elastic foundation problem is one of these soil-structure interaction problems [1, 2]. Its solution requires the conceptualization of one, the mechanical behavior of the beam, two, the mechanical behavior of the soil as elastic subgrade and three, the form of interaction between the beam and the soil [3].

The idea of beams on elastic foundation has been widely applied in the sub-disciplines of civil engineering such as geotechnical engineering, structural engineering [4], highway pavement engineering, railway engineering [5], machine building factory [6] and retaining structures [7]. Past studies are basically interested in elastic deformation of the beam on elastic foundation model using mathematical methods solved using different numerical and analytical methods.

Winkler [8] was the first to present the analysis of a beam on an elastic foundation for the analysis of railroad track deflection, based on the premise that the foundation reaction forces are proportionate to the deflection of the beam at that location. The relationship between the deflection θ and the load q can be described using equation 1.

$$EI\frac{d^4\theta}{dx^4} + k\theta = q \quad (1)$$

$$\text{For } 0 < x < L$$

The parameter for the spring constant k is called the modulus of subgrade reaction of the soil in this equation, and El is the beam stiffness. Figure 1 depicts the deformation pattern of a loaded beam for this model.

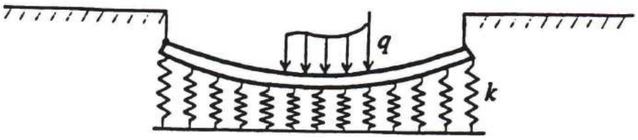

Figure 1. *Winkles model for deformation loaded beams.*

An elastic material recovers its original dimensions on unloading whereas plastic material do not; an elastoplastic material undergoes both elastic (recoverable) and plastic (unrecoverable) deformations during loading and unloading. Soils are truly elastoplastic material [9-11]. At small strains soil behave like an elastic material. Succinctly put, stresses below the yield stresses cause the soil to respond elastically, wheresas, stresses beyong yield stress cause soil to respond elastoplastically [12].

To the best of the researchers' knowledge, no research has been done on a beam on elastoplastic soil for a laterally positioned foundation. As a result, this study focuses on the analysis of a beam on an elastoplastic foundation, which differs from the traditional analysis of a beam on an elastic foundation**.**

2. Methodology

2.1. Model Assumption

With respect to modelling beam on elastoplastic foundation the following assumptions are made:

i. Winkler model is adopted excepted that soil is assumed to be elastoplastic.
ii. The sub-structural soil is homogeneous and isotropic.
iii. The system of plate on elastoplastic is symmetrical.
iv. The principle of superposition is valid.

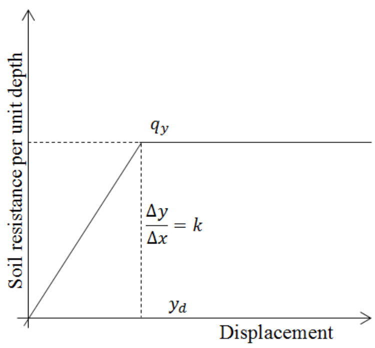

Figure 2. *Elasto-plastic Winkler spring model.*

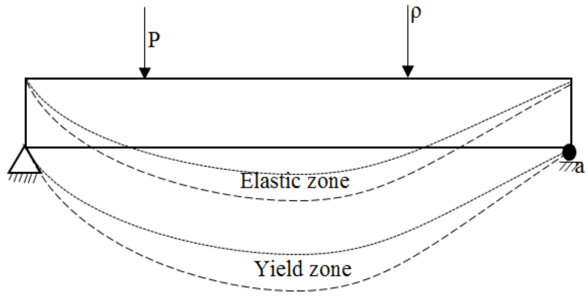

Figure 3. *Beam on elastoplastic foundation.*

2.2. Model Derivation

For beam on elastic foundation, the stresses in the soil and deformation can be derived by using an elastoplastic winkler spring model proposed by [13]. The model is shown in Figure 2 in which K is the modulus of sub grade reaction; q_y = yield soil resistance per unit depth of the beam; and y_d = yield displacement of the soil.

Considering infinitesimal elements enclosed between two vertical cross sections a distance d_x apart on the beam being considered. Assuming that this element was taken from a part where beam was acted upon by a distributed loading q. The forces transferred to such element are shown in Figure 4.

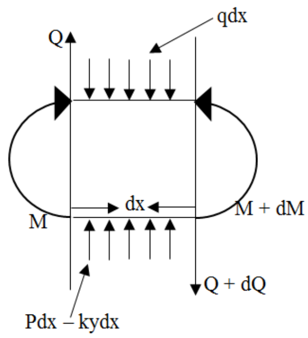

Figure 4. *Elemental section of the beam.*

The shearing force acting upward, Q, to the left of the element is taken to be positive, same is the corresponding bending moment, M, that is a clockwise moment acting from the left on the element. These positive directions for Q and M will be observed in all derivations taking into account the equilibrium of the element in Figure 4, we find that the summation of the vertical forces is

$$Q - (Q + dQ) + k_y d_x - q d_x = 0$$

Differentiating we have

$$\frac{dQ}{dx} = ky - q \quad (2)$$

Recall that $Q = \frac{dM}{dx}$

$$\text{We have that } \frac{dQ}{dx} = \frac{d^2M}{dx^2} = ky - q \quad (3)$$

Using the known differential equation of a beam in bending

$$EI\left(\frac{d^2y}{dx^2}\right) = -M$$

And differentiating twice, we have

$$EI\left(\frac{d^4y}{dx^4}\right) = \frac{d^2M}{dx^2} \quad (4)$$

Substituting equation (3) into equation (4) we have:

$$EI\left(\frac{d^4y}{dx^4}\right) = -ky + q \quad (5)$$

According to [14] equation (5) is the differential equation for the deflection curve of a beam supported on an elastic foundation. Along the unloaded parts of the beam, where there is no distributed load q = 0, and the equation takes the form

$$EI\left(\frac{d^4y}{dx^4}\right) = -ky \quad (6)$$

It will be adequate to consider only the general solution of equation (6) from which solutions will be obtained as well for cases implied in (5) by applying suitable particular integral corresponding to q in (5).

Substituting $y = e^{mx}$ in (2), we have the characteristic equation

$$m^4 = -\frac{K}{EI}$$

Which has a complex root

$$m_1 = -m_3 = \left(\frac{K}{4EI}\right)^{1/4}(1+i) = \lambda(1+i)$$

$$m_2 = -m_4 = \left(\frac{K}{4EI}\right)^{1/4}(-1+i) = \lambda(-1+i)$$

The general solution of (6) gives

$$y = A_1e^{m_1x} + A_2e^{m_2x} + A_3e^{m_3x} + A_4e^{m_4x} \quad (7)$$

Where

$$\lambda = \left(\frac{K}{4EI}\right)^{1/4} \quad (8)$$

Multiplying both sides by L

$$\lambda L = \left(\frac{KL^4}{4EI}\right)$$

$$e^{i\lambda x} = cos\lambda x + isin\lambda x$$

$$e^{-i\lambda x} = cos\lambda x - isin\lambda x$$

Then introducing new constants C_1, C_2, C_3, C_4, where

$$(A_1 + A_4) = C_1, i(A_1 - A_4) = C_2$$

$$(A_2 + A_3) = C_3, i(A_2 - A_3) = C_4$$

Rewriting equation (7)

$$y = e^{\lambda x}(C_1cos\lambda x + C_2sin\lambda x) + e^{-\lambda x}(C_3cos\lambda x + C_4sin\lambda x) \quad (9)$$

λ include the flexural rigidity of the beam as well as elasticity of the supporting soil. Owing to this, λ is referred to as the characteristic of the model, and since its dimension is inverse of length, the parameter $1/\lambda$ is called dicharacteristic length. Consequently λx will be a unity.

Equation (9) represents the general solution for the deflection line of a beam supported on an elastic foundation and subjected to transverse bending forces, but without loading. An additional term is required where there is distributed load.

Differentiating equation (9) we have

$$\frac{1}{\lambda}\frac{dy}{dx} = e^{\lambda x}[C_1(cos\lambda x + sin\lambda x) + C_2(cos\lambda x + sin\lambda x)] - e^{\lambda x}[C_3(cos\lambda x + sin\lambda x) - C_4(cos\lambda x - sin\lambda x)]$$

$$\frac{1}{2\lambda^2}\frac{d^2y}{dx^2} = -e^{\lambda x}(C_1sin\lambda x - C_2cos\lambda x) + e^{\lambda x}(C_3sin\lambda x - C_4cos\lambda x)$$

$$\frac{1}{2\lambda^4}\frac{d^4y}{dx^4} = -e^{\lambda x}[C_1(cos\lambda x + sin\lambda x) - C_2(cos\lambda x - sin\lambda x)] + e^{-\lambda x}[C_3(cos\lambda x - sin\lambda x) + C_4(cos\lambda x + sin\lambda x)] \quad (10)$$

From basic strength of material

$$\frac{dy}{dx} = \tan\theta, -EI\frac{d^4y}{dx^4} = M \; and \; -EI\frac{d^3y}{dx^3} = Q \quad (11)$$

The general expression for the slope θ, the bending moment M and the shearing force Q can be obtained from equation (10). The magnitude of stress in the foundation can be found from equation (9) to be $P = ky$.

In applying the general equation, the next step is to obtain the constants of integration C_1, C_2, C_3 and C_4. The integration constant depends on how the beam is subjected to the loading, having constant value along each portion of the beam. Their values can be derived from the conditions at the end of the continuous part.

Out of $y, \theta, M \; and \; Q$ characterizing the condition at the end, two are normally known at each end from which adequate results are provided for the determination of the constants C.

Where

$$C_1 = e^{-\lambda x}(cos\lambda x + \sin\lambda x) \quad (12)$$

$$C_2 = e^{-\lambda x}\sin\lambda x \quad (13)$$

$$C_3 = e^{-\lambda x}(cos\lambda x - \sin\lambda x) \quad (14)$$

$$C_4 = e^{-\lambda x}\cos\lambda x \quad (15)$$

a) For concentrated load V at centre:

$$Deflection\ (\theta) = \frac{V\lambda C_1}{2Kb} \tag{16}$$

b) Moment M_o at centre

$$Deflection\ (\theta) = \frac{M_o\lambda^2 C_2}{Kb} \tag{17}$$

c) Concentrated load V at free end:

$$Deflection\ (\theta) = \frac{2V\lambda C_4}{Kb} \tag{18}$$

d) Moment M_o at free end

$$Deflection\ (\theta) = \frac{-2M_o\lambda^2 C_3}{Kb} \tag{19}$$

2.3. Considering Elastoplastic Condition

When unloaded, an elastic material returns to its original proportions; when loaded, an elastomeric material deforms both elastically (recoverable) and plastically (unrecoverable). Elastoplastic materials include soils. The behavior of soil changes from being elastic at low strains to becoming elastic at higher strains. The yield surface is the point at which a soil gives under the applied loads. The soil responds elastically to stresses that are lower than the yield stress. The soil reacts in an elastoplastic manner to stresses above the yield stress [12].

Elasto-plastic behaviour of soil can be modelled using 2 different models, that is, simple constitutive model and advanced constitutive model [15]. Three types of plastic behaviour are known: perfect plasticity, strain hardening and softening plasticity. The models assume elastic behaviour before yield and can therefore couple the benefits of elastic and plastic behaviour.

The stress, q, due to soil on unit width of the foundation is given as

$$q = kx^a y^b \tag{20}$$

Where k, a and b are constants.

According to [13], the recommended values of c and d respectively are 1 and 0.5.

Solution of equation (1) with q given by equation (20) is tedious, but plausible due to non-linear nature of the relationship.

The form of the equation (3.12) is similar to the one suggested by [16] for cohesive soils (curve a of Figure 5). The curve is defined by

$$\epsilon = \sigma(c + d\epsilon) \tag{21}$$

Where σ is the stress at any strain. ε, a and b are also constants, where $c = {}^{1}\!/_{E}$ and $d = \frac{1}{\sigma_{max}}$

Simple forms as equation (20) and (21) when incorporated in equation (1) results to

$$q = kxy^{0.5} \tag{22}$$

$$\sigma = \varepsilon/(c + d\varepsilon) \tag{23}$$

Such mathematical complexities that provides direct solution of equation (1) is trivial. Linearizing curve ‘a’ from stress-strain behaviour of cohesive soil in Figure 5 gives rise to curve ‘b’ as previously demonstrated by [17] which is described by

$$q = ky\ if\ y \le y_o or\ q > q_{max} \tag{24}$$

$$q = q_{max}\ if\ y > y_o$$

Where q_{max} is the yield stress and$y_o = q_{max}/k$. Equations (24) describe an elasto-plastic material. A material that is elastic upto a stress level, q_{max}, and deforms plastically once the stress level attains q_{max}. Interestinly, as the gradient of the curve tends to zero, the rigidity of the soil tends to zero, and that is the major reason EI is not present in the equation.

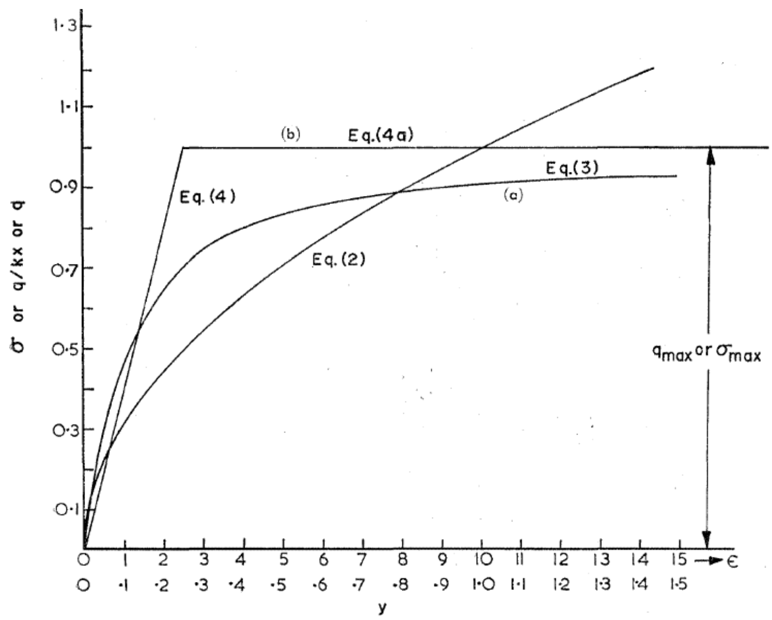

Figure 5. *Stress-strain relations of soil (Source: Madhav, 1971).*

3. Results

Using Buckingham Pi theorem for dimensional analysis to check if Equation 24 can take another important parameter E.

$$\Pi = q^a y^b E^c K \tag{25}$$

$$\Pi = (ML^{-1}T^{-2})^a (L)^b (ML^{-1}T^{-2})^c (ML^{-2}T^{-2}) \tag{26}$$

$$M: a + c + 1$$

$$L: -a + b - c$$

$$T: -2a - 2c - 2 \tag{27}$$

Equation 27 has no solution showing that at this point, elastic modulus cannot be part of the equation.

Then subjecting Equation 24 to the same Buckingham Pi theorem for dimensional analysis for quality control and homogeneity test, we have;

$$\Pi = q^a y^b K \tag{28}$$

$$\Pi = (ML^{-1}T^{-2})^a (L)^b (ML^{-2}T^{-2}) \tag{29}$$

$$M: a + 1$$

$$L: -a + b - 2$$

$$T: -2a - 2 \tag{30}$$

Solving equation 30 we have that $a = -1 \, and \, b = 1 \, resulting \, to$:

$$\Pi = \frac{ky}{q} \tag{31}$$

Equation 31 confirms the basic validity of Equation 24.

4. Conclusion

Given that soil is an elastoplastic by nature, taking into account the plate-foundation system as a beam on an elastoplastic foundation ensures better geotechnical structure design that will withstand the test of time. Consequently, designing foundation as a beam on elastic foundation results to false economy. Anything that happens to the system beyond its yield stress will result to settlement beyond acceptable limit or even lead to total collapse. Therefore, to avoid these consequences, it is relatively economical and safe to design foundation as beam on elastoplastic foundation.

Conflict of Interest

The Authors declare that they have do conflict of interest.

Authors' Contributions

Emmanuel Arinze contributed to conceptualization and manuscript writing whereas Emeka Otti contributed to Literature review and review of the manuscript.

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