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Sharpness of the Segre's Upper Bound for the Regularity Index of Fat Points

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Abstract: In this paper we show some results about estimating the regularity index of fat points and study when the Segre's upper bound is sharp for arbitrary fat points in $\mathbb{P}^n$. We show that the Segre's upper bound is sharp for fat points where the points are constrained by geometric conditions in $\mathbb{P}^n$ (Corollary 2.1 and Proposition 2.1). We show that if $s \leq 4$, the Segre's upper bound is sharp for s arbitrary fat points in $\mathbb{P}^n$ (Theorem 3.1), and the Segre's upper bound is sharp for 5 equimultiple fat points in $\mathbb{P}^n$ (Theorem 3.2). We also show that if $s \geq 6$ and $n \geq 2$, then there exists always a set of s fat points in $\mathbb{P}^n$ whose the Segre's upper bound is not sharp (Proposition 3.1). We predict that Segre's upper bound is sharp for 5 non-equimultiple fat points, but we can not prove this prediction nor we can find an example to show that the prediction is incorrect.

Keywords: Segre's Upper Bound, Fat Points, Regularity Index

1. Introduction

Let K be an algebraically closed field of characteristic 0, let $\mathbb{P}^n$ be the n-dimensional projective space over K. Let $P_1, \ldots, P_s$ be s distinct points in $\mathbb{P}^n$, and let $m_1, \ldots, m_s$ be s positive integers. If $\wp_1, \ldots, \wp_s$ are the defining prime ideals in $R = K[X_0, \ldots, X_n]$ corresponding to the points $P_1, \ldots, P_s$, we let

$$\mathcal{Z} := \{(P_1, m_1), \ldots, (P_s, m_s)\}$$

be the zero-scheme defined by the ideal $I = \wp_1^{m_1} \cap \cdots \cap \wp_s^{m_s}$. We may view $\mathcal{Z}$ as fat points in $\mathbb{P}^n$, and write $\mathcal{Z} \subset \mathbb{P}^n$.

The ring R/I is the homogeneous coordinate ring of $\mathcal{Z}$, it is graded, $R/I = \bigoplus_{t\geq 0} (R/I)_t$. For each t, the t-th graded part $(R/I)_t$ is a finite dimensional K-vector space. The function

$$H_{R/I}(t) = \dim_K (R/I)_t$$

is called the Hilbert function of R/I (or of $\mathcal{Z}$), we also denote $H_{R/I}(t)$ by $H_{\mathcal{Z}}(t)$. The ring R/I has the multiplicity to be $e(R/I) = \sum_{i=1}^{s} \binom{m_i+n-1}{n}$. It is well known that $H_{R/I}(t)$ strictly increases until it reaches the multiplicity $e(R/I)$, and it keeps constant thereafter. The number

$$r(R/I) = \min\{t \in \mathbb{N} \mid H_{R/I}(t) = e(R/I)\}$$

is called the regularity index of R/I (or of $\mathcal{Z}$), we also denote $r(R/I)$ by $r(\mathcal{Z})$.

It is difficult to estimate $r(\mathcal{Z})$, so one tries to find an upper bound for it. There have been results in finding upper bounds for $r(\mathcal{Z})$ (see [1]-[6], [8]-[16]).

For a rational number b, denote by $[b]$ the greatest integer less than or equal to b. In [12] N.V. Trung (and, independently, G. Fatabbi and A. Lorenzini in [6]) conjectured that

$$r(\mathcal{Z}) \leq Seg(\mathcal{Z})$$

where

$$Seg(\mathcal{Z}) = \max\{T_j(Z) | j = 1, \ldots, n\},$$

$$T_j(\mathcal{Z}) = \max\left\{\left[\frac{\sum\limits_{P_i\in\beta} m_i + j - 2}{j}\right] \mid \beta \text{ is a linear } j\text{-subspace in } \mathbb{P}^n\right\}.$$

The number $Seg(\mathcal{Z})$ is called the Segre's bound because Segre [10] proved the conjecture for case where no three points are collinear in $\mathbb{P}^2$. The above conjecture was successfully proven by Nagel in general case ([11, Theorem 5.3]). The Segre's upper bound is called to be sharp if $r(\mathcal{Z}) = Seg(\mathcal{Z})$.

In this paper we show some results about estimating the regularity index $r(\mathcal{Z})$ and study when the Segre's upper bound is sharp for arbitrary fat points $\mathcal{Z} \subset \mathbb{P}^n$.

2. Preliminaries

From now on, denote by $\wp_j \subset R$ the defining prime ideal of $P_j \in \mathbb{P}^n$.

Lemma 2.1. Let $\mathcal{Z} = \{(P_1, m_1), \ldots, (P_s, m_s)\} \subset \mathbb{P}^n$ be fat points. Then

$$T_1(\mathcal{Z}) \le r(\mathcal{Z}) \le Seg(\mathcal{Z}).$$

Proof. By the result of Nagel and Trok [11, Theorem 5.3] we have

$$r(\mathcal{Z}) \le Seg(\mathcal{Z}).$$

Let γ be the linear 1-subspace of $\mathbb{P}^n$ such that

$$T_1(\mathcal{Z}) = \sum_{P_i\in\gamma\cap X} m_i - 1.$$

Put $J = \bigcap\limits_{P_i\in\gamma\cap X} \wp_i^{m_i}$. Since the points of $\gamma \cap X$ lie in a line, by [4, Corollary 2.3]

$$r(R/J) = \sum_{P_i\in\gamma\cap X} m_i - 1.$$

Put $I = \bigcap\limits_{P_i\in X} \wp_i^{m_i}$. By [13, Lemma 3.3] we have

$$r(R/J) \le r(R/I).$$

Note that $r(R/I) = r(\mathcal{Z})$. From above results we get $T_1(\mathcal{Z}) \le r(\mathcal{Z}) \le Seg(\mathcal{Z})$.

Corollary 2.1. Let $\mathcal{Z} = \{(P_1, m_1), \ldots, (P_s, m_s)\} \subset \mathbb{P}^n$ be fat points such that $P_1, \ldots, P_s$ are collinear. Then

$$r(\mathcal{Z}) = Seg(\mathcal{Z}).$$

Proof. We have $T_1(\mathcal{Z}) = m_1 + \cdots + m_s - 1 = Seg((\mathcal{Z})$. By the above lemma

$$r(\mathcal{Z}) = T_1(\mathcal{Z}) = Seg((\mathcal{Z}).$$

The points $P_1, \ldots, P_s$ in $\mathbb{P}^n$ are called to be in linearly general position if no $i+2$ points of them lie on a linear i-subspace for every $i < n$. If $P_1, \ldots, P_{n+3}$ are in linearly general position in $\mathbb{P}^n$, then there exists a rational normal curve of $\mathbb{P}^n$ containing these points (see [7, Theorem 1.18]). This implies the following remark.

Remark 2.1. If $s \le n+3$ and $P_1, \ldots, P_s$ are in linearly general position in $\mathbb{P}^n$, then there exists a rational normal curve of $\mathbb{P}^n$ containing these points.

Proposition 2.1. Let $\mathcal{Z} = \{(P_1, m_1), \ldots, (P_s, m_s)\} \subset \mathbb{P}^n$ be fat points such that all P_j are on a linear r-subspace $\alpha \cong \mathbb{P}^r$, $s \le r+3$. Consider α as a r-dimensional projective space containing the points $P_{1\alpha} := P_1, \ldots, P_{s\alpha} := P_s$, and consider $\mathcal{Z}_\alpha = \{(P_{1\alpha}, m_1), \ldots, (P_{s\alpha}, m_s)\} \subset \mathbb{P}^r$ as fat points. Denote by $r(\mathcal{Z}_\alpha)$ the regularity of $\mathcal{Z}_\alpha$ in $\mathbb{P}^r$. If no $i+2$ points of $\{P_1, \ldots, P_s\}$ are on a linear i-subspace for every $i < r$, then

$$r(\mathcal{Z}) = r(\mathcal{Z}_\alpha) = Seg(\mathcal{Z}_\alpha) = Seg(\mathcal{Z}).$$

Proof. By [14, Theorem 3.6]

$$r(\mathcal{Z}) = r(\mathcal{Z}_\alpha).$$

Since $s \le r+3$ and no $i+2$ points of $\{P_1, \ldots, P_s\}$ are on a linear i-subspace for every $i < r$, by Remark 2.1 there exists a rational normal curve of $\mathbb{P}^r$ passing through these points. By [3, Proposition 7]

$$r(\mathcal{Z}_\alpha) = \max\{T_1(\mathcal{Z}_\alpha), T_r(\mathcal{Z}_\alpha)\}.$$

In this case we have

$$T_i(\mathcal{Z}_\alpha) = T_i(\mathcal{Z}), i = 1, \ldots, r,$$

$$T_i(\mathcal{Z}_\alpha) < T_1(\mathcal{Z}_\alpha), i = 2, \ldots, r-1,$$

and

$$T_r(\mathcal{Z}) > T_i(\mathcal{Z}), i = r+1, \ldots, n.$$

So, $Seg(\mathcal{Z}) = \max\{T_1(\mathcal{Z}_\alpha), T_r(\mathcal{Z}_\alpha)\} = Seg(\mathcal{Z}_\alpha) = r(\mathcal{Z}_\alpha)$.

3. Sharpness of the Segre's Upper Bound

We see that in Corollary 2.1 and Proposition 2.1, the Segre's upper bound is sharp for fat points $\{(P_1, m_1), \ldots, (P_s, m_s)\} \subset \mathbb{P}^n$ where the points $P_1, \ldots, P_s$ are constrained by geometric conditions in $\mathbb{P}^n$. In this section we study the sharpness of Segre's upper bound for fat points $\{(P_1, m_1), \ldots, (P_s, m_s)\}$ where the points $P_1, \ldots, P_s$ are arbitrary.

Theorem 3.1. Let $\mathcal{Z} = \{(P_1, m_1), \ldots, (P_s, m_s)\} \subset \mathbb{P}^n$ be s arbitrary fat points. If $s \le 4$, then

$$r(\mathcal{Z}) = Seg(\mathcal{Z}).$$

Proof. There are the following 4 cases for s.

Case s=1. It is well known that $r(R/\wp_1^{m-1}) = m_1 - 1 = Seg(\mathcal{Z})$.

Case s=2. Then P_1, P_2 lie on a line. by Corollary 2.1

$$r(\mathcal{Z}) = m_1 + m_2 - 1 = T_1(\mathcal{Z}) = Seg(\mathcal{Z}).$$

Case s=3. If P_1, P_2, P_3 lie on a line, then by Corollary 2.1

$$r(\mathcal{Z}) = m_1 + m_2 + m_3 - 1 = T_1(\mathcal{Z}) = Seg(\mathcal{Z}).$$

If P_1, P_2, P_3 do not lie on a line, then by Proposition 2.1

$$r(\mathcal{Z}) = Seg(\mathcal{Z}).$$

Case s=4. We consider the three following cases:

1) If P_1, P_2, P_3, P_4 lie on a line, then by Corollary 2.1

$$r(\mathcal{Z}) = m_1 + m_2 + m_3 + m_4 - 1 = T_1(\mathcal{Z}) = Seg(\mathcal{Z}).$$

2) If P_1, P_2, P_3, P_4 do not lie on any linear 2-subspace of $\mathbb{P}^n$, then by Remark 2.1 there exists a rational normal curve of $\mathbb{P}^n$ passing through P_1, P_2, P_3, P_4. By [3, Proposition 7]

$$r(\mathcal{Z}) = Seg(\mathcal{Z}).$$

3) If P_1, P_2, P_3, P_4 do not lie on a line, but they lie on a linear 2-subspace $\alpha \subset \mathbb{P}^n$. We assume that $\alpha \cong \mathbb{P}^2$. Consider two following subcases for $\{P_1, \dots, P_4\}$:

Case 3.a). There are 3 points of $\{P_1, \dots, P_4\}$ on a line, say l: Since there is only one point not on l, we have

$$T_1(\mathcal{Z}) \geq \max\left\{\sum_{P_i \in l} m_i - 1, \sum_{P_i \notin l} m_i - 1\right\}$$

and

$$T_2(\mathcal{Z}) = \left[\Big(\sum_{i=1}^{s} m_i\Big)/2\right].$$

Then $T_1(\mathcal{Z}) \geq T_2(\mathcal{Z})$ and $T_1(\mathcal{Z}) = Seg(\mathcal{Z})$. By Lemma 2.1

$$r(\mathcal{Z}) = T_1(\mathcal{Z}) = Seg(\mathcal{Z}).$$

Case 3.b). There are not 3 points of $\{P_1, \dots, P_4\}$ on a line: Note that P_1, P_2, P_3, P_4 lie on the linear 2-subspace α, so in this case by Proposition 2.1 we get

$$r(\mathcal{Z}) = Seg(\mathcal{Z}).$$

Fat points $\{(P_1, m_1), \dots, (P_s, m_s)\}$ is called equimultiple if $m_1 = \cdots = m_s = m$.

Theorem 3.2. Let $\mathcal{Z} = \{(P_1, m), \dots, (P_5, m)\} \subset \mathbb{P}^n$ be 5 arbitrary equimultiple fat points. Then

$$r(\mathcal{Z}) = Seg(\mathcal{Z}).$$

Proof. Consider the two following cases for $\{P_1, \dots, P_5\}$:

Case 1. $\{P_1, \dots, P_5\}$ do not lie any linear 3-subspace of $\mathbb{P}^n$: Then $n \geq 4$ and these points P_i are in linearly general position of $\mathbb{P}^n$. So, by Proposition 2.1

$$r(\mathcal{Z}) = Seg(\mathcal{Z}) = \max\left\{2m-1, \left[\frac{5m+n-2}{n}\right]\right\} = 2m-1.$$

Case 2. $\{P_1, \dots, P_5\}$ lie on a linear 3-subspace of $\mathbb{P}^n$, say β: Then we consider three following subcases.

Case 2.1: There are no 4 points of $\{P_1, \dots, P_5\}$ on a linear 2-subspace: By Proposition 2.1 we get

$$r(\mathcal{Z}) = r(\mathcal{Z}_\beta) = Seg(\mathcal{Z}_\beta)$$

$$= Seg(\mathcal{Z}) = \max\left\{2m-1, \left[\frac{5m+1}{3}\right]\right\}.$$

Case 2.2. There are 4 points of $\{P_1, \dots, P_5\}$ on a linear 2-subspace, say α, and $\{P_1, \dots, P_5\} \not\subseteq \alpha$: If there are 3 points of $\{P_1, \dots, P_5\}$ on a line, then $T_1(\mathcal{Z}) = 3m - 1 = Seg(\mathcal{Z})$. By Lemma 2.1

$$r(\mathcal{Z}) = T_1(\mathcal{Z}) = Seg(\mathcal{Z}).$$

If there are no 3 points of $\{P_1, \dots, P_5\}$ on a line, then $T_2(\mathcal{Z}) = 2m = Seg(\mathcal{Z})$. Put

$$\mathcal{U} = \{(P_i, m) | P_i \in \alpha\}.$$

By Proposition 2.1 we have $r(\mathcal{U}) = r(\mathcal{U}_\alpha) = Seg(\mathcal{U}_\alpha) = Seg(\mathcal{U}) = 2m$. By [13, Lemma 3.3] we have $r(\mathcal{U}) \leq r(\mathcal{Z})$. By [11, Theorem 5.3] we have $r(\mathcal{Z}) \leq Seg(\mathcal{Z})$. Therefore

$$r(\mathcal{Z}) = r(\mathcal{U}) = 2m = Seg(\mathcal{Z}).$$

Case 2.3. $P_1, \dots, P_5$ lie on a linear 2-subspace, say α: If there are 3 points of $\{P_1, \dots, P_5\}$ on a line, then $T_1(\mathcal{Z}) = 3m - 1 = Seg(\mathcal{Z})$. By Lemma 2.1

$$r(\mathcal{Z}) = T_1(\mathcal{Z}) = Seg(\mathcal{Z}).$$

If there are no 3 points of $\{P_1, \dots, P_5\}$ on a line, then by Proposition 2.1 we have

$$r(\mathcal{Z}) = r(\mathcal{Z}_\alpha) = Seg(\mathcal{Z}_\alpha) = [\frac{5m}{2}] = Seg(\mathcal{Z}).$$

Theorem 3.1 and Theorem 3.2 show that if $s \leq 5$, the Segre's upper bound is sharp for any set of s equimultiple fat points in $\mathbb{P}^n$. But this is not still true for $s \geq 6$.

Proposition 3.1. Let $n \geq 2$. If $s \geq 6$, then there exists always s equimultiple fat points $\mathcal{Z} \subset \mathbb{P}^n$ such that

$$r(\mathcal{Z}) < Seg(\mathcal{Z}).$$

Proof.

Let α be a linear 2-subspace of $\mathbb{P}^n$ and let l_1, l_2 be two distinct lines on α. Let $\mathcal{Z} = \{(P_1, m), \dots, , (P_s, m)\} \subset \mathbb{P}^n$ be s equimultiple fat points such that $P_1, \dots, P_{[\frac{s}{2}]} \in l_1$, $P_{[\frac{s}{2}]+1}, \dots, P_{s-1} \in l_2 \setminus l_1$, and $P_s \notin l_1 \cup l_2$. We have

$$T_1(\mathcal{Z}) = \sum_{P_i \in l_1} m_i - 1 = \left[\frac{sm}{2}\right] - 1,$$

$$T_2(\mathcal{Z}) = \left[\frac{sm}{2}\right] > T_i(\mathcal{Z}),\ i = 3, \dots, n.$$

So,

$$Seg(\mathcal{Z}) = \left[\frac{sm}{2}\right]. \tag{1}$$

Put $P_s = (1, 0, \dots, 0)$, $J = \cap_{i=1}^{s-1} \wp_i^m$. For $j = 1, 2$, since $P_s \notin l_j$, there is a hyperplane of $\mathbb{P}^n$, say L_j, containing the line l_j and avoiding P_s. We coincide the hyperplane L_j with the linear form in $K[X_0, \dots, X_n]$ defining it. Since L_1 contains $P_1, \dots, P_{[\frac{s}{2}]}$, we have

$$L_1^m \in \cap_{i=1}^{[\frac{s}{2}]} \wp_i^m.$$

Since L_2 contains $P_{[\frac{s}{2}]+1}, \dots, P_{s-1}$, we have

$$L_2^m \in \cap_{i=[\frac{s}{2}]+1}^{s-1} \wp_i^m.$$

These imply that for every monomial M in $X_1, \dots, X_n$, $\deg(M) = i$, $i = 0, \dots, m-1$, we have

$$L_1^m L_2^m M \in J.$$

Since L_j does not contain P_s, we can write $L_j = X_0 + H_j$ for some $H_j \in (X_1, \dots, X_n) = \wp_s$, $j = 1, 2$. Therefore,

$$(X_0 + H_1)^m (X_0 + H_2)^m M \in J.$$

This implies $X_0^{3m-1-i} M \in J + \wp_s^{i+1}$, $i = 0, \dots, m-1$, because $H_1, H_2 \in \wp_s$ and $M \in \wp_s^i$. By [12, Lemma 2.2]

$$r(R/(J + \wp_s^m)) \leq 3m - 1. \tag{2}$$

Put $\mathcal{Y} = \{(P_1, m), \dots, (P_{s-1}, m)\}$. Then $r(R/J) = r(\mathcal{Y})$,

$$T_1(\mathcal{Y}) = \sum_{P_i \in l_1} m_i - 1 = \left[\frac{sm}{2}\right] - 1,$$

$$T_2(\mathcal{Y}) = \left[\frac{(s-1)m}{2}\right] > T_i(\mathcal{Y}),\ \ i = 3, \dots, n.$$

So, $Seg(\mathcal{Y}) = \left[\frac{sm}{2}\right] - 1$. By [11, Theorem 5.3] we have $r(\mathcal{Y}) \leq Seg(\mathcal{Y})$. Hence

$$r(R/J) = r(\mathcal{Y}) \leq Seg(\mathcal{Y}) = \left[\frac{sm}{2}\right] - 1. \tag{3}$$

Put $I = J \cap \wp_s^m$. Then $r(\mathcal{Z}) = r(R/I)$. By [3, Lemma 1]

$$r(R/I) = \max\{m - 1, r(R/J), r(R/(J + \wp_s^m))\}. \tag{4}$$

Since $s \geq 6$ and $m \geq 1$, from (2), (3), (4) and (1) we get

$$r(\mathcal{Z}) \leq \left[\frac{sm}{2}\right] - 1 < Seg(\mathcal{Z}).$$

Now we consider 5 fat points $\mathcal{V} = \{(P_1, m_1), \dots, (P_5, m_5)\} \subset \mathbb{P}^n$, $n \geq 2$, and $\mathcal{V}$ is not equimultiple. If $P_1, \dots, P_5$ are in linearly general position in $\mathbb{P}^n$, then by Proposition 2.1 we get $r(\mathcal{V}) = Seg(\mathcal{V})$. If $P_1, \dots, P_5$ are not in linearly general position in $\mathbb{P}^n$ and $T_1(\mathcal{V}) = Seg(\mathcal{V})$, then by Lemma 2.1 we get $r(\mathcal{V}) = Seg(\mathcal{V})$. In case of $P_1, \dots, P_5$ are not in linearly general position in $\mathbb{P}^n$ and $T_1(\mathcal{V}) < Seg(\mathcal{V})$ we predict that

$$r(\mathcal{V}) = Seg(\mathcal{V}),$$

but we cannot prove that the prediction is correct, nor can we find an example to prove that the prediction is wrong.

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Conflicts of Interest

The author declare no conflicts of interest.

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