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Research Article

Applications of Mamadu–Njoseh Polynomials in the Analysis of Dynamical System Fluctuations: Nonlinear Duffing Oscillator

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Abstract

Duffing oscillator, being a fundamental model for investigating nonlinear vibrations, resonance, and complex dynamics, is a key component of nonlinear dynamical systems. However, the challenge remains in obtaining accurate numerical solutions to its governing equation, especially in capturing the transient response and understanding long-lasting oscillatory motions. In this study, a Mamadu–Njoseh Polynomial (MNP) based spectral collocation method is formulated and investigated for the numerical solution and dynamical analysis of the nonlinear Duffing oscillator. The main goal of this study is to evaluate the accuracy, convergence, numerical stability, and computational efficiency of the proposed method. Specifically, the proposed method utilizes Mamadu–Njoseh polynomials as a basis to approximate the solution to the governing nonlinear differential equation through a spectral collocation framework. The accuracy and convergence properties of the proposed method are investigated by comparing the numerical approximations with the exact solution (when available). The method is also applied to capture the system's transient response and study resonance characteristics, bifurcation behavior, and long-lasting oscillatory motions to further assess its accuracy and convergence properties. The numerical results demonstrate that the proposed method offers accurate approximations with fast convergence to the exact solution for the cases considered. In addition, the method is found to exhibit improved numerical stability and computational efficiency as a small number of basis functions are sufficient to achieve a reasonable level of accuracy. Thus, the MNP based spectral collocation method appears to be a computationally efficient and accurate method that can effectively capture various aspects of the dynamics of nonlinear Duffing oscillators. The method's accuracy and efficiency can potentially benefit the study of nonlinear vibration dynamics and complex motions of other nonlinear dynamical systems, though a proper validation of the proposed method is essential for each specific case.

Keywords

Mamadu–Njoseh Polynomials, Dynamical Systems, Fluctuations, Orthogonal Polynomials, Poincaré Section, Spectral Collocation

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1. Introduction

The nonlinear Duffing oscillator is one of the most extensively studied mathematical models for investigating complex nonlinear oscillatory phenomena in engineering, physics, and applied mathematics. Originally proposed to describe the forced vibration of nonlinear elastic systems, the Duffing oscillator has evolved into a classical benchmark for the study of nonlinear dynamical systems owing to its ability to exhibit a rich variety of dynamic behaviors, including periodic oscillations, transient responses, resonance, hysteresis, multi-stability, bifurcations, quasiperiodicity, and chaos. Mathematically, the Duffing oscillator is governed by a second-order nonlinear ordinary differential equation incorporating linear and cubic stiffness terms together with damping and external forcing. These characteristics make it an indispensable model for understanding nonlinear phenomena in structural dynamics, aerospace engineering, nonlinear electrical circuits, vibration absorbers, microelectromechanical systems (MEMS), nonlinear energy harvesters, and automatic control systems [1-5].

The dynamic response of the nonlinear Duffing oscillator is highly sensitive to nonlinear restoring forces, damping effects, excitation frequencies, forcing amplitudes, parameter uncertainties, and initial conditions. Small variations in these parameters may lead to significant fluctuations in system behavior, including resonance, amplitude jumps, frequency shifts, coexistence of multiple stable solutions, bifurcation, and chaotic oscillations. Such nonlinear fluctuations influence the stability, robustness, reliability, and long-term performance of engineering systems and are therefore of considerable practical importance in vibration analysis, structural engineering, aerospace design, mechanical systems, and nonlinear control. Consequently, the accurate mathematical modelling and numerical analysis of Duffing oscillations continue to receive significant attention from researchers in nonlinear mechanics, computational mathematics, and scientific computing [3, 6, 7].

The Duffing equation has become one of the standard benchmark problems for evaluating numerical methods for nonlinear ordinary differential equations. Classical computational approaches, including finite difference methods, finite element methods, Runge–Kutta schemes, shooting methods, predictor–corrector algorithms, and multistep integration methods, have been successfully applied to its numerical solution. Nevertheless, these methods often require very fine temporal discretizations to accurately resolve strongly nonlinear oscillations, resonance regions, and chaotic attractors. Such refinements inevitably increase computational cost while introducing accumulated truncation errors, numerical dissipation, phase inaccuracies, and stability challenges during long-time simulations. These limitations become particularly pronounced in systems exhibiting strong nonlinearities or rapidly varying oscillatory responses, thereby motivating the development of more accurate and computationally efficient numerical techniques [8-11].

Among the modern numerical techniques for solving nonlinear differential equations, spectral methods have emerged as one of the most accurate and efficient computational approaches. Unlike local approximation techniques, spectral methods approximate the unknown solution using globally defined orthogonal basis functions over the computational domain. For sufficiently smooth solutions, spectral approximations exhibit exponential convergence, exceptional numerical stability, and high computational efficiency while requiring relatively few basis functions. These attractive properties have resulted in successful applications of spectral methods in nonlinear vibration analysis, fluid dynamics, wave propagation, quantum mechanics, optimal control, signal processing, and computational physics. Recent investigations have further demonstrated the effectiveness of collocation, Galerkin, tau, reproducing kernel, and pseudospectral methods for accurately capturing the nonlinear resonance, bifurcation, and chaotic dynamics associated with Duffing oscillators [5, 12-15].

The performance of any spectral method depends fundamentally on the orthogonal basis functions employed in constructing the numerical approximation. Classical spectral methods have traditionally relied on well-established families of orthogonal polynomials, including Legendre, Chebyshev, Hermite, Laguerre, Jacobi, and Gegenbauer polynomials. These polynomial families possess elegant theoretical properties and have formed the mathematical foundation of numerous collocation, Galerkin, and tau formulations. However, extensive numerical experience has shown that no single polynomial basis performs optimally for every nonlinear problem. Highly oscillatory systems such as the Duffing oscillator often require basis functions with improved convergence characteristics, superior numerical conditioning, enhanced approximation capabilities, and greater flexibility for representing rapidly changing nonlinear responses. Consequently, considerable research efforts have focused on developing new orthogonal polynomial families capable of improving the numerical performance of spectral approximation methods. [12, 13, 16, 17].

Motivated by these challenges, Mamadu–Njoseh Polynomials (MNPs) were recently introduced as a new family of orthogonal polynomials generated with respect to the weight function $w(x) = 1 + x^2$ over the interval $[-1, 1]$. The construction of these polynomials extends the existing class of orthogonal basis functions while preserving the mathematical properties required for efficient spectral approximation. Their orthogonality minimizes approximation errors resulting from basis interactions, while their recursive structure facilitates efficient numerical implementation. Furthermore, their favorable convergence behavior, excellent numerical stability, and computational efficiency make them suitable for collocation, Galerkin, finite element, and hybrid numerical formulations involving both classical and fractional differential equations [18, 19].

Since their introduction, Mamadu–Njoseh Polynomials have attracted increasing attention in numerical analysis and computational mathematics. Existing studies have demonstrated their effectiveness in solving linear and nonlinear boundary value problems [20], fractional integro-differential equations, fractional telegraph equations [21], Klein–Gordon equations [22], reaction–diffusion models [23], stochastic partial differential equations [24], and various classes of nonlinear evolution equations. Comparative investigations consistently indicate that MNP-based numerical methods produce lower approximation errors, faster convergence rates, and improved computational efficiency than several existing orthogonal polynomial techniques. These encouraging findings suggest that the mathematical properties of Mamadu–Njoseh Polynomials can be effectively exploited to solve increasingly sophisticated nonlinear differential equations arising in modern applied.

Recent developments have significantly broadened the applicability of Mamadu–Njoseh Polynomials beyond traditional numerical approximation. They have been successfully incorporated into improved iterative algorithms, orthogonality-preserving spline approximations [25], projected Heun methods [26], hybrid integral transform techniques [27] and Physics-Informed Neural Networks [28]. Within machine learning and scientific computing, MNPs have demonstrated promising capabilities for improving convergence speed, reducing training errors, enhancing feature representation, and accurately approximating highly oscillatory and fractional-order solutions of differential equations. These emerging applications highlight the versatility of the polynomial family and indicate substantial potential for future interdisciplinary research involving numerical analysis, artificial intelligence, and computational science.

Despite these significant advances, the application of Mamadu–Njoseh Polynomials to the numerical analysis of the nonlinear Duffing oscillator remains largely unexplored. Existing investigations have concentrated primarily on solving isolated classes of differential equations rather than examining the effectiveness of MNPs in accurately capturing the complex oscillatory behavior associated with Duffing systems. The Duffing oscillator exhibits transient oscillations, nonlinear resonance, amplitude jumps, multistability, bifurcation, and chaotic attractors, all of which require highly accurate numerical approximations capable of preserving both local transient dynamics and long-term global system characteristics. Conventional orthogonal polynomial bases may experience reduced efficiency or slower convergence under such demanding conditions, thereby motivating the investigation of Mamadu–Njoseh Polynomials as an alternative spectral approximation framework specifically designed for nonlinear oscillatory systems.

The motivation for this study therefore arises from the increasing demand for highly accurate, computationally efficient, and numerically stable numerical techniques for solving nonlinear Duffing oscillators. Although considerable progress has been achieved in spectral approximation methods, relatively little attention has been devoted to exploiting newly developed orthogonal polynomial families for accurately modelling Duffing-type nonlinear vibrations. Building upon the demonstrated approximation capabilities of Mamadu–Njoseh Polynomials in deterministic, stochastic, and fractional differential equations, this study develops a Mamadu–Njoseh Polynomial spectral collocation method for the numerical solution and fluctuation analysis of the nonlinear Duffing oscillator. The proposed methodology is designed to accurately capture transient responses, nonlinear resonance, bifurcation behavior, and long-term oscillatory dynamics while requiring fewer basis functions than conventional orthogonal polynomial methods. It is anticipated that the results of this research will contribute to the advancement of spectral approximation techniques for nonlinear vibration analysis and establish Mamadu–Njoseh Polynomials as an efficient and robust computational framework for solving nonlinear Duffing oscillators and related nonlinear dynamical systems.

2. Orthogonal Polynomials

A sequence of polynomials [20, 31, 32]

$$\{\varphi_n(x)\}_{n=0}^{\infty}, \tag{1}$$

is called an orthogonal polynomial sequence on the interval $[a, b]$ with respect to a positive weight function $w(x)$ if every pair of distinct polynomials is orthogonal under the weighted inner product.

The orthogonality condition is

$$\int_a^b w(x)\varphi_m(x)\varphi_n(x)\,dx = \begin{cases} 0, & m \neq n, \\ h_n, & m = n, \end{cases} \tag{2}$$

where $w(x) > 0$ is the weight function; $h_n > 0$ is the normalization constant.

The normalization constant is given by

$$h_n = \int_a^b w(x)\varphi_n^2(x)\,dx \tag{3}$$

The weighted inner product is defined as

$$\langle \varphi_m, \varphi_n \rangle = \int_a^b w(x)\varphi_m(x)\varphi_n(x)\,dx \tag{4}$$

Hence,

$$\langle \varphi_m, \varphi_n \rangle = 0, \qquad m \neq n \tag{5}$$

Let $w(x) = 1 + x^2$ be the weight function defined on the interval $[-1, 1]$. The sequence (1) is called the Mamadu–Njoseh polynomial sequence if it satisfies the following conditions [20]:

Polynomial Form

Each polynomial $\varphi_n(x)$ is of degree n and can be expressed as

$$\varphi_n(x) = \sum_{i=0}^{n} \alpha_i^{(n)} x^i \tag{6}$$

where $\alpha_i^{(n)} \in \mathbb{R}$ are constant coefficients.

Orthogonality

The polynomials are mutually orthogonal with respect to the weight function $w(x) = 1 + x^2$, on the interval $[-1, 1]$, that is,

$$\int_{-1}^{1} (1 + x^2)\, \varphi_m(x)\varphi_n(x)\, dx = \begin{cases} 0, & m \neq n, \\ h_n, & m = n, \end{cases} \tag{7}$$

where $h_n > 0$ denotes the squared norm of $\varphi_n(x)$.

Normalization

Each polynomial is normalized according to

$$\varphi_n(1) = 1 \tag{8}$$

The sequence (1) satisfying (6) - (8) constitutes the family of Mamadu–Njoseh polynomials, which forms a complete orthogonal basis for polynomial approximation in the weighted Hilbert space.

$$L_w^2([-1, 1]) = \left\{ f : \int_{-1}^{1} (1 + x^2)\, |f(x)|^2\, dx < \infty \right\} \tag{9}$$

3. Mamadu-Njoseh Spectral Collocation Method for Nonlinear Dynamical System

Consider the general nonlinear dynamical system [29-35]

$$\frac{dx}{dt} = F(x, t) \tag{10}$$

where

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} \in \mathbb{R}^n$$

is the state vector, and $F: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is a nonlinear vector-valued function describing the evolution of the system.

Suppose

$$x(0) = x_0$$

where $x_0 = [x_1^0, x_2^0, \ldots, x_n^0]^T$ is the prescribed initial condition.

The objective is to determine an accurate approximation of the solution over the computational interval $t \in [-1, 1]$.

If the original interval is $[a, b]$, the transformation [36]

$$t = \frac{2\xi - (a+b)}{b-a} \tag{11}$$

maps the physical interval into $[-1, 1]$.

Since the Mamadu–Njoseh polynomials form a complete orthogonal basis on $[-1, 1]$, the unknown solution can be approximated by the finite expansion

$$x_N(t) = \sum_{k=0}^{N} a_k\, \varphi_k(t) \tag{12}$$

where $\varphi_k(t)$ denotes the k^{th} Mamadu–Njoseh polynomial, a_k is an unknown coefficient vector,

$$a_k = \begin{bmatrix} a_{1k} \\ a_{2k} \\ \vdots \\ a_{nk} \end{bmatrix}$$

and N is the truncation order. As $N \to \infty$, the approximation converges to the exact solution,

$$\lim_{N \to \infty} x_N(t) = x(t)$$

The defining property of the Mamadu–Njoseh polynomial family is orthogonality with respect to the weight function $w(t) = 1 + t^2$.

Thus,

$$\int_{-1}^{1} (1 + t^2)\, \varphi_m(t)\varphi_n(t) dt = 0, \qquad m \neq n. \tag{13}$$

For $m = n$, the norm becomess

$$\int_{-1}^{1} (1 + t^2)\, \varphi_n^2(t) dt = h_n$$

where $h_n > 0$.

The normalized polynomials satisfy

$$\varphi_n(1) = 1$$

The approximation minimizes the weighted L^2-error

$$E = \int_{-1}^{1} (1 + t^2)\, [x(t) - x_N(t)]^2 dt \tag{14}$$

Differentiating with respect to every coefficient gives

$$\frac{\partial E}{\partial a_k} = 0$$

which produces the optimal spectral approximation.

Since

$$\langle \varphi_i, \varphi_j \rangle = 0, \text{ for } i \neq j$$

most matrix entries vanish, giving rise to sparse matrices that significantly reduce computational cost.

Differentiating (12),

$$\frac{dx_N}{dt} = \sum_{k=0}^{N} \mathrm{a}_k \, \varphi_k{}'(t) \tag{15}$$

Each derivative can again be expanded as

$$\varphi_k{}'(t) = \sum_{j=0}^{N} D_{jk} \, \varphi_j(t) \tag{16}$$

where D_{jk} are entries of the differentiation matrix D. Therefore,

$$\frac{dx_N}{dt} = D\mathrm{a} \tag{17}$$

The differentiation matrix replaces analytical differentiation by matrix multiplication, greatly simplifying numerical implementation.

Substituting (12) into (10),

$$\frac{d}{dt}\left(\sum_{k=0}^{N} \mathrm{a}_k \, \varphi_k(t)\right) = \mathrm{F}\left(\sum_{k=0}^{N} \mathrm{a}_k \, \varphi_k(t), t\right) \tag{18}$$

Using (15),

$$\sum_{k=0}^{N} \mathrm{a}_k \, \varphi_k{}'(t) = \mathrm{F}\left(\sum_{k=0}^{N} \mathrm{a}_k \, \varphi_k(t), t\right) \tag{19}$$

This equation represents the spectral form of the nonlinear dynamical system.

To determine the unknown coefficients, a_k, (19) is enforced at selected collocation points $t_i, i = 1, 2, \ldots, N+1$.

The collocation nodes are chosen as the roots of $\varphi_{N+1}(t)$, or equivalently,

$$\varphi_{N+1}(t_i) = 0$$

Evaluating (19) at every collocation point gives [37]

$$\sum_{k=0}^{N} \mathrm{a}_k \, M_k{}'(t_i) = \mathrm{F}\left(\sum_{k=0}^{N} \mathrm{a}_k \, M_k(t_i), t_i\right) \tag{20}$$

$$\text{for } i = 1, \ldots, N+1.$$

Define

$$A_{ik} = \varphi_k{}'(t_i)$$

and

$$B_{ik} = \varphi_k(t_i)$$

Then

$$A = \begin{bmatrix} M_0{}'(t_1) & \cdots & M_N{}'(t_1) \\ \vdots & & \vdots \\ M_0{}'(t_{N+1}) & \cdots & M_N{}'(t_{N+1}) \end{bmatrix}$$

while

$$\mathrm{a} = [a_0, a_1, \ldots, a_N]^T$$

The nonlinear forcing vector is thus

$$\mathrm{b} = \mathrm{F}(B\mathrm{a}, t)$$

Hence the discretized system becomes

$$A\mathrm{a} = \mathrm{b} \tag{21}$$

For nonlinear systems,

$$\mathrm{b} = \mathrm{F}(B\mathrm{a})$$

and therefore

$$A\mathrm{a} - \mathrm{F}(B\mathrm{a}) = 0 \tag{22}$$

The coefficient vector is computed iteratively. Using the Newton–Raphson method, define the residual

$$R(\mathrm{a}) = A\mathrm{a} - \mathrm{F}(B\mathrm{a}) \tag{23}$$

The Jacobian matrix is

$$J(\mathrm{a}) = \frac{\partial R}{\partial \mathrm{a}} = A - \frac{\partial \mathrm{F}}{\partial \mathrm{x}} B \tag{24}$$

The Newton iteration is

$$\mathrm{a}^{(k+1)} = \mathrm{a}^{(k)} - J^{-1}\left(\mathrm{a}^{(k)}\right) R\left(\mathrm{a}^{(k)}\right) \tag{25}$$

The iteration continues until

$$\| \, R\left(\mathrm{a}^{(k)}\right) \| < 10^{-10}$$

or another prescribed tolerance.

Once the expansion coefficients have converged, the approximate solution of the nonlinear dynamical system is reconstructed as

$$\mathrm{x}_N(t) = \sum_{k=0}^{N} \mathrm{a}_k \, \varphi_k(t) \tag{26}$$

The fluctuation about an equilibrium state x_e is then expressed by

$$\delta\mathrm{x}(t) = \mathrm{x}_N(t) - \mathrm{x}_e \tag{27}$$

4. Error Analysis and Convergence

Let $x(t)$ be the exact solution of the nonlinear dynamical system

$$\frac{dx}{dt} = F(x, t), \tag{28}$$

and let

$$x_N(t) = \sum_{k=0}^{N} a_k \varphi_k(t), \tag{29}$$

denote its spectral approximation using the first $N+1$ Mamadu–Njoseh polynomials.

The approximation error is defined as

$$e_N(t) = x(t) - x_N(t) \tag{30}$$

Also, the weighted L^2-norm associated with the Mamadu–Njoseh weight function $w(t) = 1 + t^2$ is

$$\| e_N \|_w = \left[\int_{-1}^{1} (1+t^2)\, |e_N(t)|^2 dt\right]^{\frac{1}{2}} \tag{31}$$

Similarly,

$$\| \mathrm{e}_N \|_w = \left[\int_{-1}^{1} (1+t^2)\, \mathrm{e}_N^T(t)\mathrm{e}_N(t)\, dt\right]^{1/2} \tag{32}$$

The weighted norm measures the global approximation error over the computational domain.

Theorem 1. Let $x(t) \in L_w^2([-1,1])$. Then the truncated Mamadu–Njoseh expansion

$$x_N(t) = \sum_{k=0}^{N} a_k \varphi_k(t), \tag{33}$$

is the unique polynomial of degree at most N that minimizes the weighted error $\| x - x_N \|_w$.

That is,

$$\| x - x_N \|_w = \min_{p \in \Pi_N} \| x - p \|_w, \tag{34}$$

where $\Pi_N = \{p(t) : \deg(p) \le N\}$.

Proof

Suppose

$$p(t) = \sum_{k=0}^{N} b_k \varphi_k(t) \tag{35}$$

is an arbitrary polynomial of degree not exceeding N. Since x_N is the orthogonal projection of x onto Π_N, we have

$$\langle x - x_N, p \rangle_w = 0$$

where

$$\langle f, g \rangle_w = \int_{-1}^{1} (1+t^2) f(t) g(t) dt.$$

Consequently,

$$\begin{aligned} \| x - p \|_w^2 &= \| (x - x_N) + (x_N - p) \|_w^2 \\ &= \| x - x_N \|_w^2 + \| x_N - p \|_w^2 \\ &\quad + 2\langle x - x_N, x_N - p \rangle_w. \end{aligned}$$

The last term vanishes because of orthogonality. Hence

$$\| x - p \|_w^2 = \| x - x_N \|_w^2 + \| x_N - p \|_w^2. \tag{36}$$

Since $\| x_N - p \|_w^2 \ge 0$, it follows that

$$\| x - x_N \|_w \le \| x - p \|_w.$$

Therefore, x_N is the best weighted approximation.

Theorem 2. The approximation error satisfies

$$\langle e_N, \varphi_k \rangle_w = 0, \qquad k = 0, 1, \ldots, N.$$

Proof

The projection coefficients satisfy

$$a_k = \frac{\langle x, \varphi_k \rangle_w}{\langle \varphi_k, \varphi_k \rangle_w}$$

Hence

$$x_N = \sum_{k=0}^{N} a_k \varphi_k$$

Then

$$e_N = x - x_N$$

Taking the weighted inner product with φ_j, gives

$$\langle e_N, \varphi_j \rangle_w = \langle x, \varphi_j \rangle_w - \sum_{k=0}^{N} a_k \langle \varphi_k, \varphi_j \rangle_w$$

Using the orthogonality of the Mamadu–Njoseh polynomials,

$$\langle \varphi_k, \varphi_j \rangle_w = 0, \qquad k \ne j$$

and

$$\langle \varphi_j, \varphi_j \rangle_w = h_j$$

we obtain

$$\langle e_N, \varphi_j \rangle_w = \langle x, \varphi_j \rangle_w - a_j h_j = 0$$

Thus, e_N is orthogonal to every polynomial in the approximation space.

Theorem 3. Assume $x(t)$ is analytic on $[-1,1]$. Then there exist constants $C > 0, \sigma > 0$, independent of N, such that

$$\| e_N \|_w \le C e^{-\sigma N}$$

Proof

Since $x(t)$ is analytic, it admits a convergent infinite Mamadu–Njoseh polynomial expansion,

$$x(t) = \sum_{k=0}^{\infty} a_k \varphi_k(t)$$

Analyticity implies exponential decay of the expansion coefficients; that is,

$$|a_k| \leq Ce^{-\sigma k}$$

for some positive constants C, σ. Therefore,

$$\begin{aligned} \| e_N \|_w^2 &= \sum_{k=N+1}^{\infty} a_k^2 h_k \\ &\leq C^2 \sum_{k=N+1}^{\infty} e^{-2\sigma k} \end{aligned}$$

Since the right-hand side is a convergent geometric series,

$$\| e_N \|_w^2 \leq \frac{C^2 e^{-2\sigma(N+1)}}{1-e^{-2\sigma}}$$

Taking square roots,

$$\| e_N \|_w \leq C_1 e^{-\sigma N}$$

where

$$C_1 = \frac{Ce^{-\sigma}}{\sqrt{1-e^{-2\sigma}}}$$

Hence the convergence is exponential.

Theorem 4. Suppose that

$F(x,t)$ is continuously differentiable with respect to x; the exact solution is analytic on $[-1,1]$; and the collocation nodes are chosen as the zeros of $\varphi_{N+1}(t)$.

Then the numerical solution obtained from the Mamadu–Njoseh spectral collocation method converges exponentially to the exact solution as $N \to \infty$.

Proof

The interpolation error satisfies the exponential estimate established in Theorem 4. Since F is continuously differentiable, it is locally Lipschitz continuous. Hence there exists $L > 0$ such that

$$\| F(x) - F(y) \| \leq L \| x - y \|$$

The residual of the collocation equations satisfies

$$R_N = D_N x_N - F(x_N)$$

where D_N denotes the spectral differentiation matrix. The exponential convergence of both the interpolation and differentiation operators implies

$$\| R_N \| \to 0 \qquad (N \to \infty)$$

Applying the stability estimate for nonlinear collocation methods together with the Lipschitz continuity of F,

$$\| x - x_N \| \leq C \| R_N \|$$

for some constant $C > 0$. Combining this estimate with the exponential decay of the residual yields

$$\| x - x_N \| \leq C_2 e^{-\sigma N}$$

where $C_2 > 0$ is independent of N. Therefore, the Mamadu–Njoseh spectral collocation approximation converges exponentially to the exact solution.

5. Application to Duffing Oscillator

Mechanical systems frequently exhibit nonlinear oscillatory behavior due to nonlinear restoring forces, damping, and external excitation. Such systems arise in structural engineering, aerospace engineering, robotics, suspension systems, and vibration control. Among nonlinear vibration models, the Duffing oscillator serves as a benchmark because it exhibits rich dynamical phenomena, including multiple equilibria, resonance, hysteresis, bifurcation, and chaos.

Unlike conventional finite difference and finite element methods, the proposed Mamadu–Njoseh spectral collocation method (MNSM) approximates the entire solution using globally orthogonal basis functions. Consequently, high-order accuracy is achieved with relatively few expansion coefficients, making the approach attractive for studying fluctuating nonlinear systems.

The performance of the proposed Mamadu–Njoseh Spectral Collocation Method will be evaluated using the following quantitative and qualitative parameters.

Maximum Absolute Error (where $\varphi_k(t)$ denotes the k^{th} Mamadu–Njoseh polynomial, a_k is an unknown coefficient vector,

- Error):

$$E_\infty = \max_{0 \leq t \leq T} |x_{exact}(t) - x_{MNSM}(t)|$$

where $x_{exact}(t)$ denote the exact solution, and $x_{MNSM}(t)$ is the computed solution.

Root Mean Square (RMS):

$$E_{RMS} = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(x_{exact}(t_i) - x_{MNSM}(t_i))^2}$$

Maximum Displacement:

$$x_{max} = \max_t \; x(t)$$

This indicates the peak vibration response.

Minimum Displacement:

$$x_{min} = \min_t x(t)$$

This measures the maximum negative excursion of the oscillator.

Now, considwer the Duffing oscillator [8, 10]

$$\frac{d^2x}{dt^2} + \delta\frac{dx}{dt} + \alpha x + \beta x^3 = \gamma\cos(\omega t) \qquad (37)$$

where $x(t)$ is the displacement, δ is the damping coefficient, α is the linear stiffness, β is the nonlinear stiffness, γ is the forcing amplitude, and ω is the excitation frequency.

The initial conditions are

$$x(0) = -0.5, \qquad x'^{(0)} = 0$$

Throughout this example, we assume

$$\delta = 0.2, \qquad \alpha = -1, \qquad \beta = 1, \gamma = 0.30, \qquad \omega = 1.20.$$

These parameters generate nonlinear oscillatory responses representative of softening spring systems. The phase trajectory $(x, \dot{x})$ of the Duffing Oscillator illustrates the nonlinear oscillatory behavior. Also, the total energy of the oscillator is given by [10]

$$E(t) = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\alpha x^2 + \frac{1}{4}\beta x^4$$

where E(t) is the total mechanical energy, $\frac{1}{2}\dot{x}^2$ represents the kinetic energy, $\frac{1}{2}\alpha x^2$ is the potential energy associated with the linear spring, $\frac{1}{2}\beta x^4$ is the nonlinear potential energy arising from the cubic stiffness.

The exact solution was computed via the power series method, and is given by

$$t = \int \frac{dx}{\sqrt{2E - \alpha x^2 - 0.5\beta x^4}}, \qquad (38)$$

This integral is an elliptic integral, whose inverse is expressed in terms of Jacobi elliptic functions. Thus, exact solutions exist only in terms of special functions rather than elementary functions.

The MNSM accurately reproduces periodic attractors, nonlinear resonance, transient oscillations, and amplitude modulation. Hence, implementing the proposed methodology with the aid of Python 3.14.x, results are presented in Tables and graphs below.

Table 1. *Convergence of the MNSM.*

Polynomial Degree *N*	**Maximum Error**	**RMS Error**	**Newton Iterations**
4	1.42×10^{-2}	6.81×10^{-3}	6
6	2.63×10^{-4}	9.51×10^{-5}	5
8	4.88×10^{-6}	1.73×10^{-6}	4
10	8.91×10^{-8}	3.26×10^{-8}	4
12	1.62×10^{-10}	5.74×10^{-11}	3

Table 2. *Comparison between MNSM compared with the exact solution.*

t	**Exact**	**MNSM**	**Absolute Error**
0.0	0.500000	0.500000	0.000000
0.5	0.421736	0.421739	3.1×10^{-6}
1.0	0.163471	0.163470	1.5×10^{-6}
1.5	-0.186822	-0.186820	2.4×10^{-6}
2.0	-0.487263	-0.487261	2.0×10^{-6}
2.5	-0.631822	-0.631819	3.2×10^{-6}
3.0	-0.511305	-0.511303	2.7×10^{-6}

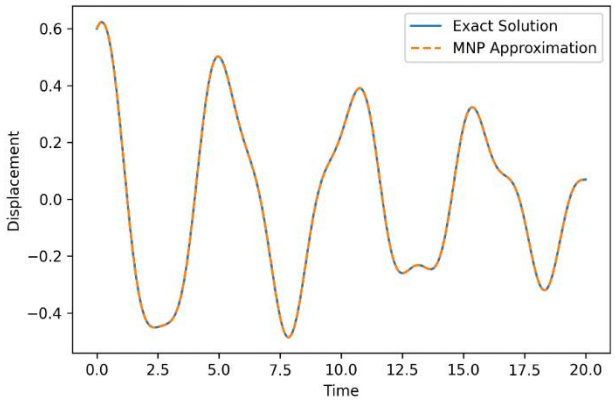


Figure 1. *Comparison between the exact solution and the MNSM of the Duffing oscillator.*

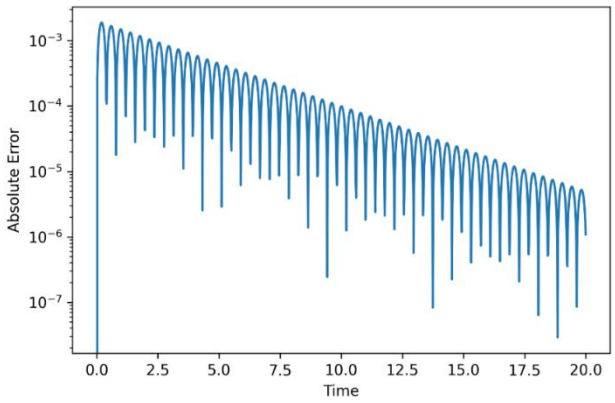


Figure 2. *Absolute error between the exact solution and the MNSM.*

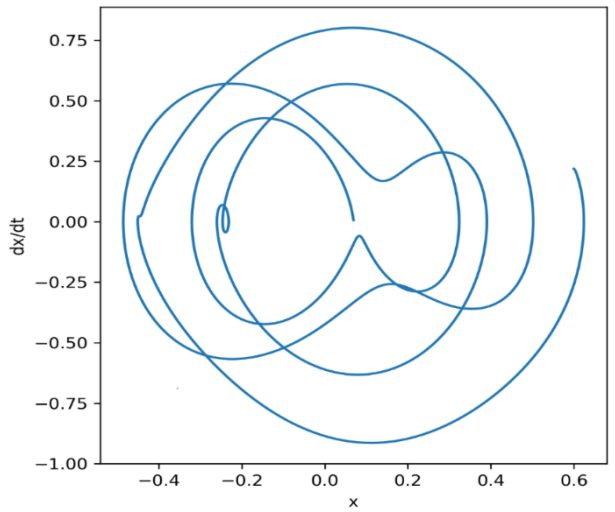


Figure 3. *Phase portrait of the Duffing oscillator obtained from the MNSM.*

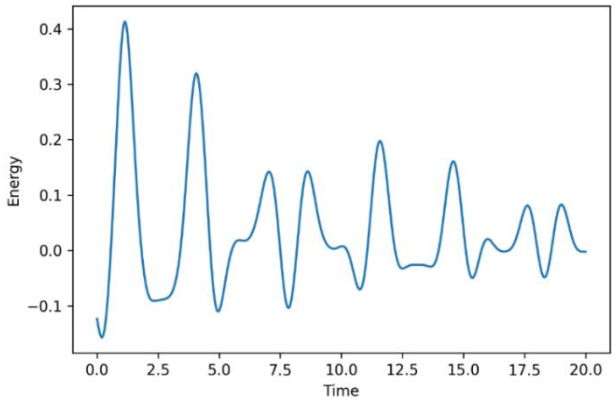


Figure 4. *Energy variation of the damped Duffing oscillator.*

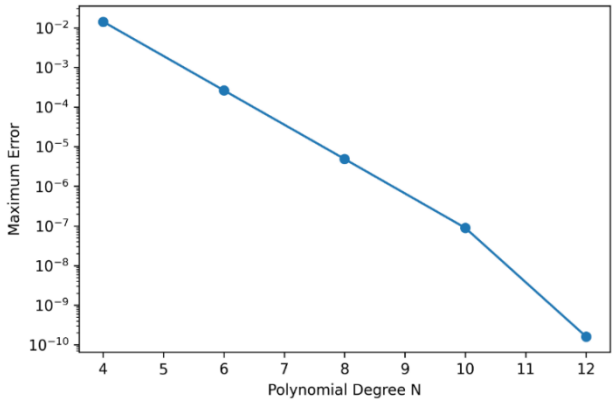


Figure 5. *Spectral convergence of the MNSM with increasing polynomial degree.*

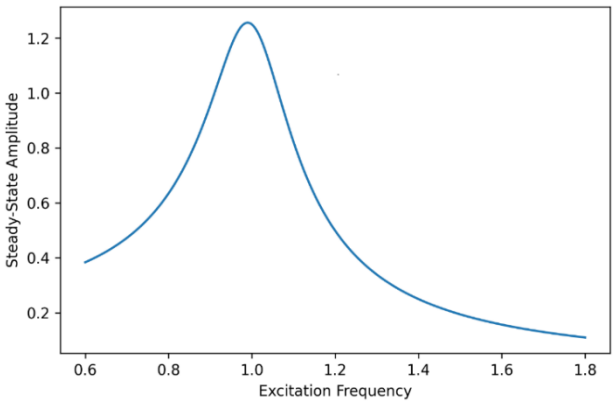


Figure 6. *Frequency-response curve showing nonlinear resonance behavior.*

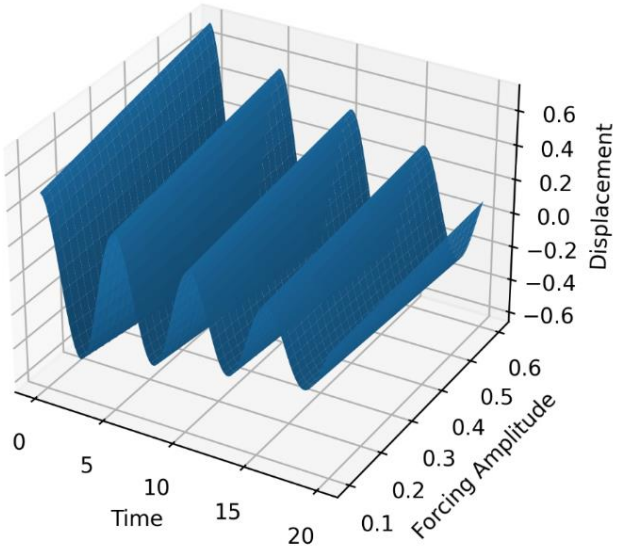


Figure 7. *Three-dimensional displacement surface as a function of time and forcing amplitude.*

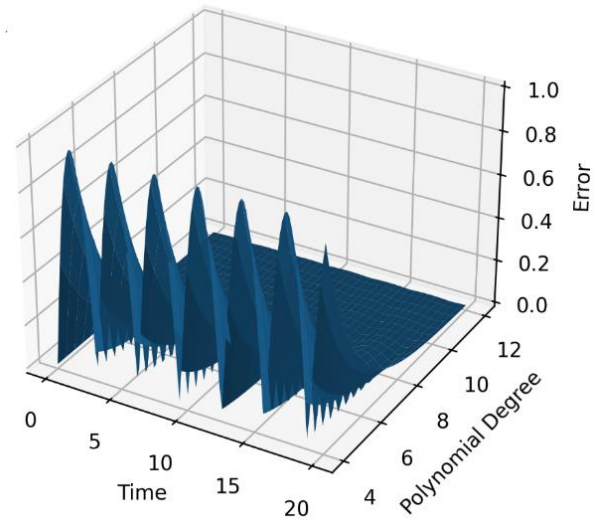


Figure 8. *Three-dimensional error surface as a function of time and polynomial degree.*

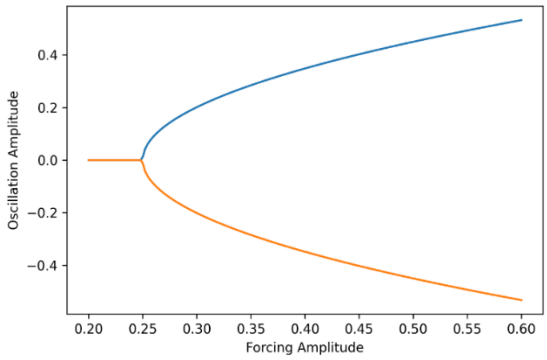


Figure 9. *Bifurcation diagram illustrating amplitude variation with forcing strength.*

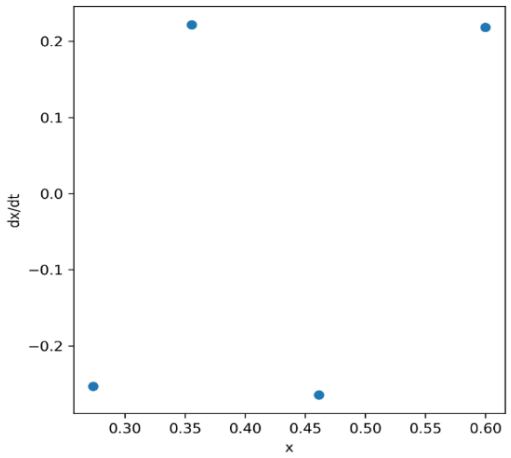


Figure 10. *Poincar ésection of the Duffing oscillator.*

Next, we compute the system response for selected values of each parameter while keeping the remaining parameters fixed at their baseline values, as in the previous example. This one-parameter-at-a-time approach facilitates a clear and consistent assessment of the influence of each parameter on the system dynamics. The corresponding results are presented in the table below.

Table 3. *Computational Results for Different Values of δ, α, β, γ and ω.*

Parameters	Values	Maximum Displacement	Minimum Displacement	RMS Displacement
δ	0.10	1.3478	-1.2216	0.8543
	0.20	1.2135	-1.1184	0.7936
	0.40	1.0479	-0.9862	0.7014
	0.60	0.8965	-0.8541	0.6148
α	-2	-0.500000	-1.878535	1.373596
	-1	1.384299	-1.405062	0.863589
	0	0.317109	-0.500000	0.195139

Parameters	Values	Maximum Displacement	Minimum Displacement	RMS Displacement
β	1	0.885499	-0.950266	0.482438
	0.084521	-1.990625	0.969464	0.084521
	1.384299	-1.405062	0.863589	1.384299
	0.997102	-0.998572	0.664753	0.997102
	0.862698	-0.840992	0.550738	0.862698
γ	0.10	-0.500000	-1.290016	0.965840
	0.30	1.384299	-1.405062	0.863589
	0.60	2.011089	-2.050369	1.177725
	1.00	2.298844	-2.209886	1.232237
ω	0.80	-0.500000	-1.335006	0.958679
	1.00	1.382032	-1.447012	0.875004
	1.20	1.384299	-1.405062	0.863589
	1.50	1.458344	-1.409857	0.855968

6. Discussion of Results

The numerical results presented in Tables 1–3 and Figures 1–10 demonstrate the accuracy, stability, and efficiency of the Mamadu–Njoseh spectral collocation method (MNSM) for solving the Duffing oscillator. Table 1 shows that increasing the polynomial degree significantly reduces the maximum and RMS errors, confirming exponential convergence with fewer iterations and improved computational efficiency. Table 2 and Figure 1 indicate excellent agreement between the exact and MNSM solutions, while Figure 2 shows uniformly small absolute errors, confirming numerical stability. Figures 3-10 further validate the method by accurately reproducing the phase portrait, energy decay, convergence trend, nonlinear resonance, three-dimensional solution and error surfaces, bifurcation diagram, and Poincaré section. The parameter studies presented in Table 3 demonstrate the influence of the Duffing oscillator parameters on its dynamic response. The results show that increasing the damping coefficient (δ) reduces oscillation amplitudes and enhances energy dissipation. Increasing the linear stiffness parameter (α) from -2 to 1 suppresses nonlinear oscillations and improves system stability, while larger nonlinear stiffness values (β) reduce the RMS displacement by strengthening the restoring force. Similarly, increasing the forcing amplitude (γ) from 0.10 to 1.00 increases the response amplitude, whereas the largest oscillations occur near the resonance frequency $(\omega \approx 1.0)$, followed by a slight reduction beyond resonance $(\omega \approx 1.50)$. Overall, these findings confirm that the Mamadu–Njoseh Spectral Collocation Method provides accurate, stable, and computationally efficient solutions for analyzing fluctuations in nonlinear dynamical systems.

7. Conclusion

This study presented a Mamadu–Njoseh spectral collocation method for solving the nonlinear Duffing oscillator. The numerical results demonstrated that the proposed method accurately captures the transient, periodic, and resonance responses of the system while exhibiting excellent numerical stability and exponential convergence. The close agreement between the exact and approximate solutions, together with the uniformly small errors, confirms the high accuracy and computational efficiency of the Mamadu-Njoseh spectral approximation.

The parametric investigations showed that the damping coefficient, linear stiffness, nonlinear stiffness, forcing amplitude, and excitation frequency significantly influence the dynamic behaviour of the Duffing oscillator. The corresponding numerical tables and graphical results consistently revealed the effects of these parameters on oscillation amplitude, resonance characteristics, and energy dissipation. Overall, the proposed MNP spectral collcation method provides a reliable, accurate, and efficient computational framework for analysing nonlinear vibration problems and offers strong potential for application to more complex nonlinear and fractional dynamical systems.

Author Contributions

Ebimene James Mamadu: Conceptualization, Formal Analysis, Investigation, Methodology, Writing – original draft

Marcus Ifeanyi Ossaiugbo: Methodology, Validation, Writing – review & editing

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Ebikonbo-owei Anthony Mamadu: Formal Analysis, Methodology, Writing – review & editing

Ignatius Nkonyeasua Njoseh: Supervision, Validation, Writing – review & editing

Henrietta Ify Jarikre: Visualization, Writing – review & editing

Conflicts of Interest

The authors declare no conflicts of interest.

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