

# On the Discrete and Continuous Harmonic Encoding of Primes

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**To cite this article:**

Ioannis Papadakis. (2026). On the Discrete and Continuous Harmonic Encoding of Primes. *Mathematics and Computer Science*, 11(3), 45-56. <https://doi.org/10.11648/j.mcs.20261103.11>

**Received:** 4 April 2026 ; **Accepted:** 15 April 2026 ; **Published:** 21 May 2026

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**Abstract:** The dichotomy between the prime-based multiplicative and additive representations of integers poses fundamental and distinct challenges in analyzing the underlying prime distribution. This paper contributes to this area of research by introducing a tripartite framework for the lossless mathematical encoding of primes and their additive partitions. First, we establish Hybrid Prime Factorization (HPF) as a bounded *a priori* structural prime-generation framework. On certified intervals, primality is forced by disjoint partitions of canonical prime bases under the stated magnitude bound, so that certain HPF configurations yield prime outputs structurally, i.e., without requiring a separate post hoc primality test of the evaluated output. Second, to address the linear expansion of additive partitions, we introduce a deterministic pairing map,  $L(N)$ , which losslessly compresses the entire additive state of even integer partitions into a single, uniquely factorizable scalar in  $\mathbb{Z}$ . Finally, recognizing the asymptotically factorial limits of discrete integer representation, we map this arithmetic complexity into the continuous domain. We derive bounded, piecewise smooth harmonic sieve functions over  $\mathbb{R} \setminus \mathbb{Z}$  that isolate prime and composite structures through the limits of indeterminate trigonometric forms. This progression establishes that prime complexity need not be confined to discrete combinatorial bounds, but can be translated into continuous harmonic functions, demonstrating that the prime counting function  $\pi(x)$  can be generated as a sum of continuous harmonic trigonometric functions.

**Keywords:** Prime Distribution, Hybrid Prime Factorization, Lossless Encoding, Continuous Harmonic Sieves, Analytic Number Theory, Additive Prime Partitions, Goldbach Pairings

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## 1. Introduction

The Fundamental Theorem of Arithmetic (FTA) guarantees the unique multiplicative decomposition of every integer, yet the generation of integers via additive prime partitions resists simple algebraic categorization. Classical approaches to additive prime problems typically rely on probabilistic models, sieve theory, and the distribution of error terms bounded by complex analytic functions [1]. However, these classical sieves operate primarily as discrete algorithmic filters, suffering from combinatorial explosion and parity barriers when attempting to isolate specific additive structures [2].

Historically, attempts to bypass probabilistic sieves via exact, closed-form expressions for primes have relied on Wilson's theorem (e.g., Willans' formula) [3, 4].

While theoretically precise, such formulas function as computationally intractable, discontinuous step-functions governed by factorial magnitude explosion, offering no meaningful insight into the structure of primes or their distribution. A structural optimization to these classical limits was achieved in previous work [5], which derived discrete algebraic representations of the prime counting function and Goldbach partitions utilizing the integer-part operation (see Appendix A of [5]). This deterministic paradigm was recently expanded through a Failure Mode Analysis (FMA) of the Goldbach state space, demonstrating that additive prime partitions function as a necessary, structurally forced residual upon the exhaustion of composite inventories [6]. However, while circumventing factorial dependence and probabilistic uncertainty, these discrete frameworks remained bound to the

discrete topology of  $\mathbb{Z}$ , unable to leverage the analytics of continuous calculus.

This paper proposes an alternative paradigm: treating prime distribution and additive partitions not as pseudo-random sequences requiring probabilistic estimation, but as structured additive configurations that can be losslessly encoded. By formalizing primes through the lens of structural representation, this framework translates discrete algebraic generation into continuous harmonic functions.

The present work is structured into three parts, each mitigating the limitations of the preceding:

1. *Discrete Structural Prime Encoding (HPF)*: In Section 2, Hybrid Prime Factorization is introduced as a bounded *a priori* structural prime-generation framework. By partitioning the primordial base into disjoint multiplicative supports linked by a single additive or subtractive operation, certain HPF configurations are shown to yield prime outputs on certified intervals. In these cases, primality follows from the structural hypotheses of the construction rather than from a separate post hoc primality test.
2. *Lossless Multiplicative State Compression*: In Section 3, the limits of local discrete generation are addressed by encoding the global additive complexity of an even integer. An injective pairing function,  $L(C_N)$ , is defined, which maps a variable-length set of additive prime partitions into a unique scalar coordinate. This establishes that binary additive prime states possess a well-defined multiplicative representation.
3. *Continuous Harmonic Sieve Functions*: Because discrete multiplicative encodings are ultimately constrained by asymptotically factorial magnitude growth, Section 4 resolves this “magnitude barrier” via dimensional translation. Symmetric discriminant kernels are constructed that are bounded, continuous, and infinitely differentiable ( $C^\infty$ ) on  $\mathbb{R} \setminus \mathbb{Z}$ . These harmonic sieves isolate prime from composite structures through the limits of indeterminate trigonometric forms, embedding discrete arithmetic logic within a smooth real-valued framework suitable for continuous analytics and optimization.

By bridging discrete factorization frameworks with real-valued harmonic analysis, this paper provides a foundation for examining additive prime complexity through continuous calculus.

## 2. Hybrid Prime Factorization (HPF) and Structural Prime Encoding

The difficulty of additive prime problems stems from the operational asymmetry between multiplicative factorization and additive partitions. To bridge this divide, we introduce the Hybrid Prime Factorization (HPF) framework, which synthesizes both arithmetic operations into a unified generative process [7].

### 2.1. Definitions and Structural Constraints

Let  $\mathbb{P} = \{2, 3, 5, 7, 11, \dots\}$  denote the set of all primes, and let  $p_i$  denote the  $i$ -th prime. For a given dimension  $k \geq 1$ , we consider the canonical contiguous prime base consisting of the first  $k$  primes,  $\{p_1, \dots, p_k\}$ .

We define two multiplicative components,  $A$  and  $B$ , such that:

$$A := \prod_{i=1}^k p_i^{a_i}, \quad B := \prod_{i=1}^k p_i^{b_i}. \quad (1)$$

The Hybrid Prime Factorization of dimension  $k$  is defined by the additive/subtractive combination of these components:

$$\text{HPF}(k) := A \pm B. \quad (2)$$

To ensure structural integrity, the exponent vectors  $a = (a_1, \dots, a_k)$  and  $b = (b_1, \dots, b_k)$  satisfy the following algebraic constraints:

$$a_i, b_i \in \mathbb{Z}_{\geq 0} \quad \text{for all } i \in \{1, \dots, k\}, \quad (3)$$

$$a_i \cdot b_i = 0 \quad \text{for all } i \in \{1, \dots, k\}, \quad (4)$$

$$\sum_{i=1}^k a_i \geq 1, \quad \sum_{i=1}^k b_i \geq 0, \quad (5)$$

$$\text{HPF}(k) > 1. \quad (6)$$

Constraint (4) ensures *disjoint prime support*: no prime  $p_i$  appears in the factorization of both  $A$  and  $B$ . Consequently,  $A$  and  $B$  are coprime. Constraint (5) establishes structural asymmetry by enforcing a non-trivial primary component ( $A \geq 2$ ) while permitting the secondary component to resolve to the multiplicative identity ( $B \geq 1$ ). Furthermore, Constraint (6) restricts the output to the domain of prime candidates, explicitly excluding the unit (1) and non-positive evaluations generated by the subtractive branch.

*Definition 2.1* (Complete HPF). An HPF( $k$ ) configuration is *Complete* if it satisfies constraints (3)–(6) alongside the coverage condition:

$$a_i + b_i \geq 1 \quad \text{for all } i \in \{1, \dots, k\}. \quad (7)$$

In a Complete HPF, every prime in the canonical base  $\{p_1, \dots, p_k\}$  is utilized in either  $A$  or  $B$  with a nonzero exponent.

Configurations where  $\exists i \leq k$  such that  $a_i = b_i = 0$  are classified as *Incomplete HPFs*, as they bypass one or more primes within the contiguous canonical base.

### 2.2. Structural Certification and Arithmetic Evolution

The synthesis of multiplicative disjointness and additive combination in a Complete HPF algebraically enforces primality for bounded outputs, bypassing the need for algorithmic sieving or probabilistic testing.

*Theorem 2.1* (Conservative Structural Certification for Complete HPF). Let  $\{p_1, \dots, p_k\}$  be the first  $k$  primes. Suppose there exist exponent vectors  $a$  and  $b$  satisfying the

constraints of a Complete HPF( $k$ ). If the output satisfies the magnitude bound:

$$\text{HPF}(k) < p_{k+1}^2, \tag{8}$$

then  $\text{HPF}(k) \in \mathbb{P}$  and  $\text{HPF}(k) > p_k$ .

*Proof:* By the definition of a Complete HPF, the expression  $\text{HPF}(k) = A \pm B$  utilizes all primes  $p_i \leq p_k$  exactly once across the disjoint supports of  $A$  and  $B$ . Therefore,  $\text{gcd}(A, B) = 1$ . By the elementary properties of divisibility, if  $p_i \mid A$ , then  $p_i \nmid (A \pm B)$ . Consequently,  $\text{HPF}(k)$  cannot be divisible by any prime  $p_i \leq p_k$ .

It follows that any prime factor  $q$  dividing  $\text{HPF}(k)$  must satisfy  $q \geq p_{k+1}$ . If  $\text{HPF}(k)$  were composite, its smallest possible value would be  $p_{k+1}^2$ . Because we are given  $\text{HPF}(k) < p_{k+1}^2$ , it cannot possess any prime factors greater than or equal to  $p_{k+1}$ . Having exhausted all possible prime divisors,  $\text{HPF}(k)$  must itself be prime.

Theorem 2.1 identifies a phenomenon of *a priori* structural prime generation. Unlike classical prime-representing constructions, which typically produce sparse subsequences or require evaluation followed by an independent primality check, the HPF framework yields structurally certified prime outputs on bounded intervals.

To illustrate, consider the Complete HPF(3) subtractive configurations:  $5^2 - 2^2 \cdot 3$ ,  $3^3 - 2 \cdot 5$ ,  $2^2 \cdot 3^2 - 5$ ,  $2^2 \cdot 3 + 5$  and  $2^3 + 3 \cdot 5$ . Even if  $p_4$  is unknown, the inequality  $p_{k+1} \geq p_k + 2$  ensures that  $(p_k + 2)^2$  serves as a conservative certification bound in Theorem 2.1. For  $k = 3$ , this bound is  $(5 + 2)^2 = 49$ . The required magnitude bound can be verified via structural inequalities without terminal evaluation. For instance, the bound for the first subtractive form is established directly by the dominant term:  $5^2 - 2^2 \cdot 3 < 5^2 < 7^2$ . Because these expressions use the canonical base  $\{2, 3, 5\}$  in disjoint multiplicative supports and satisfy the certification bound, Theorem 2.1 certifies that their outputs are prime. By definition, the output must be coprime to the base elements, forcing a unidirectional generative sequence: *smaller primes combine algebraically to generate larger primes*. Thus, primality is derived exclusively from algebraic structure, bypassing post hoc testing of the evaluated scalar.

However, as  $k$  increases, this structural containment encounters a "magnitude barrier." For the additive branch ( $\text{HPF}_+ := A + B$ ), the product  $A \cdot B$  must be a multiple of the primorial  $p_k\# = \prod_{i=1}^k p_i$ . Because the primorial grows exponentially, the search space to satisfy the bound  $A + B < p_{k+1}^2$  rapidly collapses, eventually making additive generation mathematically impossible for larger primes. While the subtractive branch ( $\text{HPF}_- := |A - B|$ ) opens an unbounded search space to circumvent this, the magnitude limits inherent in discrete integer arithmetic necessitate a broader representational framework. This discrete limitation motivates the transition to the global integer mappings discussed in the following section.

To illustrate this limitation and the eventual encounter with the magnitude barrier, we examine the complete additive

decompositions for specific primes.

Table 1 details the structural evaluation of all additive partitions for  $p = 17$ . Of the eight available additive decompositions, four are Complete HPFs, satisfying the required certification bounds.

*Definition 2.2 (HPF Prime Set).* For any given HPF configuration  $A \pm B$ , the *HPF Prime Set*, denoted  $P_{\text{HPF}}$ , is defined as the set of all distinct prime factors utilized in the supports of  $A$  and  $B$ :

$$P_{\text{HPF}} := \{p_i \in \mathbb{P} \mid a_i + b_i > 0\}. \tag{9}$$

By definition, a Complete HPF( $k$ ) satisfies  $P_{\text{HPF}} = \{p_1, p_2, \dots, p_k\}$ , forming a contiguous, gapless sequence of the first  $k$  primes.

Table 1. Structural HPF evaluation of all additive decompositions for  $p = 17$ .

#	Sum	HPF ( $A + B$ )	$k$	Type	$P_{\text{HPF}}$
1	1 + 16	1 + 2 <sup>4</sup>	1	Complete	{2}
2	2 + 15	2 + 3 · 5	3	Complete	{2, 3, 5}
3	3 + 14	3 + 2 · 7	–	Incomplete	{2, 3, 7}
4	4 + 13	2 <sup>2</sup> + 13	–	Incomplete	{2, 13}
5	5 + 12	5 + 2 <sup>2</sup> · 3	3	Complete	{2, 3, 5}
6	6 + 11	2 · 3 + 11	–	Incomplete	{2, 3, 11}
7	7 + 10	7 + 2 · 5	–	Incomplete	{2, 5, 7}
8	8 + 9	2 <sup>3</sup> + 3 <sup>2</sup>	2	Complete	{2, 3}

To demonstrate the generative sequence of the HPF framework, Table 2 shows the smallest Complete HPF configurations that structurally certify the first 20 odd primes. By scaling the canonical dimension  $k$ , every target prime is represented via an additive HPF while satisfying the  $p_{k+1}^2$  certification limit.

Table 2. Structurally Certified Complete Additive HPF Configurations for the First 20 Odd Primes.

Prime ( $p$ )	Complete HPF ( $A + B$ )	Dimension ( $k$ )	Bound ( $p_{k+1}^2$ )
3	2 + 1	1	9
5	2 <sup>2</sup> + 1	1	9
7	2 <sup>2</sup> + 3	2	25
11	2 <sup>3</sup> + 3	2	25
13	2 <sup>2</sup> + 3 <sup>2</sup>	2	25
17	2 <sup>3</sup> + 3 <sup>2</sup>	2	25
19	2 <sup>4</sup> + 3	2	25
23	2 <sup>3</sup> + 3 · 5	3	49
29	2 <sup>3</sup> · 3 + 5	3	49
31	2 <sup>4</sup> + 3 · 5	3	49
37	3 <sup>3</sup> + 2 · 5	3	49
41	2 <sup>2</sup> · 3 <sup>2</sup> + 5	3	49
43	2 <sup>3</sup> · 5 + 3	3	49
47	2 <sup>5</sup> + 3 · 5	3	49
53	2 · 3 <sup>2</sup> + 5 · 7	4	121
59	2 <sup>3</sup> · 3 + 5 · 7	4	121

Prime ( $p$ )	Complete HPF ( $A + B$ )	Dimension ( $k$ )	Bound ( $p_{k+1}^2$ )
61	$2^3 \cdot 5 + 3 \cdot 7$	4	121
67	$2 \cdot 3 \cdot 7 + 5^2$	4	121
71	$2^2 \cdot 3^2 + 5 \cdot 7$	4	121
73	$2^2 \cdot 7 + 3^2 \cdot 5$	4	121

Through systematically testing primes  $> 73$ , this additive completeness holds sequentially until  $p = 149$ . The prime 151 is the first integer to fail, as no additive partition exists that yields a contiguous canonical base satisfying  $p < p_{k+1}^2$ . Thus,  $p = 149$  represents the limit of the additive magnitude barrier. As shown in Table 3, out of its 74 possible additive partitions, exactly one yields a Complete HPF, representing the final deterministic configuration before the structural collapse of (8).

Table 3. Condensed additive HPF evaluation for  $p = 149$  (The Magnitude Barrier).

#	Sum	HPF ( $A + B$ )	$k$	Type	$P_{\text{HPF}}$
1	$1 + 148$	$1 + 2^2 \cdot 37$	–	Incomplete	$\{2, 37\}$
2	$2 + 147$	$2 + 3 \cdot 7^2$	–	Incomplete	$\{2, 3, 7\}$
⋮	⋮	⋮	⋮	⋮	⋮
44	$44 + 105$	$2^2 \cdot 11 + 3 \cdot 5 \cdot 7$	5	Complete	$\{2, 3, 5, 7, 11\}$
⋮	⋮	⋮	⋮	⋮	⋮
73	$73 + 76$	$73 + 2^2 \cdot 19$	–	Incomplete	$\{2, 19, 73\}$
74	$74 + 75$	$2 \cdot 37 + 3 \cdot 5^2$	–	Incomplete	$\{2, 3, 5, 37\}$

Because  $149 < p_6^2 = 13^2 = 169$ , partition #44 certifies primality. However, the exhaustion of the additive branch at this specific value is not a computational artifact, but a structural boundary caused by the primorial.

*Remark 2.1* (Upper Bound of the Additive HPF). For any Complete HPF additive partition  $N = A + B$  spanning the canonical base up to  $p_k$ , the disjoint support requires the product  $A \cdot B \geq p_k \#$ , where  $p_k \#$  denotes the primorial of  $p_k$ . By the Arithmetic Mean-Geometric Mean (AM-GM) inequality, the additive output is bounded below by  $A + B \geq 2\sqrt{A \cdot B} \geq 2\sqrt{p_k \#}$ . Simultaneously, HPF prime certification requires  $N < p_{k+1}^2$ .

Because the primorial  $p_k \#$  grows exponentially while the certification bound  $p_{k+1}^2$  scales polynomially, the theoretical minimum sum overtakes the certification window at  $k = 6$ . Specifically,  $2\sqrt{p_6 \#} = 2\sqrt{30,030} \approx 346.58$ , which exceeds  $p_7^2 = 17^2 = 289$ . Thus, the additive branch mathematically collapses for all  $k \geq 6$ , capping additive certification below  $N = 169$ . Consequently, structural certification beyond this point necessitates the transition to the subtractive branch ( $|A - B|$ ), where the output represents a relative distance rather than an absolute sum, and is therefore not bounded below by the AM-GM inequality.

While the additive branch encounters a definitive magnitude barrier at  $p = 149$ , the subtractive branch ( $\text{HPF}_- := |A - B|$ ) escapes this constraint by opening an unbounded Diophantine search space. To illustrate, consider the prime  $p = 151$ , which fails additive Complete HPF representation. By utilizing the

subtractive branch with dimension  $k = 5$  (canonical base  $\{2, 3, 5, 7, 11\}$ ), we achieve the following Complete HPF [8]:

$$A = 3^1 \cdot 5^1 \cdot 11^1 = 165, \quad B = 2^1 \cdot 7^1 = 14. \quad (10)$$

$$\text{HPF}_-(5) = A - B = 165 - 14 = 151. \quad (11)$$

This configuration disjointly utilizes the entire canonical base. Furthermore, because the output satisfies the structural certification bound  $151 < p_6^2 = 13^2 = 169$ , Theorem 2.1 deterministically guarantees its primality.

Due to the unbounded nature of the subtractive difference, Complete HPF representations for  $p = 151$  using larger dimensions (e.g.,  $k \geq 6$ ) are also mathematically possible, effectively bypassing the exponential lower bound of the primorial. However, as  $k$  increases, the rapidly increasing magnitudes of  $A$  and  $B$  eventually trigger computational overflow. This reliance on unbounded discrete magnitudes underscores the fundamental limits of algebraic integer generation, motivating the global multiplicative mappings developed in Section 3.

To systematically explore this subtractive Diophantine space while preserving the structural certification of Theorem 2.1, the search can be framed as a bounded enumeration targeting the upper limit of the structural window,  $p_{k+1}^2$ .

A demonstration of this capability is found by expanding the dimension to  $k = 11$ , utilizing the canonical base of the first eleven primes  $P_{\text{HPF}} = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31\}$ . The structural certification bound for this dimension is  $p_{12}^2 = 37^2 = 1369$ , making the prime 1367 the absolute theoretical supremum for structural certification in this space.

While generating maximal primes via the additive branch is structurally impossible due to the exponential magnitude of the primorial, the subtractive branch ( $\text{HPF}_-$ ) provides an unbounded Diophantine space to approach this limit. By programmatically traversing a finite subset of this space—restricting maximum prime exponents to  $\leq 3$  and searching for disjoint components bounded near  $1.34 \times 10^6$ —we isolate the following exact Complete HPF configuration landing firmly within the structural barrier:

$$A = 2^1 \cdot 3^3 \cdot 7^1 \cdot 11^1 \cdot 17^1 \cdot 19^1 = 1,343,034, \quad (12)$$

$$B = 5^1 \cdot 13^1 \cdot 23^1 \cdot 29^1 \cdot 31^1 = 1,344,005. \quad (13)$$

$$\begin{aligned} \text{HPF}_-(11) &= |A - B| \\ &= |1,343,034 - 1,344,005| = 971. \end{aligned} \quad (14)$$

This configuration disjointly utilizes all 11 primes in the canonical base up to  $p_{11} = 31$ . Utilizing the conservative bound  $(p_{11} + 2)^2 = 33^2 = 1089$ , the output satisfies  $\text{HPF}_-(11) = 971 < 1089$ . Therefore, the integer 971 is structurally certified as prime without requiring knowledge of any prime larger than 31. This demonstrates how the subtractive HPF framework leverages large multiplicative components to deterministically certify primes right up to the theoretical boundary, bridging discrete algebraic constraints with computational number theory. This demonstrates how the subtractive HPF framework leverages large multiplicative

components to deterministically certify primes right up to the theoretical boundary, bridging discrete algebraic constraints with computational number theory.

### 2.3. Upper Bound of Subtractive HPF

Although a Complete HPF utilizing the subtractive branch extends the structural certification of primes beyond  $p = 149$ , such a representation cannot be extended indefinitely.

Because a Complete HPF requires the disjoint product  $A \cdot B$  to be a multiple of  $p_k \#$ , forcing the difference  $|A - B|$  to remain below the narrow certification window of  $p_{k+1}^2$  requires both components to closely approximate  $\sqrt{p_k \#}$ . As  $k$  scales toward cosmologically significant primes, the scale of the required component magnitudes causes rapid computational overflow.

The exhaustion of this branch is governed by the limitations of exponential Diophantine equations. By Mahler’s theorem—in conjunction with Tijdeman’s (1973) computable lower bounds for the distance between integers sharing a finite prime base [9, 10]—the absolute difference between any two such integers tends to infinity as their magnitudes increase. Specifically, if  $A$  and  $B$  are composed exclusively from the fixed canonical base  $\{p_1, \dots, p_k\}$ , their minimum distance is bounded below by a function of the form  $B/(\log B)^{C_1}$  for some computable constant  $C_1$ . Because linear growth dominates logarithmic growth, as  $B \rightarrow \infty$ , the required gap  $|A - B|$  inevitably diverges.

Consequently, once the components  $A$  and  $B$  exceed a specific, finite magnitude threshold, their difference will permanently violate the  $p_{k+1}^2$  certification window. Thus, the number of admissible subtractive Complete HPF configurations is finite for any given dimension  $k$ .

This establishes a fundamental boundary condition: discrete algebraic generation—whether constrained additively by the AM-GM inequality or subtractively by Diophantine divergence—ultimately collapses under the sheer magnitude of its own multiplicative components. To preserve and analyze the additive state of primes without triggering this magnitude inflation, we transition from local sequential generation to the global, lossless multiplicative encodings developed in Section 3.

## 3. Lossless Encoding of Goldbach Primes

While Hybrid Prime Factorization (Section 2) establishes the local, deterministic forcing of primality, analyzing the global additive structure of integers requires examining sets of prime pairs. For any even integer  $2N \geq 6$ , the binary Goldbach conjecture asserts the existence of at least one pair of primes  $(p, q)$  such that  $p + q = 2N$  [11].

The study of additive number theory frequently focuses on the cardinality of these partitions through scalar counting functions. However, mapping a set of coordinate pairs into a single integer count is not necessarily a lossless operation,

since it may not preserve the exact value of each prime pair within the set of additive partitions [5]. To preserve the complete additive state of an even integer within a multiplicative framework, we define an injective mapping capable of transforming a variable-length set of additive pairs into a single, uniquely recoverable scalar.

### 3.1. The Goldbach Pairing Map $L(C_N)$

Traditional pairing functions, such as the Cantor pairing function  $\pi(x, y) = \frac{1}{2}(x+y)(x+y+1)+y$ , establish a bijection from  $\mathbb{N}^2 \rightarrow \mathbb{N}$ . However, extending such functions to encode a varying number of  $m(N)$  distinct pairs requires deep recursive nesting, leading to rapid polynomial growth and obscuring the underlying arithmetic structure [12].

Instead, we construct a pairing map that leverages the uniqueness properties of the FTA. By shifting the encoding mechanism from additive combinations to disjoint prime factorization, we achieve a lossless state encoding for the Goldbach partitions, structurally analogous to foundational Gödel numbering [13].

*Definition 3.1* (Global Goldbach Partition Space and Pairing Map). We explicitly restrict our domain to admissible even integers. Let  $2N \geq 6$  be an even integer for which the binary Goldbach Conjecture holds true, meaning there exist  $m(N) \geq 1$  prime pairs  $(x_i, y_i)$  such that  $x_i + y_i = 2N$ . We define the ordered partition state  $C_N$  as the tuple  $C_N = ((x_1, y_1), \dots, (x_{m(N)}, y_{m(N)}))$ , ordered such that  $3 \leq x_1 < x_2 < \dots < x_{m(N)} \leq y_{m(N)}$ .

Let  $\mathcal{G}$  be the global class of all such ordered partition states across all admissible even integers:

$$\mathcal{G} := \{C_N \mid 2N \geq 6, \text{ bGC holds for } 2N\}. \quad (15)$$

We define the Goldbach Pairing Map  $L : \mathcal{G} \rightarrow \mathbb{N}$  by the product:

$$L(C_N) := y_{m(N)} \cdot \prod_{i=1}^{m(N)} x_i. \quad (16)$$

The product consists of all the smaller prime components  $x_i$ , appended with exactly one larger component,  $y_{m(N)}$ , which serves as the structural *anchor*.

### 3.2. Injectivity and State Recovery

To qualify as an admissible encoding,  $L(C_N)$  must be injective; no two distinct sets of Goldbach partitions can map to the same integer, and the original additive coordinates must be recoverable from the scalar output.

*Theorem 3.1* (Injectivity of the Pairing Map). The encoding map  $L : \mathcal{G} \rightarrow \mathbb{N}$  is injective.

*Proof:* Let  $K = L(C_N)$ . By the Fundamental Theorem of Arithmetic,  $K$  decomposes into a unique multiset of prime factors. Let this multiset, containing  $m + 1$  primes, be sorted in ascending order:  $q_1 \leq q_2 \leq \dots \leq q_{m+1}$ .

By the definition of  $L(C_N)$  and the ordering condition  $x_1 < x_2 < \dots < x_m \leq y_m$ , the elements of this multiset correspond exactly to  $\{x_1, \dots, x_m, y_m\}$ . Therefore,

the maximal element of the sorted multiset is always  $y_{m(N)} = q_{m+1}$ . The second-to-maximal element (which accounts for prime multiplicity in the case where  $x_m = y_m$ ) is always  $x_{m(N)} = q_m$ .

From these two extracted elements, the target even integer is uniquely reconstructed:

$$2N = q_m + q_{m+1}. \quad (17)$$

For all remaining prime factors  $q_i$  ( $1 \leq i \leq m-1$ ), their corresponding Goldbach partners are uniquely recovered by subtraction:

$$y_i = 2N - q_i. \quad (18)$$

Because the prime factorization is unique and the sorting algebra recovers exactly one even integer and its exact ordered sequence of partition pairs, no two distinct states  $C_A, C_B \in \mathcal{G}$  can map to the same scalar  $K$ . Thus,  $L$  is globally injective.

*Example 3.1.* Let  $2N = 20$ . The  $m(10) = 2$  Goldbach partitions are  $(3, 17)$  and  $(7, 13)$ . Following the ordering  $3 < 7 < 13 < 17$ , the anchor is  $y_2 = 13$ , and the smaller components are  $x_1 = 3, x_2 = 7$ . The encoding is:

$$L(C_{10}) = 3 \cdot 7 \cdot 13 = 273. \quad (19)$$

To decode  $L(C_{10}) = 273$ , we factor it to retrieve the multiset  $\{3, 7, 13\}$ . The sorted elements are  $q_1 = 3, q_2 = 7, q_3 = 13$ . The anchor pair is  $(q_2, q_3) = (7, 13)$ , which reconstructs the target sum  $7 + 13 = 20$ . The remaining factor is  $q_1 = 3$ , yielding its partner  $20 - 3 = 17$ . The original state  $C_{10} = ((3, 17), (7, 13))$  is recovered.

*Remark 3.1.* By the FTA, the mapping  $L(C_N)$  provides a lower bound for the magnitude of a purely multiplicative structural encoding. Consequently, the primary utility of this mapping is structural rather than algorithmic, since it encounters an asymptotically factorial magnitude barrier. This inherent combinatorial growth necessitates the transition to the continuous harmonic approach developed in Section 4.

The theoretical significance of  $L(C_N)$  lies in demonstrating that the additive complexity of prime partitions can be mapped to a single value in  $\mathbb{Z}$ . However, attempting to parameterize this factorial growth using integer polynomials is mathematically bounded. Specifically, by Mahler's theorem, the greatest prime factor of any non-constant integer polynomial sequence tends to infinity. This ensures that no discrete algebraic polynomial can generate admissible configurations without eventually introducing unwanted prime factors [14]. This establishes a boundary condition: discrete integer representations—whether local algebraic constraints like HPF or global multiplicative maps like  $L(C_N)$ —inevitably encounter a magnitude barrier. To analyze prime partitions without combinatorial growth, the arithmetic structure is shifted from the discrete integers into the continuous, real-valued domain developed in Section 4.

## 4. Continuous Harmonic Sieve Functions

The discrete representations established in prior sections—both the local algebraic bounds of HPF and the global multiplicative mapping of  $L(C_N)$ —ultimately encounter magnitude limitations. As  $N \rightarrow \infty$ , the combinatorial explosion of discrete sets and the asymptotically factorial growth of their encodings exceed the bounds of practical computational representation. To circumvent this "magnitude barrier," the arithmetic complexity of the primes is translated out of the discrete domain  $\mathbb{Z}$  and into the real continuum.

By mapping prime distribution onto the real line  $\mathbb{R}$ , we shift from discrete combinatorial filtering (classical sieving) to the calculus of continuous variations. This section constructs a harmonic Goldbach discriminant kernel,  $J(x)$ , which is bounded and piecewise smooth ( $C^\infty$ ) everywhere except at the integers, where it resolves additive prime partitions through the evaluation of  $0/0$  indeterminate limits governed by zero-mode cancellations.

### 4.1. Foundational Kernels

The construction of a continuous sieve requires generating functions that vanish precisely at specific arithmetic coordinates. We utilize trigonometric functions, specifically sine waves, whose roots coincide with the integers.

Let  $2N \geq 6$  be a fixed even integer. We define the prime-detecting kernel  $GP(x)$  over the reals as:

$$GP(x) := \prod_{j=2}^{\max(2, \lfloor x/2 \rfloor)} \left( \sin^2 \left( \frac{\pi x}{j} \right) \right)^{1/j}, \quad (20)$$

for all real  $x \geq 3$ .

By defining the upper bound of the product as  $\max(2, \lfloor x/2 \rfloor)$ , the function dynamically restricts divisors to the strict proper subset of  $x$ . This structurally prevents prime self-division while ensuring that the index set remains constant in the open intervals between even integers. To target the symmetric Goldbach pairs of  $2N$ , the multiplicative composite-vanishing kernel  $CP(x)$  is defined as:

$$CP(x) := GP(x) \cdot GP(2N - x). \quad (21)$$

By definition,  $CP(x) = 0$  if and only if either  $x$  or  $2N - x$  is a composite integer (having a divisor  $j \geq 2$ ). If both  $x$  and  $2N - x$  are prime (a Goldbach pair), the sine arguments do not evaluate to integer multiples of  $\pi$ , and  $CP(x)$  remains positive.

To isolate the integers from the surrounding real numbers, we introduce the modified integer detection kernel:

$$GI_2(x) := (\sin^2(\pi x))^{1/x}. \quad (22)$$

This function satisfies  $\lim_{x \rightarrow z} GI_2(x) = 0$  for all integers  $z \in \mathbb{Z}$ , while remaining positive for all  $x \in \mathbb{R} \setminus \mathbb{Z}$ .

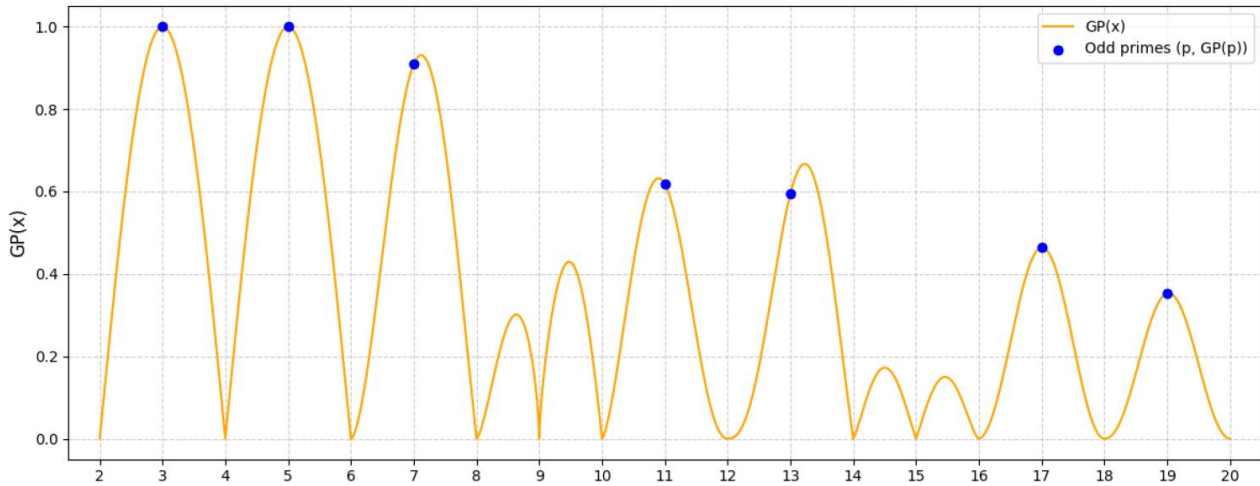


Figure 1. Graph of the prime-detecting kernel  $GP(x)$  over the interval  $x \in [2, 20]$ . The continuous scalar field is positive at all odd primes (marked by blue disks) and evaluates to 0 at all composite integers, embedding the structural exclusions of the Sieve of Eratosthenes into a harmonic, real-valued function.

### 4.2. The Goldbach Discriminant Kernel $J(x)$

To create a bounded, continuous function that discriminates Goldbach primes from all other real and integer values, the symmetric composite kernel and the integer detection kernel are combined into a single rational function.

Definition 4.1 (Goldbach Discriminant Kernel). For a fixed even integer  $2N \geq 6$ , the continuous harmonic discriminant  $J(x)$  is defined for all  $x \geq 3$  as:

$$J(x) := \begin{cases} \frac{GI_2(x)}{GI_2(x)+CP(x)} & \text{if } x \in \mathbb{R} \setminus \mathbb{Z}, \\ \lim_{t \rightarrow x} \frac{GI_2(t)}{GI_2(t)+CP(t)} & \text{if } x \in \mathbb{Z}. \end{cases} \quad (23)$$

The well-posedness of this definition at integer coordinates requires the explicit existence of the limit. As established in Lemma 4.1 and Theorem 4.1, for all  $z \in \mathbb{Z}_{\geq 3}$ , fractional exponent dominance ensures this limit converges to either 0 or 1.

### 4.3. Limit Evaluation and Zero-Mode Cancellation

To resolve the structural singularity at the integers analytically, we must determine the relative rate at which the numerator and denominator approach zero. By transitioning to the infinitesimal domain, the zero-modes generated by the trigonometric roots can be isolated and compared via their fractional exponents.

Lemma 4.1 (Convergence of  $CP(x)/GI_2(x)$  Under Zero-Mode Cancellation). Let  $2N \geq 6$  be a fixed even integer, and let  $z \in \mathbb{Z}_{\geq 3}$ .

1. If  $z$  and  $2N - z$  are both prime,  $\lim_{x \rightarrow z} \frac{CP(x)}{GI_2(x)} = +\infty$ .
2. If either  $z$  or  $2N - z$  is composite,  $\lim_{x \rightarrow z} \frac{CP(x)}{GI_2(x)} = 0$ .

Proof: We evaluate the limit as  $x \rightarrow z$  by substituting  $x = z + \epsilon$  as  $\epsilon \rightarrow 0$ .

Case 1 (Both Prime): If  $z$  and  $2N - z$  are prime, neither possesses a proper divisor within their variable product bounds. Consequently, as  $\epsilon \rightarrow 0$ , no sine term in  $CP(z + \epsilon)$

evaluates to zero, yielding  $\lim_{\epsilon \rightarrow 0} CP(z + \epsilon) = C > 0$ . Conversely, the integer detection kernel evaluates to  $GI_2(z + \epsilon) = (\sin^2(\pi z + \pi \epsilon))^{1/z}$ . Because  $z$  is an integer,  $\sin(\pi z) = 0$ , leaving  $(\sin^2(\pi \epsilon))^{1/z}$ . As  $\epsilon \rightarrow 0$ , this approaches 0. Therefore, the ratio  $C/0$  diverges to  $+\infty$ .

Case 2 (Composite Evaluation): Suppose  $z$  is composite, possessing at least one proper divisor  $j \geq 2$  such that  $j < z$ . As  $\epsilon \rightarrow 0$ , the vanishing trigonometric term corresponding to  $j$  in  $GP(z + \epsilon)$  is  $(\sin^2(\frac{\pi(z+\epsilon)}{j}))^{1/j}$ . Because  $j$  divides  $z$ ,  $\frac{z}{j}$  is an integer, and the term simplifies to  $(\sin^2(\frac{\pi \epsilon}{j}))^{1/j}$ .

Applying the first-order Taylor expansion  $\sin(\theta) \sim \theta$  for small angles, the zero-modes behave as follows:

$$\text{Numerator } CP(z + \epsilon) \propto \left(\frac{\pi \epsilon}{j}\right)^{2/j}, \quad (24)$$

$$\text{Denominator } GI_2(z + \epsilon) \sim (\pi \epsilon)^{2/z}.$$

Isolating the infinitesimal  $\epsilon$ , the convergence of the ratio is dictated by the limit:

$$\lim_{\epsilon \rightarrow 0} \frac{\epsilon^{2/j}}{\epsilon^{2/z}}. \quad (25)$$

To evaluate this algebraically without fractional exponents, we raise the ratio to the positive integer power  $z \cdot j$ :

$$\lim_{\epsilon \rightarrow 0} \left(\frac{\epsilon^{2/j}}{\epsilon^{2/z}}\right)^{z \cdot j} = \lim_{\epsilon \rightarrow 0} \frac{\epsilon^{2z}}{\epsilon^{2j}} = \lim_{\epsilon \rightarrow 0} \epsilon^{2z-2j}. \quad (26)$$

Because  $j$  is a proper divisor of  $z$ , it is true that  $z > j$ . Therefore, the exponent  $(2z - 2j)$  is a positive integer. As  $\epsilon \rightarrow 0$ , any positive power of  $\epsilon$  converges to 0. Because the powered ratio converges to 0, the original limit ratio  $\frac{CP(x)}{GI_2(x)}$  must also converge to 0. By structural symmetry, the identical zero-mode cancellation forces the limit to 0 if  $2N - z$  is composite.

#### 4.4. Convergence of the Discriminant Kernel $J(x)$

With the limit behavior of the component kernels analytically resolved in Lemma 4.1, the structural classification of the integer domain can be executed dynamically without requiring probabilistic estimation or integer factorization.

**Theorem 4.1** (Structural Convergence of  $J(x)$ ). Let  $2N \geq 6$  be a fixed even integer. The discriminant kernel  $J(x)$  evaluates as follows:

1. *Goldbach Primes*: If  $x \in \mathbb{Z}_{\geq 3}$  such that both  $x$  and  $2N - x$  are prime,  $J(x) = 0$ .
2. *Composite / Non-Pairs*: If  $x \in \mathbb{Z}_{\geq 3}$  such that either  $x$  or  $2N - x$  is composite,  $J(x) = 1$ .
3. *Real Continuum*: For all non-integer real values  $x \in \mathbb{R} \setminus \mathbb{Z}$ ,  $J(x) \in (0, 1)$ .

*Proof*: By algebraically rewriting the piecewise definition of  $J(x)$  at integer boundaries as the mathematically equivalent limit ratio:

$$J(z) = \lim_{x \rightarrow z} \frac{1}{1 + \frac{CP(x)}{GI_2(x)}}, \quad (27)$$

the structural classification reduces directly to the evaluations established in Lemma 4.1.

*Case 1 (Goldbach Primes)*: By Lemma 4.1, if both  $x$  and  $2N - x$  are prime, the limit of the ratio diverges to  $+\infty$ . Substituting this into the equivalent form yields  $J(p) = \frac{1}{1+\infty} = 0$ .

*Case 2 (Composite / Non-Pairs)*: By Lemma 4.1, if either integer coordinate is composite, the limit of the ratio converges to 0. Substituting this evaluation yields  $J(c) = \frac{1}{1+0} = 1$ .

*Case 3 (Real Continuum)*: For all  $x \in \mathbb{R} \setminus \mathbb{Z}$ , the argument of  $GI_2(x)$  is not an integer multiple of  $\pi$ . Therefore,  $\sin^2(\pi x) > 0$ , ensuring  $GI_2(x) > 0$ . Similarly, because  $x$  is not an integer,  $CP(x) > 0$ . Because both components are positive, the ratio  $\frac{GI_2(x)}{GI_2(x)+CP(x)}$  evaluates directly to a proper fraction in the open interval  $(0, 1)$ .

To demonstrate the algebraic behavior of  $J(x)$  described

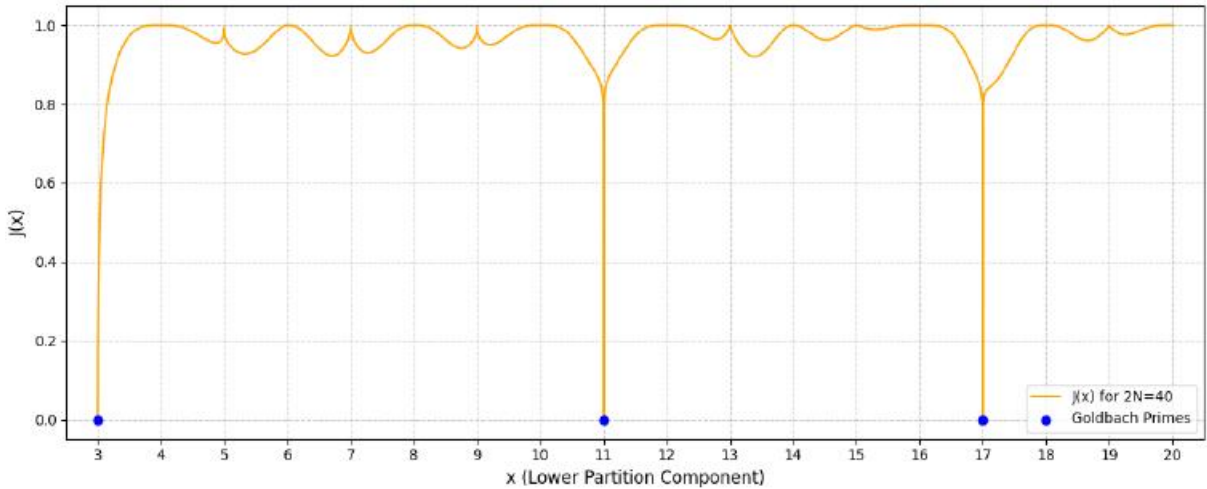
in Theorem 4.1, we analyze the 0/0 indeterminate forms for  $2N = 40$  by evaluating the mathematically equivalent limit ratio.

*Case 1 (Goldbach Primes)*: Let  $x = 17$ . Its complement is  $40 - 17 = 23$ . Because  $17 \in \mathbb{Z}$ , the integer kernel  $GI_2(x) \rightarrow 0$ , generating one zero-mode in the denominator of the ratio. However, because 17 and 23 are prime, neither possesses a proper divisor within their variable product bounds ( $\lfloor x/2 \rfloor$ ). Consequently, no trigonometric sine term in  $CP(x)$  evaluates to zero, yielding  $CP(17) = C > 0$  (no zero-modes in the numerator). The ratio  $\frac{CP(x)}{GI_2(x)}$  thus diverges to  $+\infty$ , forcing the evaluation  $J(17) = \frac{1}{1+\infty} = 0$ . This exact global minimum is visually confirmed in Figure 2 at  $x = 17$ .

*Case 2 (Composite Integers)*: Let  $x = 15$ . Its partition complement is  $40 - 15 = 25$ . As  $x \rightarrow 15$ , the integer kernel  $GI_2(x) \rightarrow 0$  again contributes one zero-mode to the denominator. Conversely, the symmetric composite kernel  $CP(x)$  evaluates the proper divisors of both 15 and 25. The product  $GP(x)$  includes zero-modes for the divisors  $j = 3$  and  $j = 5$ . Concurrently,  $GP(2N - x)$  includes a zero-mode for its divisor  $j = 5$ . Therefore,  $CP(x)$  accumulates three zero-modes in the numerator. As established in Lemma 4.1, because the proper divisors ( $j = 3, 5$ ) are less than the target integers (15, 25), the algebraic multiplicity of these zero-modes dominates the single zero-mode of  $GI_2(x)$ . This forces the ratio to 0, yielding  $J(15) = \frac{1}{1+0} = 1$ . This is confirmed in Figure 2 at  $x = 15$ .

*Case 3 (Real Continuum)*: Let  $x = 15.5$ . Because  $x \notin \mathbb{Z}$ , the argument of  $GI_2(x)$  is not an integer multiple of  $\pi$ , ensuring  $GI_2(15.5) > 0$ . Similarly, no terms in  $CP(15.5)$  vanish. The ratio  $\frac{CP(x)}{GI_2(x)}$  evaluates to a finite positive constant  $C > 0$ , yielding a continuous trajectory  $J(15.5) \in (0, 1)$ .

To visualize these structural limits across the entire Goldbach window, Figure 2 illustrates the values of the discriminant kernel  $J(x)$  for  $2N = 40$ .



**Figure 2.** Graph of the Goldbach Discriminant Kernel  $J(x)$  for  $2N = 40$ , evaluating to 0 at all Goldbach prime pairs ( $x = 3, 11, 17$ ). The function resolves to 1 at all composite or non-pair integer coordinates, and to  $(0, 1)$  for  $x \in \mathbb{R} \setminus \mathbb{Z}$ .

As shown in Figure 2, the graph of  $J(x)$  demonstrates a continuous, non-convex scalar field that evaluates to fixed global minima exclusively at the additive prime partitions. Since, by definition, the function exhibits reflective symmetry around the midpoint  $x = N = 20$ , it also encodes the commutativity of the binary Goldbach pairings.

**4.5. Harmonic Properties and Implications**

The function  $J(x)$  is continuous ( $C^0$ ) and bounded on the interval  $[0, 1]$ . It is piecewise infinitely differentiable ( $C^\infty$ ) on the open intervals between consecutive integers ( $\mathbb{R} \setminus \mathbb{Z}$ ). At the integer coordinates themselves, differentiability ceases due to two distinct mathematical mechanisms. At even integers, the floor function  $\lfloor x/2 \rfloor$  in the upper bound of  $GP(x)$  induces structural step transitions in the product sequence. At odd integers—which contain all Goldbach prime candidates—the sequence remains structurally static, but the fractional exponents applied to the vanishing sine terms force the first derivative to diverge to  $\pm\infty$ . This creates infinitely sharp, non-differentiable cusps at prime coordinates, yet the function preserves  $C^0$  continuity by yielding exact evaluation limits.

The theoretical novelty of  $J(x)$  lies in its dimensional translation of additive number theory. Traditional sieves dictate that primes must be isolated through iterative algorithmic exclusion. The discriminant  $J(x)$  proves that specific prime configurations—namely, symmetric Goldbach pairs—can instead be isolated as the global minima of a continuous scalar field.

Consequently, the discrete Goldbach parity problem is transposed into the domain of continuous real analysis. By embedding the Sieve of Eratosthenes within a harmonic trigonometric framework,  $J(x)$  establishes a foundation for applying the calculus of variations, gradient-based optimization, and Fourier-analytic bounds to the prime distribution [15].

It must be noted that while  $J(x)$  bypasses the discrete “magnitude barrier,” it introduces a corresponding continuous “precision barrier.” The exact evaluation of the 0/0 indeterminate form large integers requires evaluating trigonometric arguments to a floating-point precision proportional to the magnitude of  $x$ . This reflects a conservation of information entropy (Kolmogorov complexity) [16]; the discrete computational workload of algorithmic sieving is translated into the requisite decimal precision of continuous trigonometric evaluation. Thus,  $J(x)$  serves as a theoretical continuous bounding function rather than an algorithmic shortcut, since the inherent computational hardness of prime detection is not circumvented.

*Remark 4.1* (Connection to the Hardy-Littlewood Circle Method). The discriminant kernel  $J(x)$  serves as a deterministic, real-line counterpart to the classical Hardy-Littlewood Circle Method. Traditionally, the Circle Method estimates the number of prime partitions by integrating exponential sums over a continuous frequency domain. In that framework, the main structural information is captured by “Major Arcs” (which represent modular divisibility rules),

while the remaining frequencies create chaotic “Minor Arc” error terms that are difficult to bound.

$J(x)$  takes an inverted approach. Rather than integrating over frequencies to find an asymptotic average, it embeds these modular divisibility rules directly into the continuous spatial coordinate  $x$  using the periodic sine terms  $\sin^2(\pi x/j)$ . Here, the proper divisors  $j$  function like the Major Arc denominators. Because  $J(x)$  is evaluated pointwise rather than through integration, it avoids Minor Arc error terms. The additive prime state is not estimated probabilistically; it is resolved through the evaluation of the 0/0 indeterminate limit at the target integer.

**4.6. Construction of a Harmonic Prime-Counting Function**

While the Goldbach Discriminant Kernel  $J(x)$  isolates the structural intersection of two primality conditions for a fixed even integer  $2N$ , constructing a prime-counting function requires a singular, continuous indicator over the full prime domain  $\mathbb{P}$ . To achieve this, we define the Single-Prime Discriminant  $J_P(x)$  by isolating the baseline prime-detecting kernel  $GP(x)$ :

$$J_P(x) := \begin{cases} \frac{1}{1 + \frac{GP(x)}{GI_2(x)}} & \text{if } x \in \mathbb{R} \setminus \mathbb{Z}, \\ \lim_{t \rightarrow x} \frac{1}{1 + \frac{GP(t)}{GI_2(t)}} & \text{if } x \in \mathbb{Z}_{\geq 2}. \end{cases} \tag{28}$$

The well-posedness of this definition at integer coordinates requires the explicit existence of the limit. As established in Lemma 4.2 and Theorem 4.2, fractional exponent dominance ensures this limit converges to either 0 or 1.

Because the symmetric  $2N - x$  boundary condition is removed, the structural classification of  $J_P(x)$  relies exclusively on the absolute number of proper divisors bounded by  $\max(2, \lfloor x/2 \rfloor)$ .

*Lemma 4.2* (Convergence of  $GP(x)/GI_2(x)$ ). Let  $z \in \mathbb{Z}_{\geq 2}$ .

1. If  $z$  is prime,  $\lim_{x \rightarrow z} \frac{GP(x)}{GI_2(x)} = +\infty$ .
2. If  $z$  is composite,  $\lim_{x \rightarrow z} \frac{GP(x)}{GI_2(x)} = 0$ .

*Proof:* The infinitesimal zero-mode cancellation algebra mirrors Lemma 4.1.

*Case 1 (Primes):* Because  $z \in \mathbb{P}$ ,  $z$  possesses no proper divisors within the dynamic upper bound. Thus,  $GP(z) = C > 0$ . As  $x \rightarrow z$ ,  $GI_2(x) \rightarrow 0$ . The ratio evaluates to  $C/0$ , diverging to  $+\infty$ .

*Case 2 (Composites):* If  $z$  is composite, it possesses at least one proper divisor  $j \geq 2$  satisfying  $j < z$ . As established in Lemma 4.1, applying the first-order Taylor expansion yields the competing fractional zero-modes  $\epsilon^{2/j}$  and  $\epsilon^{2/z}$ . The convergence of the isolated ratio reduces to  $\lim_{\epsilon \rightarrow 0} \epsilon^{2z-2j}$ .

Because  $j$  is a proper divisor of  $z$ , the inequality  $z > j$  must hold, guaranteeing that the exponent  $(2z - 2j)$  is a positive integer. This algebraic dominance proves that the zero-mode from  $GP(x)$  approaches zero at a slower algebraic rate than  $GI_2(x)$ , regardless of the integer’s total divisor count. Thus, even for squares of primes (e.g.,  $z = 9, 25$ ) where zero-mode multiplicity counting fails, the single fractional exponent

dominance guarantees the ratio converges to 0.

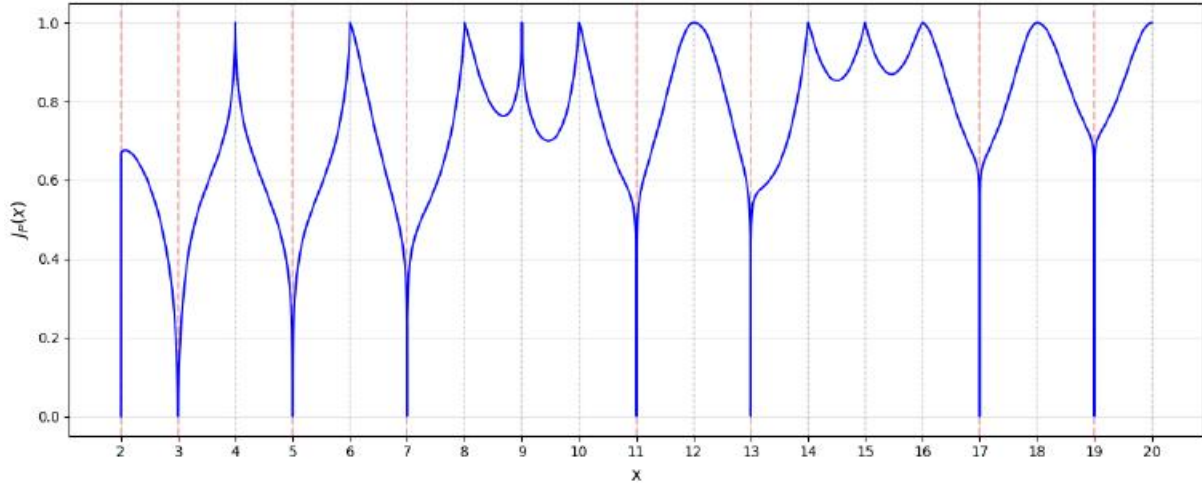
*Theorem 4.2* (Structural Convergence of  $J_P(x)$ ). The continuous single-prime discriminant evaluates as follows:

1. *Primes:* If  $x \in \mathbb{P}$ ,  $J_P(x) = 0$ .
2. *Composites:* If  $x \in \mathbb{Z}_{\geq 2}$  is composite,  $J_P(x) = 1$ .
3. *Real Continuum:* For all  $x \in \mathbb{R} \setminus \mathbb{Z}$ ,  $J_P(x) \in (0, 1)$ .

*Proof:* By substituting the limit evaluations established in Lemma 4.2 directly into the piecewise definition of  $J_P(x)$ , the

structural classifications follow directly:  $J_P(p) = \frac{1}{1+\infty} = 0$ , and  $J_P(c) = \frac{1}{1+0} = 1$ . For  $x \notin \mathbb{Z}$ , the ratio evaluates to a finite positive constant, yielding a proper fraction.

To visualize the algebraic resolution of these limit boundaries across the prime domain, Figure 3 illustrates the continuous scalar field of the Single-Prime Discriminant  $J_P(x)$  over the interval  $x \in [2, 20]$ .



**Figure 3.** Graph of the Continuous Harmonic Single-Prime Discriminant  $J_P(x)$  for  $x \in [2, 20]$ . The curve evaluates to 0 at all prime numbers (indicated by red dashed vertical lines) via non-differentiable cusps. Conversely, the fractional exponent dominance of the proper divisors forces the curve to evaluate to 1 at all composite integers (indicated by gray dotted vertical lines), validating the limit behavior established in Theorem 4.2.

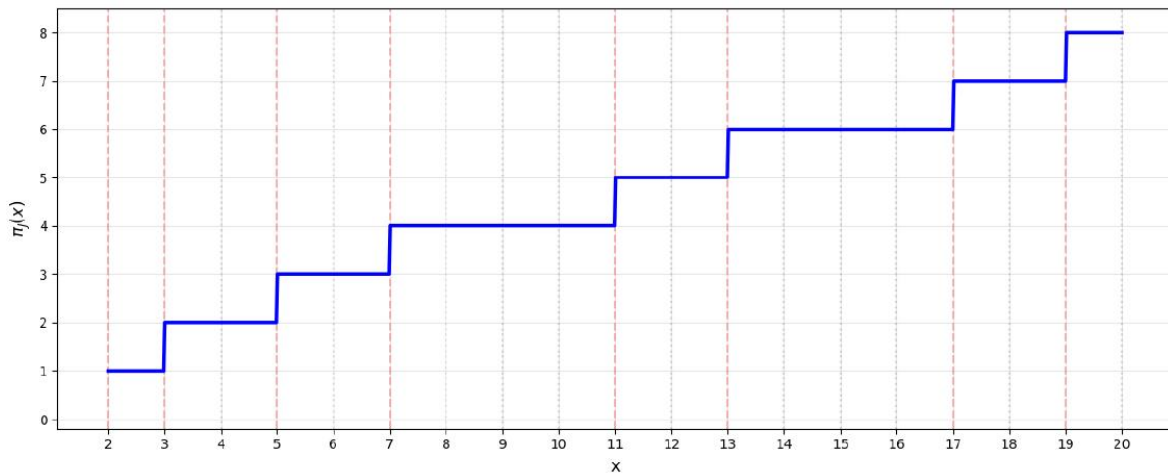
As demonstrated in Figure 3, the discriminant  $J_P(x)$  isolates individual primes without relying on the symmetric boundary of a fixed even integer, and serves as the foundational indicator necessary to construct the prime-counting function.

Because  $J_P(x)$  provides an exact, continuous classification of individual primes, it enables the construction of a deterministic prime indicator function  $I_P(x) := 1 - J_P(x)$ , where  $I_P(p) = 1$  for primes and 0 for composites.

Consequently, the prime-counting function  $\pi(x)$  can be formulated through trigonometric limits:

$$\pi_J(x) = \sum_{n=2}^{\lfloor x \rfloor} (1 - J_P(n)). \tag{29}$$

Figure 4 illustrates the continuous evaluation of Equation (29) across the interval  $x \in [2, 20]$ .



**Figure 4.** Graph of  $\pi_J(x)$  for  $x \in [2, 20]$ . The function evaluates the sum  $\sum (1 - J_P(n))$  incrementally across the integer domain. Because the continuous single-prime discriminant evaluates to 0 at primes and 1 at composites, the summation dynamically generates the discrete step-distribution of  $\pi(x)$ . The function increases at the prime coordinates (red dashed lines) and remains constant across all composite intervals (gray dotted lines), translating discrete prime cardinality into a continuous trigonometric summation.

As visualized in Figure 4, the macroscopic interference pattern of the trigonometric zero-modes systematically constructs the exact staircase geometry of the prime distribution. Because  $J_P(x)$  generates  $\pi_J(x)$  entirely through periodic trigonometric kernels, the macroscopic interference pattern of these continuous waves recovers the discrete prime cardinality. This provides a direct real-valued harmonic translation of the discrete sieve, demonstrating that prime distribution can be analyzed entirely through the continuous integration of bounded trigonometric forms.

## 5. Conclusions

The study of prime distribution and additive partitions has historically been constrained by the dichotomy between discrete multiplicative generation and additive sequences. By treating the representation of primes as an evolving problem of structural density and multiplicative encoding, this paper provides a mathematical bridge from discrete algebra to continuous real analysis.

The progression of this framework is tripartite: First, Hybrid Prime Factorization (HPF) is established as a bounded *a priori* structural prime-generation framework. By partitioning the primorial base into disjoint multiplicative supports linked by a single additive or subtractive operation, certain HPF configurations are shown to yield prime outputs on certified intervals. In these cases, primality follows from the algebraic hypotheses of the construction rather than from a separate post hoc primality test. Second, the Goldbach Pairing Map,  $L(C_N)$ , proves that the complete additive complexity of an even integer's prime partitions can be losslessly encoded into a unique, recoverable multiplicative value. While  $L(C_N)$  structurally compresses additive prime states, its asymptotically factorial growth highlights the inherent "magnitude barrier" of discrete integer representations. Finally, to bypass this discrete limitation, the arithmetic structure of the primes is shifted onto the real continuum. The Goldbach Discriminant Kernel,  $J(x)$ , embeds the Sieve of Eratosthenes into a bounded, piecewise smooth harmonic framework. By collapsing the parity problem into the evaluation of 0/0 indeterminate limits at integer coordinates,  $J(x)$  isolates symmetric prime pairs without combinatorial explosion.

Rather than isolating primes through discrete algorithmic exclusion, prime partitions can be identified as the global minima of continuous scalar fields. This dimensional translation opens promising paths for future research, inviting the application of continuous optimization, the calculus of variations, and real-valued harmonic analysis to the classical problems of prime distribution.

While the theoretical translation of prime distribution into a continuous real-valued framework is mathematically robust, it preserves the computational hardness of prime isolation. Because the evaluation of 0/0 indeterminate forms at massive integer coordinates requires a floating-point precision proportional to the magnitude of the target,

harmonic sieves like  $J(x)$  cannot be utilized as algorithmic shortcuts. Consequently, the physical search for world-record primes remains fundamentally bound to discrete algebraic algorithms [1]. Practical High-Performance Computing (HPC) implementations must continue to rely on techniques such as the Fast Fourier Transform (FFT) operating within discrete limits.

This dimensional translation reveals a conservation of computational entropy: the discrete magnitude barrier of algebra is exchanged for the continuous precision barrier of harmonic analysis. The computational hardness of prime isolation cannot be bypassed, only shifted between domains. By constructing a continuous indicator function for primality, this framework demonstrates that the discrete distribution of primes can be recovered through the continuous integration of bounded harmonic forms, offering a geometric perspective on additive prime complexity.

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## Abbreviations

bGC	Binary Goldbach Conjecture
FTA	Fundamental Theorem of Arithmetic
GC	Goldbach Conjecture
HPF	Hybrid Prime Factorization

## Acknowledgments

The author acknowledges the engineering and research teams at Google and OpenAI for developing high-level formal reasoning tools utilized during the computational validation phase of this research, serving as a valuable productivity tool for independent mathematical research.

## Author Contributions

**Ioannis Papadakis:** Conceptualization, Methodology, Software, Formal Analysis, Writing - original draft

## Data Availability Statement

The Python code, plotting scripts, and numerical search algorithms utilized to generate the harmonic sieve graphs (Figures 1–4) and to verify the HPF configurations are available upon request from the corresponding author.

## Funding

This research received no external funding.

## Conflicts of Interest

The author declares no conflicts of interest.

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