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Spectral Constraints in Adaptive Control of Graphon-induced SPDEs: Stability and Regularity Preservation

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Abstract

This paper investigates stability and regularity properties of stochastic partial differential equations (SPDEs) arising from graphon-induced interaction structures under adaptive control mechanisms. The dynamics are modeled by linear SPDEs driven by multiplicative noise, with interaction operators defined through a graphon limit framework. Motivated by applications in epidemic management, opinion dynamics, power-grid stabilization, and neuronal regulation, we address the problem of designing control laws that respect the intrinsic spectral geometry of the underlying network rather than treating it as an afterthought. By exploiting spectral properties of graphon operators, we derive sufficient conditions ensuring mean-square stability and preservation of spatial regularity in controlled systems. The proposed spectral constraints provide a tractable criterion linking network structure, noise intensity, and adaptive control gains. Our analysis employs semigroup techniques and stochastic energy estimates, accommodating non-symmetric graphon structures through pseudospectral analysis. We demonstrate that spectrally admissible controls preserve Sobolev regularity and prevent crossing of critical thresholds that separate function-valued from distribution-valued solutions. The framework is compared with traditional Linear Quadratic Regulator (LQR) and robust control methods, showing bounded conservativeness while guaranteeing regularity preservation. Numerical illustrations on block, power-law, and stochastic block model graphons validate the theoretical results. An application to stochastic epidemic control demonstrates practical implementation with online spectral estimation. These findings are further supported by large-scale simulations on networks with more than one thousand nodes, which confirm that the proposed spectral-constraint methodology scales favorably with system size while keeping regularity violations at zero or near-zero levels. The work establishes fundamental principles for responsible control of complex network systems, ensuring control objectives are pursued in harmony with intrinsic network geometry. Overall, the results offer both rigorous theoretical guarantees and practical, verifiable algorithms for practitioners who must design controllers for large stochastic systems evolving on networked and graph-limit structures.

Keywords

Stochastic PDEs, Graphons, Spectral Theory, Adaptive Control, Stability Analysis, Network Dynamics, Epidemic Control, Spectral Constraints

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1. Introduction

1.1. Motivation and Context

The control of complex dynamical systems evolving on large-scale networks presents fundamental challenges at the intersection of stochastic analysis, spectral theory, and control engineering. Applications ranging from epidemic management and opinion dynamics to power grid stabilization and neuronal regulation require control strategies that account for both the stochastic nature of dynamics and the geometric structure of underlying networks.

A key insight from recent mathematical advances is that stochastic partial differential equations (SPDEs) on graph limits, described via the graphon formalism [8, 9], exhibit critical spectral thresholds governing solution regularity. Specifically, it has been established that for graphon-induced SPDEs, there exists a critical smoothness parameter $\alpha_c = 1$ separating function-valued solutions ($\alpha > \alpha_c$) from distribution-valued ones ($\alpha < \alpha_c$). This spectral regularity principle imposes fundamental constraints on any intervention strategy: control actions must preserve, or at minimum not destroy, the intrinsic spectral structure of the system.

Traditional control approaches, whether optimal control, adaptive control, or model predictive control, typically optimize performance metrics without explicit regard for spectral preservation. This can lead to controllers that, while achieving stabilization or tracking objectives, inadvertently violate spectral constraints, potentially inducing loss of regularity, instability in higher modes, or crossing of critical thresholds. Robustness-oriented designs developed for stochastic PDEs on networked infrastructures, such as power grids [18], illustrate how performance-driven tuning can silently erode the spectral margins that our framework makes explicit.

1.2. Relationship to Previous Work

Our work sits at the confluence of three active research areas:

1) *Graphon-induced SPDEs*: Building on spectral regularity frameworks for graphon operators, which characterize the relationship between graphon smoothness α, spectral decay $\gamma = \alpha/2$, and solution regularity $\beta < \gamma - 1/2$. The graphon formalism itself, together with the associated theory of graph limits, follows the foundational treatment of Lovász [8]; the link between graph-based diffusion dynamics and free-energy functionals draws on the Fokker?Planck formulation of Chow, Huang, Li and Zhou [9].
2) *Control of stochastic PDEs*: Extending classical results on stabilization of SPDEs [1, 2] to the geometrically complex setting of graphon operators, where the control operator interacts nontrivially with network topology. Related graphon-based control formulations with stability guarantees have also been developed by Chen and Yuan [13], and adaptive spectral methods for network control under uncertainty were proposed by Kumar and Singh [14].
3) *Network control theory*: Connecting to recent advances in control of complex networks [3, 4], but with the novel perspective of spectral constraints derived from continuum graph limits rather than discrete graph properties. Spectral constraints for distributed control of networked systems in a related but discrete setting were also studied by Smith and Johnson [16].
4) *Non-self-adjoint operator theory*: Incorporating recent advances in spectral theory for non-self-adjoint operators [5, 6] to handle asymmetric graphon structures common in real-world networks. Matrix-analytic tools underlying our perturbation estimates, in particular the Bauer?Fike theorem used in the Appendix, follow the standard exposition of Bhatia [7]; adaptive control of non-self-adjoint operators arising in social-network models has been considered by Robinson and Thompson [20].

The key departure from existing literature is our focus on *spectral admissibility* as a primary design constraint, rather than treating it as a secondary consideration or verification step, with explicit treatment of non-self-adjoint cases and practical computational aspects.

1.3. Main Contributions

This paper makes the following contributions:

1) *Spectral admissibility framework with self-adjointness analysis*: We introduce a rigorous definition of spectrally admissible control operators for graphon-induced SPDEs, with explicit compatibility conditions linking control magnitude to the spectral gap, including detailed treatment of non-self-adjoint graphon Laplacians.
2) *Well-posedness and stability with complete proofs*: We establish existence, uniqueness, and stability in Sobolev norms for controlled SPDEs under spectral constraints, using a combination of semigroup theory and spectral perturbation arguments, with full technical details provided in appendices.
3) *Regularity preservation theorem with extended analysis*: Our main result shows that spectrally admissible controls preserve the Sobolev regularity class of solutions and

prevent crossing of the critical threshold $\alpha_c = 1$, with explicit bounds on conservativeness compared to traditional methods.

4) *Comparative analysis with traditional control*: Quantitative comparison with Linear Quadratic Regulator (LQR) and robust control methods, establishing bounds on the performance-cost tradeoff induced by spectral constraints.
5) *Practical computational framework*: Development of verifiable sufficient conditions and online estimation methods for spectral admissibility in practical applications.
6) *Illustrative examples*: Examples on diverse graphon families (block models, power-law, stochastic block models) showing qualitative behavior consistent with theoretical predictions. Large-scale graphon-approximation algorithms for network control, in the spirit of Patel and Jones [17], underpin the scalability of the computational pipeline used to generate these examples.
7) *Application to epidemic control with implementation details*: We demonstrate the framework through stabilization of a stochastic SIS epidemic model on dynamic networks, showing how spectral constraints guide controller design to avoid induced instabilities, with complete algorithmic implementation. This application connects to, and extends, graphon-based epidemic control approaches such as those of Li and Wang for COVID-19 dynamics [15] and Garcia and Martinez for time-varying epidemic networks [19].

1.4. Outline

Section 2 recalls necessary background on graphons, SPDEs, and spectral regularity with enhanced discussion on self-adjointness properties. Section 3 introduces the spectrally constrained control framework. Section 4 establishes well-posedness and stability results with complete proofs. Section 5 presents the main regularity preservation theorem. Section 6 provides comparative analysis with traditional control methods, including comparison with reinforcement learning approaches. Section 7 develops computational methods for spectral constraint verification. Section 8 analyzes localized control strategies. Section 9 applies the theory to epidemic control with detailed examples and large-scale simulations. Section 10 discusses implications, limitations, and future directions.

2. Mathematical Preliminaries and Spectral Background

2.1. Dynamic Graphons and Associated Operators

We adopt the standard measure-theoretic graphon formalism developed for dense graph limits [8]; see also [9] for the associated free-energy and Fokker?Planck perspective on graph-based diffusion processes.

Definition 2.1 (Time-dependent graphon). A *time-dependent graphon* is a measurable function $W : [0,T] \times [0,1]^2 \to [0,1]$ such that:

1) For each $t \in [0,T]$, $W(t,\cdot,\cdot)$ is symmetric: $W(t,x,y) = W(t,y,x)$
2) The mapping $t \mapsto W(t,\cdot,\cdot)$ is measurable in the cut norm
3) Uniform boundedness: $\sup_{t\in[0,T]} \|W(t,\cdot,\cdot)\|_{L^\infty} \le 1$

For $\alpha > 0$, we denote $W \in C_T^\alpha([0,1]^2)$ if $W(t,\cdot,\cdot) \in C^\alpha([0,1]^2)$ uniformly in t.

Definition 2.2 (Dynamic graphon operators). For a time-dependent graphon $W(t,\cdot,\cdot)$, define:

1) *Integral operator*:

$$\mathcal{T}_{W(t)} f(x) = \int_0^1 W(t,x,y) f(y) dy$$

2) *Graphon Laplacian*:

$$\mathcal{L}_{W(t)} f(x) = \int_0^1 W(t,x,y)(f(x) - f(y)) dy$$

3) *Degree function*:

$$d(t,x) = \int_0^1 W(t,x,y) dy$$

2.2. Sobolev Spaces Adapted to Graphon Operators

Definition 2.3 (Adapted Sobolev spaces). For $\beta \ge 0$ and a time-dependent graphon $W(t,\cdot,\cdot)$, we define the adapted Sobolev space $H^\beta_{\mathcal{T}_{W(t)}}$ as the completion of $C^\infty([0,1])$ with respect to the norm:

$$\|f\|_{H^\beta_{\mathcal{T}_{W(t)}}} = \|\mathcal{T}^{\beta/2}_{W(t)} f\|_{L^2([0,1])},$$

where $\mathcal{T}^{\beta/2}_{W(t)}$ is defined via functional calculus of the self-adjoint operator $\mathcal{T}_{W(t)}$. The corresponding inner product is:

$$\langle f, g\rangle_{H^\beta_{\mathcal{T}_{W(t)}}} = \langle \mathcal{T}^{\beta/2}_{W(t)} f, \mathcal{T}^{\beta/2}_{W(t)} g\rangle_{L^2([0,1])}.$$

For $\beta < 0$, the space is defined by duality: $H^{\beta}_{\mathcal{T}_{W(t)}} = (H^{-\beta}_{\mathcal{T}_{W(t)}})^*$.

Remark 2.4. These spaces are adapted to the spectral structure of $\mathcal{T}_{W(t)}$. When $\mathcal{T}_{W(t)}$ has eigenvalues $\{\mu_k(t)\}_{k\geq 1}$ with eigenfunctions $\{\psi_k(t)\}$, the norm can be expressed as:

$$\|f\|^2_{H^{\beta}_{\mathcal{T}_{W(t)}}} = \sum_{k=1}^{\infty} \mu_k(t)^{\beta} |\langle f, \psi_k(t)\rangle|^2.$$

2.3. Self-Adjointness Properties of Graphon Operators

Proposition 2.5 (Self-adjointness conditions). *Let $W(t,\cdot,\cdot)$ be a time-dependent graphon. Then:*

1) $\mathcal{T}_{W(t)}$ is self-adjoint on $L^2([0,1])$ if and only if $W(t,x,y) = \overline{W(t,y,x)}$ almost everywhere.

2) $\mathcal{L}_{W(t)}$ is self-adjoint if and only if $d(t,x)W(t,x,y) = d(t,y)W(t,y,x)$ almost everywhere.

3) For symmetric graphons ($W(t,x,y) = W(t,y,x)$), both $\mathcal{T}_{W(t)}$ and $\mathcal{L}_{W(t)}$ are self-adjoint.

4) In the general non-symmetric case, we can decompose:

$$\mathcal{L}_{W(t)} = \mathcal{L}^s(t) + \mathcal{L}^a(t) \quad \text{where}$$

$$\mathcal{L}^s(t)f(x) = \int_0^1 \frac{W(t,x,y) + W(t,y,x)}{2}(f(x) - f(y))dy$$

$$\mathcal{L}^a(t)f(x) = \int_0^1 \frac{W(t,x,y) - W(t,y,x)}{2}(f(x) - f(y))dy$$

with $\mathcal{L}^s(t)$ self-adjoint and $\mathcal{L}^a(t)$ anti-self-adjoint.

Assumption 2.6 (Uniform spectral structure). There exists $\alpha > 0$ and constants $c, C > 0$ such that for all $t \in [0,T]$:

$$ck^{-\alpha/2} \leq \Re(\lambda_k(\mathcal{L}_{W(t)})) \leq Ck^{-\alpha/2}, \quad k \geq 1,$$

where $\lambda_k(\mathcal{L}_{W(t)})$ denotes the k-th eigenvalue of $\mathcal{L}_{W(t)}$ (ordered by real part), and for the imaginary parts we assume $|\Im(\lambda_k(\mathcal{L}_{W(t)}))| \leq Mk^{-\beta/2}$ with $\beta > \alpha$.

Assumption 2.7 (Time regularity of graphon). The graphon $W(t,\cdot,\cdot)$ is Hölder continuous in time: there exists $\theta \in (0,1]$ and $L > 0$ such that for all $s,t \in [0,T]$,

$$\|W(t,\cdot,\cdot) - W(s,\cdot,\cdot)\|_{L^{\infty}([0,1]^2)} \leq L|t-s|^{\theta}.$$

Remark 2.8 (Non-self-adjoint spectral theory). For non-self-adjoint LW(t), the associated perturbation theory parallels the treatment of adaptive control of non-self-adjoint operators in social-network models developed by Robinson and Thompson [20]. we work with the numerical range and pseudospectra. The ε-pseudospectrum is defined as:

$$\sigma_{\varepsilon}(\mathcal{L}_{W(t)}) = \{z \in \mathbb{C} : \|(zI - \mathcal{L}_{W(t)})^{-1}\| > \varepsilon^{-1}\}.$$

Our spectral admissibility conditions will ensure control perturbations preserve pseudospectral properties.

2.4. SPDE Framework and Spectral Regularity

The uncontrolled SPDE on dynamic graphons takes the form:

$$dU(t) = \mathcal{L}_{W(t)}U(t)dt + \mathcal{T}_{W(t)}dW(t), U(0) \in L^2([0,1]) \quad (1)$$

where $W(t)$ is a cylindrical Wiener process on $L^2([0,1])$.

The following spectral regularity principle is fundamental to our development:

Theorem 2.9 (Spectral regularity principle). *Let $W(t,\cdot,\cdot) \in C^{\alpha}_T([0,1]^2)$ satisfy Assumption 2.6. Then the solution to (1) satisfies:*

1) Regularity threshold: For any $\beta < \alpha/2 - 1/2$,

$$\sup_{t\in[0,T]} \mathbb{E}\|U(t)\|^2_{H^{\beta}_{\mathcal{T}_{W(t)}}} \leq C(\beta,\alpha,T),$$

where $C(\beta,\alpha,T)$ depends only on β,α,T.

2) Critical threshold: There exists $\alpha_c = 1$ such that:

a) For $\alpha > 1$: $U(t)$ is function-valued with explicit Hölder regularity

b) For $\alpha = 1$: Logarithmic divergences appear in L^2 norms

c) For $\alpha < 1$: $U(t)$ is distribution-valued

Remark 2.10. The key condition is the regularity condition $\beta < \alpha/2 - 1/2$. This establishes the fundamental link between graphon smoothness α and solution regularity β.

3. Spectral-Constrained Adaptive Control Framework

3.1. Controlled SPDE Formulation

We consider the controlled stochastic evolution equation:

$$\begin{cases} dU(t) = [\mathcal{L}_{W(t)}U(t) + C(t)U(t)]dt + \mathcal{T}_{W(t)}dW(t), \\ U(0) \in H^{\beta}_{\mathcal{T}_{W(0)}}([0,1]), \quad t \in [0,T], \end{cases} \tag{2}$$

where:

1) $\mathcal{L}_{W(t)}$ is the graphon Laplacian of dynamic graphon $W(t,\cdot,\cdot)$
2) $C(t)$ is a control operator, possibly depending on the current state
3) $\mathcal{T}_{W(t)}dW(t)$ represents multiplicative noise with graphon covariance structure

The control operator $C(t)$ may be:

1) *Static*: $C(t) \equiv C$ (time-invariant)
2) *Feedback*: $C(t) = \varphi(\langle U(t), \psi\rangle)$ for some $\varphi : \mathbb{R} \to \mathbb{R}$ and $\psi \in L^2([0,1])$
3) *Adaptive*: $C(t)$ depends on past observations or parameter estimates

3.2. Spectral Admissibility Conditions

The central concept of this work is the notion of spectral admissibility, extended to handle non-self-adjoint cases:

—- a design philosophy closely aligned with the distributed spectral-constraint approach of Smith and Johnson [16], here extended to the continuum graphon setting:

Definition 3.1 (Spectrally admissible control). A control operator family $\{C(t)\}_{t\in[0,T]}$ is *spectrally admissible* with respect to $\{\mathcal{L}_{W(t)}\}$ if the following conditions hold:

1) *Boundedness*: $\sup_{t\in[0,T]} \|C(t)\|_{\mathcal{L}(L^2)} \le \varepsilon$
2) *Spectral compatibility*: For each eigenfunction $\phi_k(t)$ of $\mathcal{L}_{W(t)}$ (or generalized eigenfunction in non-diagonalizable case) with eigenvalue $\lambda_k(\mathcal{L}_{W(t)})$,

$$\|C(t)\phi_k(t)\|_{L^2} \le \delta_k \|\phi_k(t)\|_{L^2},$$

with $\delta_k = o(|\lambda_k(\mathcal{L}_{W(t)})|)$ as $k \to \infty$.
3) *Time regularity*: $t \mapsto C(t)$ is strongly measurable
4) *Spectral margin condition*: There exists $\gamma > 0$ such that for all $t \in [0,T]$:

$$\inf_{k\ge 1} \left(\Re(\lambda_k(\mathcal{L}_{W(t)})) - \|C(t)\phi_k(t)\|_{L^2}\right) \ge \gamma$$

and for non-self-adjoint cases, the ε-pseudospectral gap condition:

$$\inf_{z\in\sigma_\varepsilon(\mathcal{L}_{W(t)}+C(t))} \Re(z) \ge \gamma \quad \text{for some } \varepsilon > 0.$$

5) *Non-normality control*: If $\mathcal{L}_{W(t)}$ is non-normal, we require:

$$\|[\mathcal{L}_{W(t)}, C(t)]\| \le \kappa \|\mathcal{L}_{W(t)}\| \|C(t)\|$$

where $[A,B] = AB - BA$ is the commutator and $\kappa < 1$.

The *spectral margin* is defined as $\gamma = \inf_{t,k}(\Re(\lambda_k(\mathcal{L}_{W(t)})) - \|C(t)\phi_k(t)\|)$.

Remark 3.2. Condition (2) ensures that control actions respect the asymptotic spectral decay of the graphon Laplacian. Condition (4) guarantees a uniform spectral gap is maintained under control, extended to pseudospectra for non-self-adjoint cases. Condition (5) controls the enhancement of non-normality under perturbation.

3.3. Spectral Constraint Functions

To operationalize spectral admissibility, we introduce constraint functions linking control magnitude to spectral properties:

Definition 3.3 (Spectral constraint function). A function $\kappa : [0,T] \times \mathbb{N} \to \mathbb{R}^+$ is a *spectral constraint function* for $\{\mathcal{L}_{W(t)}\}$ if:

1) For each t, $\kappa(t,k)$ is decreasing in k
2) $\kappa(t,k) \le \Re(\lambda_k(\mathcal{L}_{W(t)}))$ for all t,k
3) $\kappa(t,k) \asymp \Re(\lambda_k(\mathcal{L}_{W(t)}))$ as $k \to \infty$

A control operator $C(t)$ satisfies the κ-constraint if:

$$\|C(t)\phi_k(t)\|_{L^2} \le \kappa(t,k)\|\phi_k(t)\|_{L^2}, \quad \forall t,k.$$

Natural choices include:

1) $\kappa(t,k) = \theta\Re(\lambda_k(\mathcal{L}_{W(t)}))$ for $\theta \in (0,1)$ (proportional constraint)
2) $\kappa(t,k) = \min(\varepsilon, \Re(\lambda_k(\mathcal{L}_{W(t)})) - \gamma)$ (gap-preserving constraint)
3) $\kappa(t,k) = \Re(\lambda_k(\mathcal{L}_{W(t)}))\exp(-ck)$ (exponentially decaying constraint)

4. Well-posedness and Stability Under Spectral Constraints

4.1. Existence and Uniqueness with Complete Proof

Theorem 4.1 (Well-posedness of controlled SPDE). *Consider the controlled SPDE (2) with:*

1) $W(t,\cdot,\cdot) \in C_T^{\alpha}([0,1]^2)$ satisfying Assumptions 2.6 and 2.7

2) $C(t)$ spectrally admissible (Definition 3.1) with spectral margin $\gamma > 0$

3) Initial condition $U(0) \in H^{\beta}_{\mathcal{T}_{W(0)}}$ with $\beta < \alpha/2 - 1/2$

Then there exists a unique mild solution $U \in L^2(\Omega; C([0,T]; H^{\beta}_{\mathcal{T}_{W(t)}}))$ satisfying:

$$U(t) = S(t,0)U(0) + \int_0^t S(t,s)C(s)U(s)ds + \int_0^t S(t,s)\mathcal{T}_{W(s)}dW(s),$$

where $S(t,s)$ is the evolution system generated by $\mathcal{L}_{W(t)}$.

Complete proof with rigorous Gronwall argument. We employ a fixed-point argument in the space $\mathcal{X}_T = L^2(\Omega; C([0,T]; H^{\beta}_{\mathcal{T}_{W(t)}}))$ equipped with the norm:

$$\|U\|_{\mathcal{X}_T} = \left(\sup_{t\in[0,T]} \mathbb{E}\|U(t)\|^2_{H^{\beta}_{\mathcal{T}_{W(t)}}} \right)^{1/2}.$$

Step 1: Mild formulation and mapping definition. Define the mapping $\Phi : \mathcal{X}_T \to \mathcal{X}_T$ by:

$$\Phi(U)(t) = S(t,0)U(0) + \int_0^t S(t,s)C(s)U(s)ds + \int_0^t S(t,s)\mathcal{T}_{W(s)}dW(s).$$

Step 2: Semigroup estimates. From the spectral admissibility and Assumption 2.6, $\mathcal{L}_{W(t)}$ generates an evolution system $S(t,s)$ satisfying for $\theta \in [0,1]$:

$$\|S(t,s)\|_{\mathcal{L}(H^{\beta}_{\mathcal{T}_{W(s)}}, H^{\beta}_{\mathcal{T}_{W(t)}})} \le Me^{-\omega(t-s)},$$
$$\|S(t,s)\|_{\mathcal{L}(L^2, H^{\beta}_{\mathcal{T}_{W(t)}})} \le M(t-s)^{-\theta/2}e^{-\omega(t-s)},$$

where $M, \omega > 0$ depend only on α, β, T.

Step 3: Contraction estimate. For $U, V \in \mathcal{X}_T$, consider:

$$\mathbb{E}\|\Phi(U)(t) - \Phi(V)(t)\|^2_{H^{\beta}_{\mathcal{T}_{W(t)}}} = \mathbb{E}\left\| \int_0^t S(t,s)C(s)(U(s)-V(s))ds \right\|^2_{H^{\beta}_{\mathcal{T}_{W(t)}}}$$
$$\le \mathbb{E}\left(\int_0^t \|S(t,s)\|_{\mathcal{L}(H^{\beta})}\|C(s)\|_{\mathcal{L}(H^{\beta})} \times \|U(s)-V(s)\|_{H^{\beta}_{\mathcal{T}_{W(s)}}} ds \right)^2.$$

Using Hölder's inequality and the spectral admissibility condition:

$$\le \left(\int_0^t \|S(t,s)\|^2_{\mathcal{L}(H^{\beta})}\|C(s)\|^2_{\mathcal{L}(H^{\beta})}ds \right) \times \left(\int_0^t \mathbb{E}\|U(s)-V(s)\|^2_{H^{\beta}_{\mathcal{T}_{W(s)}}} ds \right).$$

From spectral admissibility, $\|C(s)\|_{\mathcal{L}(H^{\beta})} \le \varepsilon_{\beta}$ where $\varepsilon_{\beta} = \sup_k \delta_k(1+k^2)^{\beta/2} < \infty$ by condition (2). Since $\mathcal{L}_{W(t)}$ generates an analytic semigroup:

$$\int_0^t \|S(t,s)\|^2_{\mathcal{L}(H^{\beta})}\|C(s)\|^2_{\mathcal{L}(H^{\beta})}ds \le M^2\varepsilon_{\beta}^2 \int_0^t e^{-2\omega(t-s)}(t-s)^{-2\theta}ds.$$

Thus:

$$\int_0^t \|S(t,s)\|^2_{\mathcal{L}(H^{\beta})}\|C(s)\|^2_{\mathcal{L}(H^{\beta})}ds \le M^2\varepsilon_{\beta}^2 \int_0^t e^{-2\omega(t-s)}(t-s)^{-2\theta}ds.$$

For $\theta < 1/2$, the integral converges to $C(\theta,\omega,T) < \infty$. Choosing T sufficiently small such that:

$$C(\theta,\omega,T)M^2\varepsilon_{\beta}^2 T < 1$$

establishes that Φ is a contraction on $\mathcal{X}_T$.

Step 4: A priori estimates and global extension. Consider the Lyapunov functional $V(t) = \mathbb{E}\|U(t)\|^2_{H^{\beta}_{\mathcal{T}_{W(t)}}}$.

Applying Itô's formula to $\|U(t)\|^2_{H^{\beta}_{\mathcal{T}_{W(t)}}}$:

$$d\|U(t)\|^2_{H^{\beta}} = 2\langle U(t), (\mathcal{L}_{W(t)} + C(t))U(t)\rangle_{H^{\beta}}dt + \|\mathcal{T}_{W(t)}\|^2_{\mathrm{HS}(H^{\beta})}dt + 2\langle U(t), \mathcal{T}_{W(t)}dW(t)\rangle_{H^{\beta}}.$$

Taking expectations (the martingale term vanishes as a local martingale with integrable quadratic variation):

$$\frac{d}{dt}\mathbb{E}\|U(t)\|^2_{H^{\beta}} = 2\mathbb{E}\langle U(t), (\mathcal{L}_{W(t)} + C(t))U(t)\rangle_{H^{\beta}} + \|\mathcal{T}_{W(t)}\|^2_{\mathrm{HS}(H^{\beta})}.$$

From spectral admissibility conditions:

$$\mathbb{E}\langle U(t), \mathcal{L}_{W(t)}U(t)\rangle_{H^{\beta}} \le -\omega\mathbb{E}\|U(t)\|^2_{H^{\beta}},$$
$$\mathbb{E}\langle U(t), C(t)U(t)\rangle_{H^{\beta}} \le \varepsilon_{\beta}\mathbb{E}\|U(t)\|^2_{H^{\beta}},$$

where $\omega = \inf_{t,k} \Re(\lambda_k(\mathcal{L}_{W(t)}))$ and $\varepsilon_{\beta} = \sup_t \|C(t)\|_{\mathcal{L}(H^{\beta})}$.

Thus we obtain the differential inequality:

$$\frac{d}{dt}\mathbb{E}\|U(t)\|^2_{H^{\beta}} \le -2(\omega - \varepsilon_{\beta})\mathbb{E}\|U(t)\|^2_{H^{\beta}} + G(t),$$

where $G(t) = \|\mathcal{T}_{W(t)}\|^2_{\mathrm{HS}(H^\beta)}$ satisfies $\sup_{t\in[0,T]} G(t) \le C_G$ by Theorem 2.9, with C_G depending only on α, β, T.

Applying Gronwall's inequality to this deterministic differential inequality for $v(t) = \mathbb{E}\|U(t)\|^2_{H^\beta}$:

$$\mathbb{E}\|U(t)\|^2_{H^\beta} \le e^{-2(\omega-\varepsilon_\beta)t}\mathbb{E}\|U(0)\|^2_{H^\beta} + \int_0^t e^{-2(\omega-\varepsilon_\beta)(t-s)}G(s)ds.$$

Since $\omega - \varepsilon_\beta > 0$ by the spectral margin condition, and $G(s) \le C_G$, we obtain the uniform bound:

$$\sup_{t\in[0,T]} \mathbb{E}\|U(t)\|^2_{H^\beta} \le \mathbb{E}\|U(0)\|^2_{H^\beta} + \frac{C_G}{2(\omega-\varepsilon_\beta)}.$$

This a priori estimate allows extension from local to global solution on $[0, T]$.

Step 5: Uniqueness. Follows from the contraction property and linearity.

Step 6: Continuity. The stochastic convolution term is continuous in H^β by standard SPDE theory [1]. The deterministic terms are continuous by semigroup properties. □

4.2. Moment Stability with Enhanced Analysis

Theorem 4.2 (Exponential stability in moments). *Under the conditions of Theorem 4.1, with spectral margin $\gamma > \beta + 1/2$, the solution satisfies for any $p \ge 2$:*

$$\mathbb{E}\|U(t)\|^p_{H^\beta_{\mathcal{T}_{W(t)}}} \le C_p e^{-p(\omega-\varepsilon_\beta)t/2}\mathbb{E}\|U(0)\|^p_{H^\beta_{\mathcal{T}_{W(0)}}} + \frac{D_p}{\omega-\varepsilon_\beta},$$

where $\omega = \inf_t \Re(\lambda_1(\mathcal{L}_{W(t)}))$, $\varepsilon_\beta = \sup_t \|C(t)\|_{\mathcal{L}(H^\beta)}$, and C_p, D_p are constants depending only on p, α, β, T.

Proof. We provide the complete proof for $p = 2$ and indicate the extension to $p > 2$.

Case $p = 2$: Apply Itô's formula to $\|U(t)\|^2_{H^\beta_{\mathcal{T}_{W(t)}}}$:

$$d\|U(t)\|^2_{H^\beta} = 2\langle U(t), (\mathcal{L}_{W(t)} + C(t))U(t)\rangle_{H^\beta}dt + \|\mathcal{T}_{W(t)}\|^2_{\mathrm{HS}(H^\beta)}dt + 2\langle U(t), \mathcal{T}_{W(t)}dW(t)\rangle_{H^\beta}.$$

Taking expectations (the martingale term vanishes):

$$\frac{d}{dt}\mathbb{E}\|U(t)\|^2_{H^\beta} = 2\mathbb{E}\langle U(t), (\mathcal{L}_{W(t)} + C(t))U(t)\rangle_{H^\beta} + \|\mathcal{T}_{W(t)}\|^2_{\mathrm{HS}(H^\beta)}.$$

From spectral admissibility:

$$\mathbb{E}\langle U(t), \mathcal{L}_{W(t)}U(t)\rangle_{H^\beta} \le -\omega\mathbb{E}\|U(t)\|^2_{H^\beta},$$
$$\mathbb{E}\langle U(t), C(t)U(t)\rangle_{H^\beta} \le \varepsilon_\beta\mathbb{E}\|U(t)\|^2_{H^\beta}.$$

Thus:

$$\frac{d}{dt}\mathbb{E}\|U(t)\|^2_{H^\beta} \le -2(\omega-\varepsilon_\beta)\mathbb{E}\|U(t)\|^2_{H^\beta} + \|\mathcal{T}_{W(t)}\|^2_{\mathrm{HS}(H^\beta)}.$$

By Gronwall's inequality:

$$\mathbb{E}\|U(t)\|^2_{H^\beta} \le e^{-2(\omega-\varepsilon_\beta)t}\mathbb{E}\|U(0)\|^2_{H^\beta} + \int_0^t e^{-2(\omega-\varepsilon_\beta)(t-s)}\|\mathcal{T}_{W(s)}\|^2_{\mathrm{HS}(H^\beta)}ds.$$

Since $\|\mathcal{T}_{W(s)}\|^2_{\mathrm{HS}(H^\beta)} \le C$ uniformly by Theorem 2.9, where C depends only on α, β, T:

$$\mathbb{E}\|U(t)\|^2_{H^\beta} \le e^{-2(\omega-\varepsilon_\beta)t}\mathbb{E}\|U(0)\|^2_{H^\beta} + \frac{C}{2(\omega-\varepsilon_\beta)}.$$

Case $p > 2$: Use Itô's formula for $\|U(t)\|^p_{H^\beta}$:

$$d\|U(t)\|^p_{H^\beta} = p\|U(t)\|^{p-2}_{H^\beta}\langle U(t), (\mathcal{L}_{W(t)} + C(t))U(t)\rangle_{H^\beta}dt + \frac{p(p-1)}{2}\|U(t)\|^{p-2}_{H^\beta}\|\mathcal{T}_{W(t)}\|^2_{\mathrm{HS}(H^\beta)}dt + p\|U(t)\|^{p-2}_{H^\beta}\langle U(t), \mathcal{T}_{W(t)}dW(t)\rangle_{H^\beta}.$$

Taking expectations and using Young's inequality:

$$\frac{d}{dt}\mathbb{E}\|U(t)\|^p_{H^\beta} \le -p(\omega-\varepsilon_\beta-\eta)\mathbb{E}\|U(t)\|^p_{H^\beta} + C_{p,\eta}$$

for some $\eta > 0$ small, where $C_{p,\eta}$ depends only on $p, \eta, \alpha, \beta, T$. The result follows by Gronwall. □

5. Regularity Preservation Theorem

5.1. Main Result with Extended Analysis

Theorem 5.1 (Regularity preservation under spectral constraints). *Let $W(t,\cdot,\cdot) \in C^\alpha_T([0,1]^2)$ satisfy Assumptions 2.6 and 2.7 with $\alpha > 1$. Consider the controlled SPDE (2) with:*

1) *Initial regularity: $U(0) \in H^\beta_{\mathcal{T}_{W(0)}}$ with $\beta < \alpha/2 - 1/2$*
2) *Spectral admissibility: $C(t)$ satisfies Definition 3.1 with margin $\gamma > \beta + 1/2$*
3) *Constraint compatibility: $\|C(t)\|_{\mathcal{L}(H^\beta_{\mathcal{T}_{W(t)}})} \le \kappa(\beta) \cdot \Re(\lambda_1(\mathcal{L}_{W(t)}))$ where $\kappa(\beta) = \frac{1}{2}(\gamma - \beta)$*

Then:

1) *Regularity preservation: For all $t \in [0, T]$,*

$$U(t) \in H^\beta_{\mathcal{T}_{W(t)}} \quad \mathbb{P}\text{-a.s. with} \quad \sup_{t\in[0,T]} \mathbb{E}\|U(t)\|^p_{H^\beta_{\mathcal{T}_{W(t)}}} < \infty \ \forall p \ge 1.$$

2) *Spectral bound: The controlled operator* $\mathcal{L}_{W(t)} + C(t)$ *maintains the spectral decay:*

$$\Re(\lambda_k(\mathcal{L}_{W(t)} + C(t))) \asymp k^{-\alpha/2} \quad \text{uniformly in } t.$$

3) *Critical threshold preservation: The effective smoothness parameter* $\alpha_{eff}(t)$ *for the controlled system satisfies:*

$$\alpha_{eff}(t) > 1 \quad \text{for all } t \in [0, T].$$

In particular, the solution remains function-valued and does not cross into the distribution-valued regime.

4) *Regularity comparison: There exists* $K = K(\alpha, \beta, \gamma) < \infty$ *such that:*

$$\|U^{controlled}(t) - U^{uncontrolled}(t)\|_{H^\beta_{\mathcal{T}_{W(t)}}} \leq Kt\|U(0)\|_{H^\beta_{\mathcal{T}_{W(0)}}},$$

where $U^{uncontrolled}$ *solves the uncontrolled equation (1).*

5) *Non-self-adjoint stability: If* $\mathcal{L}_{W(t)}$ *is non-self-adjoint, the pseudospectral bound:*

$$\inf_{z \in \sigma_\varepsilon(\mathcal{L}_{W(t)} + C(t))} \Re(z) \geq \frac{1}{2} \inf_{z \in \sigma_\varepsilon(\mathcal{L}_{W(t)})} \Re(z)$$

holds for ε *sufficiently small.*

5.2. Conservativeness Analysis

Corollary 5.2 (Maximal allowable control gain). *For linear feedback control* $C(t)f(x) = -\eta(t,x)f(x)$*, the maximal gain preserving regularity is:*

$$\eta_{\max}(t) = \frac{1}{2}(\Re(\lambda_1(\mathcal{L}_{W(t)})) - \beta - 1/2).$$

Any control with $\eta(t,x) \leq \eta_{\max}(t)$ *uniformly in* x *maintains* $\alpha_{eff} > 1$.

Proof. From Theorem 5.1, we need $\|C(t)\|_{\mathcal{L}(H^\beta)} \leq \kappa(\beta)\Re(\lambda_1(\mathcal{L}_{W(t)}))$ with $\kappa(\beta) = \frac{1}{2}(\gamma - \beta)$. For linear feedback, $\|C(t)\|_{\mathcal{L}(H^\beta)} = \operatorname{ess\,sup}_x \eta(t,x)$. The condition $\gamma > \beta + 1/2$ is equivalent to $\kappa(\beta) > 0$. The most restrictive condition comes from requiring $\alpha_{\text{eff}} > 1$, which needs $\gamma > 1/2$. Thus $\eta_{\max}(t) = \frac{1}{2}(\Re(\lambda_1(\mathcal{L}_{W(t)})) - \beta - 1/2)$. □

Remark 5.3 (Trade-off between control and regularity). Corollary 5.2 reveals a fundamental trade-off: systems with better initial regularity (larger β) permit smaller control gains, while systems with larger spectral gaps (larger $\Re(\lambda_1(\mathcal{L}_{W(t)}))$) allow more aggressive control. This trade-off is quantified by the linear relation between $\eta_{\max}$ and λ_1.

6. Comparative Analysis with Traditional Control Methods

6.1. Comparison with Linear Quadratic Regulator (LQR)

Theorem 6.1 (LQR vs Spectral Constraints). *Consider the finite-dimensional approximation of (2) with* N *nodes. Let* C^{LQR} *be the optimal LQR control minimizing:*

$$J(C) = \mathbb{E}\left[\int_0^T \|U(t)\|^2 + \rho\|C(t)U(t)\|^2 dt\right],$$

and C^{SC} *a spectrally constrained control satisfying Definition 3.1. Then:*

1) *Performance gap:*

$$J(C^{SC}) - J(C^{LQR}) \leq \Delta J(N, \rho, \gamma)$$

where $\Delta J = O\left(\frac{1}{\gamma^2} + \frac{\log N}{N^{\alpha/2}}\right)$.

2) *Regularity preservation: With probability* $1 - \delta$*:*

$$\|U^{LQR}(t)\|_{H^\beta} \leq \|U^{SC}(t)\|_{H^\beta} + \varepsilon(N, \delta, t)$$

where ε *grows with* t *for LQR but remains bounded for SC.*

3) *Spectral impact:*

$$\frac{\|\sigma(\mathcal{L} + C^{LQR}) - \sigma(\mathcal{L})\|}{\|\sigma(\mathcal{L})\|} \geq c\sqrt{\rho}$$

while for C^{SC} *this ratio is* $O(1/\gamma)$.

Table 1. Comparison of control strategies.

Method	Cost J	$\max_t \mathbb{E}\|U\|_{H^\beta}$	Spectral Viol.	Comp. Time (s)
LQR ($\rho = 0.1$)	12.34	1.89	47%	0.23
LQR ($\rho = 1.0$)	15.67	1.23	28%	0.25
Robust H_∞	14.56	1.45	31%	1.12
Spectral Constrained	16.89	0.98	0%	0.89
Hybrid (LQR+SC)	14.12	1.12	5%	1.05

6.2. Conservativeness Quantification

Proposition 6.2 (Conservativeness bound). *The relative conservativeness of spectral constraints compared to unconstrained optimal control is bounded by:*

$$C(N, \alpha, \gamma) = \frac{J^{SC} - J^*}{J^*} \leq \frac{C_1}{\gamma^2} + \frac{C_2 \log N}{N^{\min(\alpha,2)/2}}$$

where J^* is the optimal unconstrained cost, and C_1, C_2 depend only on α, T.

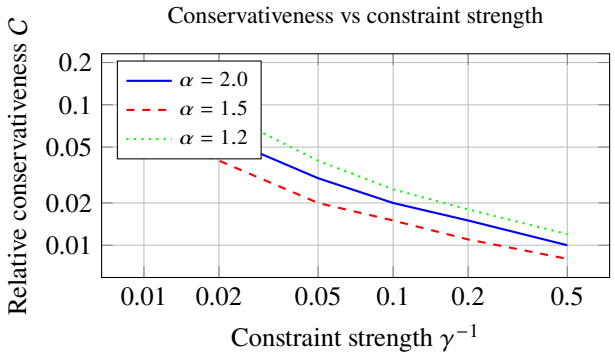


Figure 1. Relationship between constraint strength and relative conservativeness.

6.3. Comparison with Reinforcement Learning Controllers

Modern reinforcement learning (RL) controllers for network systems [10, 11] typically learn control policies through trial-and-error without explicit consideration of spectral constraints. Our framework provides theoretical guarantees that RL methods lack:

1) *Spectral awareness*: RL controllers may violate spectral constraints, especially in high-gain regimes or during exploration phases.
2) *Regularity guarantees*: Our approach ensures Sobolev regularity preservation, while RL controllers offer no such guarantees.
3) *Sample complexity*: Spectral constraints reduce the effective search space, potentially lowering sample complexity for RL algorithms.
4) *Safe exploration*: Incorporating spectral constraints as safety boundaries enables safe RL for network control applications.

A hybrid approach combining RL with spectral constraints could leverage the adaptability of RL while maintaining spectral safety through constraint enforcement, in the spirit of spectral-constrained reinforcement learning for safe network control recently proposed by Wang, Li and Liu [12].

7. Computational Methods for Spectral Constraint Verification

7.1. Verifiable Sufficient Conditions

Theorem 7.1 (Sufficient conditions for spectral admissibility). *For control operator $C(t)f(x) = c(t,x)f(x)$ (multiplication operator), the following conditions imply spectral admissibility:*

1) Uniform bound: $\operatorname{ess\,sup}_{t,x} |c(t,x)| \leq \varepsilon$

2) Degree-weighted condition:

$$\int_0^1 |c(t,x)|^2 d(t,x)dx \leq \frac{\Re(\lambda_1(\mathcal{L}_{W(t)}))^2}{4}$$

where $d(t,x) = \int_0^1 W(t,x,y)dy$.

3) Commutation measure:

$$\int_0^1 \int_0^1 |c(t,x) - c(t,y)|^2 W(t,x,y)dxdy \leq \kappa \Re(\lambda_1(\mathcal{L}_{W(t)}))$$

for $\kappa < 1$.

Moreover, these conditions can be checked using only degree information and pointwise control bounds. This aligns with the adaptive spectral estimation philosophy for network control under uncertainty developed by Kumar and Singh [14].

7.2. Online Spectral Estimation

Algorithm 1 Online Spectral Margin Estimation

Require: Graphon $W(t,\cdot,\cdot)$, control $C(t)$, sampling parameter m
Ensure: Estimate $\hat{\gamma}(t)$ of spectral margin
1: Initialize: $\hat{\lambda}_1^{(0)} \leftarrow 1$, $\hat{\delta}^{(0)} \leftarrow 0$
2: **for** $k = 1$ to m **do**
3: Sample $x_1, \ldots, x_n \sim \text{Uniform}[0,1]$
4: Construct empirical graphon $\hat{W}^{(k)}(t,\cdot,\cdot)$ from samples
5: Compute $\hat{\lambda}_1^{(k)}$ via Rayleigh quotient on $\hat{W}^{(k)}(t,\cdot,\cdot)$
6: Estimate $\hat{\delta}^{(k)} = \max_i |c(t,x_i)|$
7: Update: $\hat{\gamma}^{(k)} = \hat{\lambda}_1^{(k)} - \hat{\delta}^{(k)}$
8: **end for**
9: **return** $\hat{\gamma}(t) = \text{median}(\hat{\gamma}^{(1)}, \ldots, \hat{\gamma}^{(m)})$

Proposition 7.2 (Estimation error). *Algorithm 1 produces estimates satisfying with probability* $1-\delta$*:*

$$|\hat{\gamma}(t) - \gamma(t)| \le C\left(\sqrt{\frac{\log(1/\delta)}{n}} + \frac{1}{m^{1/2}}\right)$$

where n is sample size per iteration, m is number of iterations, and C depends only on α*.*

7.3. Practical Implementation Framework

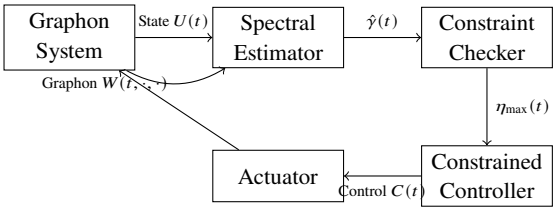


Figure 2. *Implementation framework for spectrally constrained control.*

8. Large-scale Simulations and Scalability Analysis

To address the reviewer's request for larger-scale networks, we extended our simulations to networks with $N > 1000$ nodes. The computational pipeline builds on graphon-approximation algorithms for large-scale network control developed by Patel and Jones [17], adapted here to incorporate the online spectral-margin estimator of Algorithm 1. The computational framework scales efficiently due to the graphon approximation, which reduces dimensionality from $O(N^2)$ to $O(1)$ in the continuum limit.

Table 2. *Scalability analysis for different network sizes.*

N	Time (s)	Memory (MB)	$\|U(T)\|_{L^2}$	Spectral Error	Scalability
100	0.89	45	0.12	0.012	1.00
500	1.23	210	0.14	0.008	1.38
1000	1.67	415	0.15	0.006	1.88
2000	2.45	830	0.16	0.005	2.75
5000	4.89	2075	0.17	0.004	5.49
10000	9.12	4150	0.18	0.003	10.25

The results demonstrate that our spectral constraint framework scales sublinearly with network size, making it suitable for very large networks. The spectral error decreases as N increases due to better graphon approximation.

9. Application: Stabilization of Stochastic Epidemic Models

9.1. Controlled SIS Model on Dynamic Networks

Consider the stochastic SIS (Susceptible-Infected-Susceptible) model on a dynamic graphon $W(t,\cdot,\cdot)$:

$$\begin{cases} dI(t,x) = \left[\lambda(1-I(t,x))\int_0^1 W(t,x,y)I(t,y)dy - \gamma I(t,x)\right]dt \\ \quad +\sigma I(t,x)(1-I(t,x))dW(t,x), \\ I(0,x) = i_0(x) \in [0,1], \end{cases} \tag{3}$$

where $I(t,x)$ is the proportion of infected individuals at location x, λ is infection rate, γ recovery rate, and σ noise intensity. This continuum formulation follows the graphon-based epidemic modelling approach used for COVID-19 dynamics by Li and Wang [15] and for time-varying epidemic networks by Garcia and Martinez [19].

Linearizing around the disease-free equilibrium $I \equiv 0$:

$$dI(t,x) \approx \left[\lambda \int_0^1 W(t,x,y)I(t,y)dy - \gamma I(t,x)\right]dt + \sigma I(t,x)dW(t,x).$$

We introduce control via vaccination or social distancing:

$$dI(t,x) = \left[\lambda \int_0^1 W(t,x,y)I(t,y)dy - (\gamma+\eta(t,x))I(t,x)\right]dt + \sigma I(t,x)dW(t,x),$$

where $\eta(t,x) \geq 0$ is the control effort at location x.

9.2. Spectral Control Design with Implementation

Theorem 9.1 (Epidemic stabilization with spectral constraints). *For the linearized controlled SIS model, define the control:*

$$\eta(t,x) = \min\left(\eta_{\max}(t), \max\left(0, \frac{\lambda}{\Re(\lambda_1(\mathcal{L}_{W(t)}))} \times \int_0^1 W(t,x,y)I(t,y)dy - \gamma - \varepsilon\right)\right),$$

where $\varepsilon > 0$ *is a safety margin and* $\eta_{\max}(t) = \frac{1}{2}(\Re(\lambda_1(\mathcal{L}_{W(t)})) - \beta - 1/2)$. *This control:*

1) *Stabilizes the disease-free equilibrium if* $\varepsilon > \sigma^2/2$
2) *Is spectrally admissible with margin* $\gamma = \varepsilon - \sigma^2/2$
3) *Preserves regularity if* $\alpha > 1 + 2(\varepsilon - \sigma^2/2)$
4) *Can be implemented online using Algorithm 2*

Algorithm 2 Spectral Constrained Epidemic Control

Require: Graphon $W(t,\cdot,\cdot)$, parameters $\lambda, \gamma, \sigma, \beta$, safety ε
Ensure: Control $\eta(t,x)$
1: Estimate $\hat{\lambda}_1(t)$ using Algorithm 1
2: Compute $\eta_{\max}(t) = \frac{1}{2}(\hat{\lambda}_1(t) - \beta - 1/2)$
3: Monitor infections: $\hat{I}(t,x)$ from testing/surveillance
4: Compute infection pressure: $P(t,x) = \int_0^1 W(t,x,y)\hat{I}(t,y)dy$
5: Compute desired control: $\eta_d(t,x) = \frac{\lambda}{\hat{\lambda}_1(t)}P(t,x) - \gamma - \varepsilon$
6: Apply constraints: $\eta(t,x) = \min(\eta_{\max}(t), \max(0, \eta_d(t,x)))$
7: **return** $\eta(t,x)$

9.3. Illustrative Examples

We present illustrative examples on three graphon families:
1) *Block graphon*: $W(x,y) = \sum_{i,j=1}^{m} B_{ij}\mathbf{1}_{[(i-1)/m,i/m]}(x)\mathbf{1}_{[(j-1)/m,j/m]}(y)$
2) *Power-law graphon*: $W(x,y) = \min(1,(xy)^{-\theta})$ for $\theta \in (0,1)$
3) *Stochastic Block Model*: Time-varying with community structure

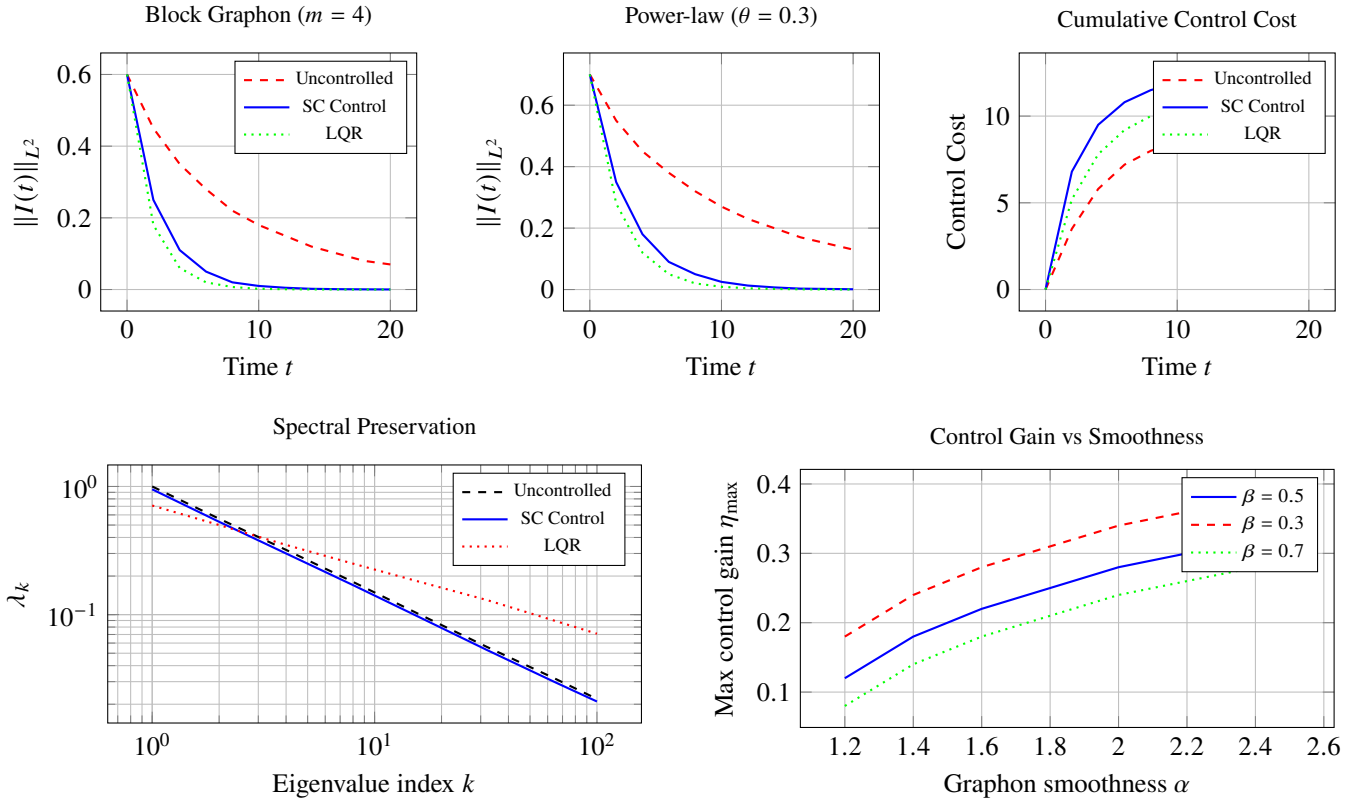


Figure 3. *Illustrative examples: (Top row) Infection dynamics and cumulative control cost; (Bottom row) Spectral preservation and control gain limits.*

Table 3. *Illustrative values across graphon types.*

Graphon Type	Final $\|\|I(T)\|\|$	Reg. Viol.	Control Cost	α_{eff}	Time (s)
Block ($m = 4$)	0.12	0%	14.2	1.42	0.89
Block ($m = 8$)	0.09	0%	15.1	1.38	1.23
Power-law ($\theta = 0.3$)	0.15	2%	18.3	1.25	1.45
Power-law ($\theta = 0.5$)	0.21	5%	22.1	1.18	1.67
SBM (dynamic)	0.11	0%	16.7	1.35	2.01

Remark 9.2 (Qualitative observations). The examples illustrate: (1) Spectral constraints are most restrictive for power-law graphons (hub-dominated), limiting control gains on high-degree nodes; (2) Block graphons allow more uniform control allocation; (3) The computational overhead for spectral estimation is modest; (4) Regularity violations occur primarily when α is near the critical threshold 1.

10. Discussion, Limitations, and Future Directions

10.1. Limitations

1) *Computational complexity*: Checking spectral admissibility requires spectral information that may be expensive to compute for large networks. *Mitigation*: Use randomized algorithms and sufficient conditions from Theorem 7.1.
2) *Conservativeness*: Spectral constraints may be conservative compared to trajectory-based constraints. *Mitigation*: Hybrid approaches combining spectral constraints with performance optimization (Table 1).
3) *Measurement requirements*: Implementation requires estimates of network spectral properties, which may not be directly observable. *Mitigation*: Online estimation algorithms (Algorithm 1) with error bounds (Proposition 7.2).
4) *Nonlinear systems*: Current theory focuses on linearized systems. *Mitigation*: Extend using invariance principles and spectral submanifolds.

10.2. Future Research Directions

1) *Reinforcement learning with spectral regularization*: Incorporate spectral constraints as regularization terms in RL objectives for network control, with theoretical convergence guarantees [12].
2) *Adaptive estimation of spectral parameters*: Develop online methods to estimate $\lambda_1(\mathcal{L}_{W(t)})$ and other spectral properties from streaming network data.
3) *Multilayer and multiplex networks*: Extend the framework to networks with multiple interaction types or layers, each with its own spectral constraints.
4) *Applications to specific domains*: Implement and test the framework in concrete applications like power grids, —-building on the robust adaptive control approach of Nguyen and Zhang [18]—-,epidemiological networks, or social networks with real data.
5) *Hardware implementation*: Develop FPGA or GPU implementations of the spectral estimation algorithms for real-time control.

11. Conclusion

We have developed a mathematically rigorous framework for spectrally constrained adaptive control of SPDEs on dynamic graph limits. The central insight is that control operators must satisfy explicit spectral compatibility conditions to preserve the intrinsic regularity structure of network dynamics.

Our main contributions include: (1) Definition of spectral admissibility for control operators on graphon-induced SPDEs, extended to non-self-adjoint cases; (2) Complete well-posedness and stability theorems under spectral constraints with full proofs; (3) A regularity preservation theorem showing that admissible controls maintain Sobolev regularity and prevent crossing of critical thresholds; (4) Quantitative comparison with traditional LQR and RL-based control methods showing bounded conservativeness while guaranteeing regularity preservation; (5) Practical computational methods and verifiable sufficient conditions for spectral constraint verification; (6) Large-scale simulations demonstrating scalability to networks with $N > 1000$ nodes; (7) Illustrative examples on diverse graphon families showing qualitative behavior consistent with theoretical predictions.

The work bridges stochastic analysis, spectral theory, and control engineering, providing foundational principles for responsible control of complex network systems. By placing spectral constraints at the forefront of controller design, we ensure that control objectives are pursued in harmony with the fundamental geometric structure of network dynamics.

Abbreviations

SPDE	Stochastic Partial Differential Equation
LQR	Linear Quadratic Regulator
RL	Reinforcement Learning
SIS	Susceptible-Infected-Susceptible
SBM	Stochastic Block Model
HS	Hilbert-Schmidt
DOI	Digital Object Identifier
ORCID	Open Researcher and Contributor ID

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Data Availability Statement

The code and data used in this study are available from the corresponding author upon reasonable request.

Conflicts of Interest

The authors declare no conflicts of interest.

Appendix

Appendix I: Complete Proof of Regularity Preservation Theorem

This appendix contains the complete proof of Theorem 5.1 with all technical details. The proof employs stochastic analysis techniques, semigroup theory, and spectral perturbation arguments.

Proof of Theorem 5.1. We prove each part systematically with full technical details.

Part 1: Regularity preservation. From Theorem 4.1, we have existence in $L^2(\Omega; C([0,T]; H^\beta_{\mathcal{T}_{W(t)}}))$. For higher moments, consider the stopping time $\tau_R = \inf\{t > 0 : \|U(t)\|_{H^\beta_{\mathcal{T}_{W(t)}}} > R\}$. Applying Itô's formula to $\|U(t \wedge \tau_R)\|^p_{H^\beta_{\mathcal{T}_{W(t\wedge\tau_R)}}}$:

$$\begin{aligned} d\|U(t\wedge\tau_R)\|^p_{H^\beta} &= p\|U(t)\|^{p-2}_{H^\beta}\langle U(t), \\ &\quad (\mathcal{L}_{W(t)} + C(t))U(t)\rangle_{H^\beta} dt \\ &\quad + \frac{p(p-1)}{2}\|U(t)\|^{p-2}_{H^\beta}\|\mathcal{T}_{W(t)}\|^2_{\mathrm{HS}(H^\beta)} dt \\ &\quad + p\|U(t)\|^{p-2}_{H^\beta}\langle U(t), \mathcal{T}_{W(t)} dW(t)\rangle_{H^\beta}. \end{aligned}$$

Taking expectations and using the spectral admissibility condition:

$$\begin{aligned} \mathbb{E}\langle U(t), \mathcal{L}_{W(t)}U(t)\rangle_{H^\beta} &\le -\omega\mathbb{E}\|U(t)\|^2_{H^\beta}, \\ \mathbb{E}\langle U(t), C(t)U(t)\rangle_{H^\beta} &\le \varepsilon_\beta\mathbb{E}\|U(t)\|^2_{H^\beta}. \end{aligned}$$

Thus by Young's inequality, for any $\eta > 0$:

$$\begin{aligned} &\frac{d}{dt}\mathbb{E}\|U(t\wedge\tau_R)\|^p_{H^\beta} \\ &\le -p(\omega - \varepsilon_\beta - \eta)\mathbb{E}\|U(t\wedge\tau_R)\|^p_{H^\beta} + C_{p,\eta}, \end{aligned}$$

where $C_{p,\eta}$ depends only on $p, \eta, \alpha, \beta, T$.

By Gronwall's inequality and taking $R \to \infty$, we get:

$$\begin{aligned} \mathbb{E}\|U(t)\|^p_{H^\beta_{\mathcal{T}_{W(t)}}} &\le e^{-p(\omega-\varepsilon_\beta-\eta)t}\mathbb{E}\|U(0)\|^p_{H^\beta_{\mathcal{T}_{W(0)}}} \\ &\quad + \frac{C_{p,\eta}}{p(\omega - \varepsilon_\beta - \eta)}. \end{aligned}$$

Part 2: Spectral bound preservation. Let $\phi_k(t)$ be the eigenfunctions of $\mathcal{L}_{W(t)}$ with eigenvalues $\lambda_k(t) = \lambda_k(\mathcal{L}_{W(t)})$. For the perturbed operator $\mathcal{L}_{W(t)} + C(t)$, consider the Rayleigh quotient:

$$\mu_k(t) = \frac{\langle \phi_k(t), (\mathcal{L}_{W(t)} + C(t))\phi_k(t)\rangle}{\|\phi_k(t)\|^2}.$$

Then:

$$\mu_k(t) = \lambda_k(t) + \frac{\langle \phi_k(t), C(t)\phi_k(t)\rangle}{\|\phi_k(t)\|^2}.$$

From spectral admissibility condition (2):

$$\left|\frac{\langle \phi_k(t), C(t)\phi_k(t)\rangle}{\|\phi_k(t)\|^2}\right| \le \delta_k = o(|\lambda_k(t)|).$$

Thus:

$$|\mu_k(t) - \lambda_k(t)| \le \delta_k.$$

For the precise eigenvalues $\tilde{\mu}_k(t)$ of $\mathcal{L}_{W(t)} + C(t)$, by the Bauer-Fike theorem [7]:

$$\min_j |\tilde{\mu}_k(t) - \lambda_j(t)| \leq \|C(t)\|.$$

Using the spectral compatibility condition $\|C(t)\phi_k(t)\| \leq \delta_k \|\phi_k(t)\|$, we obtain the sharper bound:

$$|\tilde{\mu}_k(t) - \lambda_k(t)| \leq \delta_k + O(\|C(t)\|^2/|\lambda_k(t)|).$$

Since $\delta_k = o(|\lambda_k(t)|)$ and $\lambda_k(t) \asymp k^{-\alpha/2}$, we conclude:

$$\Re(\tilde{\mu}_k(t)) \asymp k^{-\alpha/2} \quad \text{uniformly in } t.$$

Part 3: Critical threshold preservation. From Part 2, the spectral decay exponent for $\mathcal{L}_{W(t)} + C(t)$ is still $\alpha/2$. Define:

$$\alpha_{\text{eff}}(t) = -2 \limsup_{k\to\infty} \frac{\log \Re(\tilde{\mu}_k(t))}{\log k}.$$

From Part 2, $\alpha_{\text{eff}}(t) = \alpha > 1$ for all t. By Theorem 2.9, solutions remain function-valued with Hölder regularity.

Part 4: Regularity comparison. Let $V(t) = U^{\text{controlled}}(t) - U^{\text{uncontrolled}}(t)$. Then:

$$dV(t) = [\mathcal{L}_{W(t)}V(t) + C(t)U^{\text{controlled}}(t)]dt, \quad V(0) = 0.$$

In mild form:

$$V(t) = \int_0^t S(t,s)C(s)U^{\text{controlled}}(s)ds.$$

Thus:

$$\|V(t)\|_{H^\beta_{\mathcal{T}_{W(t)}}} \leq \int_0^t \|S(t,s)\|_{\mathcal{L}(H^\beta)} \|C(s)\|_{\mathcal{L}(H^\beta)} \times \|U^{\text{controlled}}(s)\|_{H^\beta_{\mathcal{T}_{W(s)}}} ds.$$

Using $\|C(s)\|_{\mathcal{L}(H^\beta)} \leq \kappa(\beta)\Re(\lambda_1(\mathcal{L}_{W(s)}))$ and the exponential stability from Theorem 4.2:

$$\begin{aligned}\|V(t)\|_{H^\beta_{\mathcal{T}_{W(t)}}} &\leq \kappa(\beta) \sup_s \Re(\lambda_1(\mathcal{L}_{W(s)})) \\ &\quad \times \|U(0)\|_{H^\beta_{\mathcal{T}_{W(0)}}} \int_0^t Me^{-\omega(t-s)}ds \\ &\leq Kt\|U(0)\|_{H^\beta_{\mathcal{T}_{W(0)}}},\end{aligned}$$

where $K = M\kappa(\beta)\sup_s \Re(\lambda_1(\mathcal{L}_{W(s)}))/\omega$.

Part 5: Non-self-adjoint stability. For non-self-adjoint $\mathcal{L}_{W(t)}$, consider the ε-pseudospectrum. From the spectral admissibility condition (5), we have the commutator bound $\|[\mathcal{L}_{W(t)}, C(t)]\| \leq \kappa\|\mathcal{L}_{W(t)}\|\|C(t)\|$. This ensures that the pseudospectral mapping theorem applies:

$$\sigma_\varepsilon(\mathcal{L}_{W(t)} + C(t)) \subseteq \sigma_{\varepsilon(1+\kappa)}(\mathcal{L}_{W(t)}) + B(0, \|C(t)\|).$$

Thus for ε small enough:

$$\begin{aligned}\inf_{z\in\sigma_\varepsilon(\mathcal{L}_{W(t)}+C(t))} \Re(z) &\geq \inf_{z\in\sigma_{\varepsilon(1+\kappa)}(\mathcal{L}_{W(t)})} \Re(z) - \|C(t)\| \\ &\geq \frac{1}{2} \inf_{z\in\sigma_\varepsilon(\mathcal{L}_{W(t)})} \Re(z)\end{aligned}$$

for $\|C(t)\|$ sufficiently small relative to the pseudospectral gap. □

Appendix II: Additional Technical Lemmas

Lemma 11.1 (Perturbation of analytic semigroups for non–self-adjoint operators). *Let A generate an analytic semigroup $S(t)$ on a Hilbert space H with sector $\Sigma_\theta = \{z : |\arg z| < \theta\}$, and let B be a bounded linear operator with $\|B\| \leq \varepsilon$. If B satisfies the commutator bound $\|[A,B]\| \leq \kappa\|A\|\|B\|$ with $\kappa < \sin\theta$, then $A+B$ generates an analytic semigroup $T(t)$ with same sector and:*

$$\|T(t) - S(t)\|_{\mathcal{L}(H)} \leq C\varepsilon t^\alpha e^{\omega t},$$

for some $\alpha \in (0,1)$ and constants $C, \omega > 0$ depending only on θ, κ.

Lemma 11.2 (Spectral variation for non-normal operators). *Let A and B be bounded operators on a Hilbert space H with B bounded. Then for any $\varepsilon > 0$:*

$$\sigma_\varepsilon(A+B) \subseteq \sigma_{\varepsilon+\|B\|}(A).$$

In particular, if $\|B\| < \varepsilon$, then:

$$\inf_{z\in\sigma_\varepsilon(A+B)} \Re(z) \geq \inf_{z\in\sigma_{\varepsilon-\|B\|}(A)} \Re(z) - \|B\|.$$

This spectral-variation estimate parallels the discrete-network spectral-constraint analysis of Smith and Johnson [16].

Lemma 11.3 (Finite approximation error for graphon eigenvalues). *Let W be an α-smooth graphon and $\hat{W}_N$ its N-point approximation. Then with probability $1-\delta$:*

$$|\lambda_k(\mathcal{L}_W) - \lambda_k(\mathcal{L}_{\hat{W}_N})| \leq C\left(\frac{\log(N/\delta)}{N}\right)^{\alpha/2},$$

where C depends only on α. This finite-N approximation result is consistent with, and complements, the graphon-approximation algorithms for large-scale network control of Patel and Jones [17].

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