International Journal of Applied Mathematics and Theoretical Physics
2026; 12(2): 65-69
http://www.sciencepublishinggroup.com/j/ijamtp
doi: 10.11648/j.ijamtp.20261202.11

Three Stage MERK Methods for Delay Differential Equations

Olaniyan Adegoke Stephen[1, *], Aleshinloye Zainab Labi[1], Omolara Fatimah Bakre[2], Akanbi Moses Adebowale[1]

[1]Department of Mathematics, Lagos State University, Lagos, Nigeria
[2]Department of Mathematics & Statistics, Federal College of Education(Technical), Akoka, Lagos, Nigeria

Email address:

gokayola@gmail.com (Olaniyan Adegoke Stephen), larabakre@gmail.com (Omolara Fatimah Bakre),
aleshinloyezainab6@gmail.com(Aleshinloye Zainab Labi), akanbima@gmail.com(Akanbi Moses Adebowale)
[*]Corresponding author

To cite this article:

Olaniyan Adegoke Stephen. (2026). Three Stage MERK Methods for Delay Differential Equations. *International Journal of Applied Mathematics and Theoretical Physics, 12*(2), 65-69. https://doi.org/10.11648/j.ijamtp.20261202.11

Received: 14 March 2026 ; **Accepted:** 26 March 2026 ; **Published:** 26 May 2026

Abstract: The application of Runge- Kutta (R-K) methods in obtaining numerical solutions to a wide class of ordinary and partial differential equations arising in various fields of applied sciences has been widely documented. Their efficiency, accuracy, and ease of implementation have made them one of the most popular techniques for solving initial value problems. This success has motivated their extension to more complex systems, particularly Delay Differential Equations (DDEs), which naturally arise in modeling real-life phenomena where time delays are inherent, such as in population dynamics, control systems, epidemiology, and engineering processes. In this paper, we present a numerical approach for solving DDEs by adapting a three-stage Multiderivative Explicit Runge-Kutta (MERK) method. The presence of delayed arguments in DDEs introduces additional computational challenges, especially in the evaluation of past states. To address this, Lagrange interpolation is employed to approximate the delayed terms. Furthermore, the stability properties of the proposed method are investigated through the derivation of the associated stability polynomials. These polynomials provide insight into the convergence behavior and robustness of the methods when applied to stiff and non-stiff delay systems. The performance of these methods are analyzed by solving DDEs, and comparisons are made with existing methods in the literature. The results demonstrate reliability of the three-stage MERK methods for solving DDEs.

Keywords: Delay Differential Equations, Runge-Kutta Methods , MERK Methods, Lagrange Interpolation, Stability Region

1. Introduction

The rapid advancement of DDEs across various fields of science and engineering have triggered an increasing interest in their numerical solutions in recent years. Several research works have been carried out to develop numerical methods to solve DDEs and even stiff DDEs. Among these numerical methods, the Runge-Kutta methods are particularly noteworthy due to their flexibility, efficiency, and accuracy. In Ismail and Read [5], a numerical solution of DDEs was obtained through the R-K method by using Hermite interpolation to estimate the delay terms. Ismail and Suleiman [6] examined the P-stability and Q-stability of singly diagonal implicit Runge-Kutta (RK) methods for DDEs. Olaniyan *et al.* [2, 9] obtained a numerical solution of DDEs with Heronian Implicit R-K methods by using Hermite interpolation to estimate the delay terms.

Due to the flexibility, efficiency and accuracy of RK method, researchers have been motivated to develop advanced variants that involved higher-order derivatives in each internal stage. Akanbi M. A. [1] derived multi-derivative explicit R-K methods involving up to second derivative. Goeken and Johnson [1] derived explicit R-K schemes of stages up to 4 with the first derivative. Wusu *et al.* [12] also derived a new 3-stage scheme by incorporating higher-order derivatives up to the second derivative. Then, Olaniyan *et al.* [10] went further to develop a 4-stage scheme also by incorporating higher-order derivative up to the second derivative. However, these variants of the R-K method were used to solve ODEs. Recently, the new variants adapted to deal with DDEs. In Kumar D. C. and

Pushpam E. K. [7], a two-stage multiderivative explicit R-K method was used to solve DDEs and Lagrange interpolation was used to estimate the delay term. In this project, we intend to improve on the work of Kumar D. C. and Pushpam E. K. [7] by extending the two-stage MERK methods to three-stage multiderivative explicit R-K methods for stiff DDEs.

2. Methodology

Three Stage MERK Method for DDEs

Consider the first-order delay differential equations (DDEs) given as:

$$y'(t) = f(t, y), \quad y(t_0) = y_0 \tag{1}$$

The 3-stage MERK method to solve equation (1) is of the form:

$$y_{n+1} = y_n + h\,\Phi(x_n, y_n, h) \tag{2}$$

where

$$\Phi(x_n, y_n, h) = d_1k_1 + d_2k_2 + d_3k_3$$

and

$$k_1 = f(y_n),$$

$$k_2 = f\left(y_n + hb_{21}k_1 + h^2C_{21}ff_y + \frac{h^3}{2}C_{22}(f^2f_yy + ff_y^2)\right),$$

$$k_3 = f\left(y_n + hb_{31}k_1 + hb_{32}k_2 + h^2C_{31}ff_y + \frac{h^3}{2}C_{32}(f^2f_{yy} + ff^2y')\right)$$

The 3-stage MERK method family according to Wusu A. S. [12] are provided in Table 1.

Table 1. *Examples of three-stage MERK methods.*

Parameter	**Method 1**	**Method 2**	**Method 3**	**3sMERK**
d_1	$\frac{169}{816}$	$\frac{1}{6}$	$\frac{1}{9}$	$\frac{1}{9}$
d_2	$\frac{5488}{7089}$	$\frac{1}{36}(16+\sqrt{6})$	$\frac{1}{36}(16-\sqrt{6})$	$\frac{1}{6}$
d_3	$\frac{125}{6672}$	$\frac{1}{36}(16-\sqrt{6})$	$\frac{1}{36}(16+\sqrt{6})$	$\frac{2}{3}$
b_{21}	$\frac{17}{28}$	$\frac{1}{10}(6-\sqrt{6})$	$\frac{1}{10}(6+\sqrt{6})$	1
b_{31}	$-\frac{91976}{10625}$	$\frac{3(-402-197\sqrt{6})}{1250}$	$\frac{3(-402+197\sqrt{6})}{1250}$	$\frac{3}{8}$
b_{32}	$\frac{108976}{10625}$	$\frac{2(489+179\sqrt{6})}{625}$	$\frac{2(489-179\sqrt{6})}{625}$	$\frac{1}{8}$
c_{21}	$\frac{289}{1568}$	$\frac{3}{100}(7-2\sqrt{6})$	$\frac{3}{100}(7+2\sqrt{6})$	$\frac{2}{5}$
c_{22}	$\frac{17}{196}$	$\frac{1}{500}(54-19\sqrt{6})$	$\frac{1}{500}(54+19\sqrt{6})$	$\frac{1}{5}$
c_{31}	$\frac{3092}{625}$	$\frac{3(-321-106\sqrt{6})}{2500}$	$\frac{3(-321+106\sqrt{6})}{2500}$	$\frac{1}{40}$
c_{32}	$\frac{1823}{625}$	$\frac{-342-37\sqrt{6}}{2500}$	$\frac{-342+37\sqrt{6}}{2500}$	$\frac{1}{40}$

The 3-stage MERK method for solving equation (1) can be adapted to solve (ODEs).

Consider the first-order DDEs with a constant delay τ:

$$\begin{aligned} y'(t) &= F(t, y(t), y(t-\tau(t, y(t))) \quad t > t_0, \\ y(t) &= \Phi(t) \end{aligned} \tag{3}$$

where $\Phi(t)$ represents the initial function.

The 3-stage MERK method adaptation to solve DDE is of the form:

$$y_{n+1} = y_n + h\,\Phi(x_n, y_n, h) \tag{4}$$

where

$$\Phi(x_n, y_n, h) = d_1k_1 + d_2k_2 + d_3k_3 = \sum_{r=1}^{3} d_rk_r$$

and

$$\begin{aligned}
k_1 &= hf(y_n, y(t_n + c_1 h - \tau)),\\
k_2 &= hf\left(y_n + hb_{21}k_1 + c_{21}h^2 ff_y + \frac{h^3}{2}c_{22}(f^2 f_{yy} + ff_y^2), y(t_n + c_2 h - \tau)\right),\\
k_3 &= hf\left(y_n + hb_{31}k_1 + hb_{32}k_2 + c_{31}h^2 ff_y + \frac{h^3}{2}c_{32}(f^2 f_{yy} + ff_y^2), y(t_n + c_3 h - \tau)\right)
\end{aligned}$$

where $y(t_n + c_1 h - \tau)$, $y(t_n + c_2 h - \tau)$ and $y(t_n + c_3 h - \tau)$ are the delayed terms. Different interpolation techniques have been applied to estimate delay terms. In this paper, the Lagrange interpolation method is employed.

3. Stability Analysis of 3 Stage MERK Method

The most frequently used test equation out of several concepts of stability of numerical methods when applied to DDEs is given as ;

$$\begin{aligned}
y'(t) &= \lambda y(t) + \mu y(t - \tau), \quad t \geq t_0\\
y(t) &= f(t), \quad -\tau \leq t \leq 0
\end{aligned} \tag{5}$$

where $\lambda, \mu \in \mathbb{C}$, $\tau > 0$, and f is continuous.

The test equation (3.5) will be used to investigate the stability of the adapted MERK method using the concept P-stability in Barewell [2]. In addition, Lagrange interpolation is employed to approximate the delay terms based on previously computed values of y and is given as

$$y(t_n + c_i h - \tau) = y(t_{n-m} + c_i h) = \sum_{l=r}^{s} L_l(c_i)\, y_{n-m+l} \tag{6}$$

having

$$L_l(c_i) = \prod_{j_1=r}^{s} \frac{c_i - j_1}{1 - j_1}, \quad \text{where } j_1 \neq 1,\ r, s > 0$$

and y_{n-m+l} is the computed value for $y(t_n - m + l)$.

Applying the 3-stage MERK method to the test equation with delay $\tau = 1$, we obtain:

$$\begin{aligned}
y_{n+1} &= y_n + h\sum_{r=1}^{3} d_r k_r\\
k_1 &= hf\left(y_n, \sum_{l=r}^{s} L_l(c_i)\, y_{n-m+l}\right),\\
k_2 &= hf\left(y_n + hb_{21}k_1 + c_{21}h^2 ff_y + \frac{h^3}{2}c_{22}(f^2 f_{yy} + ff_y^2), \sum_{l=r}^{s} L_l(c_i)\, y_{n-m+l}\right),\\
k_3 &= hf\left(y_n + hb_{31}k_1 + hb_{32}k_2 + c_{31}h^2 ff_y + \frac{h^3}{2}c_{32}(f^2 f_{yy} + ff_y^2), \sum_{l=r}^{s} L_l(c_i)\, y_{n-m+l}\right)
\end{aligned} \tag{7}$$

Where $u = (1, \ldots, 1)^T, k = (k^{(1)}, \ldots, k^{(q)})^T, d = (d_1, \ldots, d_q)^T$ and $L_l(c_l) = (l_l(c_1), \ldots, l_l(c_q))^T$. Considering f as in equations (3.5) and (3.7) given $n \geq m$, we have:

$$k = \lambda(y_n u + hAk) + \mu\left(\sum L_l(c_i) y_{n-m+l}\right) \tag{8}$$

$$y_{n+1} = y_n + hd^T k \tag{9}$$

From equation (8) , we obtained the following:

$$\begin{aligned}
k &= \lambda y_n u[I - \lambda hA]^{-1}\\
&\quad + \mu[I - \lambda hA]^{-1}\sum L_l(c_i) y_{n-m+l}
\end{aligned} \tag{10}$$

$$hk = \alpha y_n u\zeta + \beta\zeta\sum L_l(c) y_{n-m+l} \tag{11}$$

where $\alpha = \lambda h$, $\beta = \mu h$, $\zeta = [I - \lambda hA]^{-1}$ and I represents the identity matrix. Substituting (11) in (9) , we have:

$$y_{n+1} = y_n + \alpha d^T \zeta y_n u + \beta d^T \zeta \sum L_l(c_i) y_{n-m+l} \tag{12}$$

The stability polynomial will be in its standard form when $n - m + l = 0$ is substituted. The recurrence relation is considered stable if the roots of the stability polynomial given below satisfy the stability condition.

$$S(\alpha, \beta, \psi) = \det\left[\psi^{n+1} I - \psi^n X - \sum_{l=r}^{s} \psi^{1+l} Z_l\right] \tag{13}$$

where the root condition $|\psi| \leq 1$ hold.

We evaluated the function $y(t_n + c_i h - 1)$ using five-point interpolation to determine the stability region.

The corresponding stability polynomial for the method is given by:

$$S(\alpha, \beta, \psi) = \psi^{n+1} - (1 - \alpha d^T \zeta u)\psi^n - \beta d^t \zeta (L_{-1}(c) + L_0(c)\psi + L_1(c)\psi^2 + L_2(c)\psi^3 + L_3(c)\psi^4)$$

Hence, the stability polynomial for the 3-stage MERK method is given by:

$$\begin{aligned} S(\alpha,\beta,\psi) = \psi^{n+1} &- \left(1 + \alpha + \frac{\alpha^2}{2} + \frac{\alpha^3}{6} + \frac{\alpha^4}{24} + \frac{\alpha^5}{120}\right)\psi^n \\ &+ \left(\frac{9}{37}\beta + \frac{1}{3}\alpha\beta\right)\psi^4 - \left(\frac{4}{21}\beta\right)\psi^3 \\ &+ \left(\frac{2}{3}\right)\psi^2 + \left(\frac{2}{23}\beta + \frac{3}{4}\alpha\beta\right)\psi - \left(\frac{4}{25}\beta\right) \end{aligned}$$

The corresponding stability region is given in the figure below:

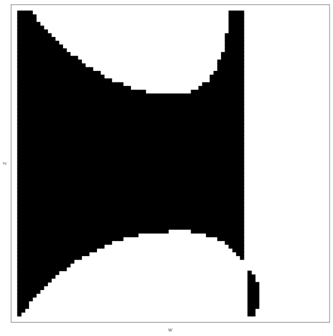

Figure 1. *Stability Region of 3sMERK.*

4. Numerical Experiment

The following are some of the problems tested. Problem 1 and 2 were taken from Kumar and Pushpam[7], while Problem 3 was taken from Ismail and Read [5].

4.1. Problem 1

Consider the nonlinear DDE given by:

$$y'(t) = 5y(t) + y(t-1), \;\; y(t) = 5$$

for $t \leq 0$ with the exact solution as $y(t) = 6e^{5t} - 1$

4.2. Problem 2

Consider the nonlinear DDE given by:

$$y'(t) = y^2(t) + y(t-1) - t^4 + t + 1, \;\; y(t) = t$$

for $t \leq 0$ with the exact solution as $y(t) = t^2$

4.3. Problem 3

Consider the nonlinear DDE given by:

$$y'(t) = \cos(t) y(y(t) - 2), \;\; y(t) = 1$$

for $t \leq 0$ with the exact solution as $y(t) = \sin(t) + 1$.

The numerical results were generated by solving the problems with the three-stage MERK methods and the classical Runge-Kutta method. Table 2 presents the absolute error results of Problems 1 to 3.

Table 2. *Absolute Error Results of Problems 1 to 3.*

Methods	Absolute Error of Problem 1	Absolute Error of Problem 2	Absolute Error of Problem 3
ERK	2.523e-6	1.331e-6	3.185e-5
3sMERK	1.289e-7	8.472e-8	5.233e-6
Method 1	3.0323-7	9.914e-7	5.883e-6
Method 2	8.386e-6	1.403e-6	1.386e-6
Method 3	7.341e-7	1.976e-6	8.112e-5

5. Conclusion

This study presented a numerical technique for solving delay differential equations using a three-stage Multiderivative Explici Runge-Kutta (MERK) method. The delay terms were approximated by using Lagrange interpolation. The stability polynomial of the method was obtained to study its stability characteristics. To evaluate the efficiency of the methods, some DDEs were solved and the results obtained were measured alongside the classical Runge-Kutta method. The comparison based on absolute error shows that the three-stage MERK methods provide accurate approximations to the solutions of DDEs. Hence, these methods prove to be a reliable numerical approach for solving DDEs .

ORCID

0000-0002-5784-5456 (Olaniyan Adegoke Stephen)
0000-0002-5260-3060 (Akanbi Moses Adebowale)
0000-0001-9122-6707 (Bakre Fatimah Omolara)
0009-0008-7245-1069 (Alehinloye Zainab Labi)

Abbreviations

DDE	Delay Differential Equation
R-K	Runge-Kutta
MERK	Multiderivative Explicit Runge-Kutta

Author Contributions

Olaniyan Adegoke Stephen: Methodology, Resources
Akanbi Moses Adebowale: Conceptualization
Omolara Fatimah Bakre: Data curation, Formal Analysis, Supervision
Aleshinloye Zainab Labi: Writing - original draft, Investigation

Conflicts of Interest

The authors declare no conflicts of interest.

References

[1] Akanbi M. A. (2005). A Family of 2-Stage Multiderivative Explicit Runge-Kutta Methods for Nonstiff Ordinary Differential Equations. *Proceedings of NMC on Recent Development in Numerical Methods for Solving Ordinary Differential Equations* , 6(2005), 87-100.

[2] Barewell V. K. (1975). Special Stability Problems for Functional Differential Equations. *BIT*, Vol. 15, pp: 130-135.

[3] Butcher J. C.(1964). On Runge-Kutta processes of high Order. *J. Austral. Math. Soc.*, Vol. 4, pp: 39–41.

[4] Goeken D. and Johnson O. (1999). Fifth-order Runge-Kutta with Higher Order Derivative Approximations. *Electronic Journal of Differential Equations*, pp. 1-9.

[5] Ismail F., Read A. A. (2002). Numerical Treatment of Delay Differential Equations by Runge-Kutta Method Using Hermite Interpolation. *Matematika*, No 2, 79-90.

[6] Ismail F. and Suleiman M. B.(1988). Embedded Singly Diagonally Implicit Runge-Kutta Method. *Int. J. Comput. Maths*, Vol. 66, pp: 325-341.

[7] Kumar D. C. and Pushpam E. K.(2020). Two Stage Multiderivative Explicit Runge-Kutta Methods for Delay Differential Equations. *Advances in Mathematics: Scientific Journal*, No. 3 1293-1299.

[8] Lambert J. D. (1973). *Computational Methods in Ordinary Differential Equations*, Jojn Wiley and Sons Inc., New York.

[9] Olaniyan A. S., Bakre O. F., Akanbi M. A. (2020). A 2-Stage Implicit Runge-Kutta method based on heronian mean for solving ordinary Differential Equations. *Pure and Applied Mathematics Journal*, Vol. 9(5).

[10] Olaniyan A. S., Akanbi M. A., Wusu A. S., Shonibare K. A.(2024). A Four-Stage Multiderivative Explicit Runge-Kutta Method for the Solution of First Order Ordinary Differential Equations*Annals of Mathematics and Computer Science*, 20, 72-81.

[11] Olaniyan A. S., Kazeem M. T., Aweda A. A. (2025). Numerical Solution of Delay Differential Equations with Heronian Implicit Runge-Kutta Method.*International Journal of Research and Innovation in Applied Science (IJRIAS)*, Vol. 9, Pages: 326-330.

[12] Wusu A. S., Akanbi M. A., Okunuga S. A. (2013). A Three-Stage Multiderivative Explicit Runge-Kutta Method.*American Journal of Computational Mathematics*, 3, 121-126.