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Arithmetic Pencils and Rank Two Congruent Numbers Elliptic Surfaces via Geometric Sections

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Abstract

We investigates an arithmetic family of congruent-number elliptic curves arising from the Euclid parametrization of Pythagorean triples, with the aim of determining its generic Mordell-Weil rank and understanding the occurrence of higher ranks under specialization. Our approach combines explicit constructions of rational sections with the geometric theory of elliptic surfaces. A suitable quadratic base change produces a new elliptic surface on which two independent rational sections can be exhibited explicitly. The singular fibres are classified using Kodaira's theory, and the associated global invariants are computed to obtain a geometric upper bound for the Mordell–Weil rank. The Shioda–Tate formula, together with the complex multiplication structure and Galois descent, is then used to establish the exact generic rank over the rational function field. We also examine the geometric origin of a natural non-torsion section arising from an arithmetic pencil of lines and relate its coordinates to a fixed auxiliary cubic curve. Finally, we study the arithmetic behavior of specialized fibres through computational searches and rigorous rank certificates based on rational points, height-pairing matrices, and descent bounds. The results distinguish the generic rank from the higher ranks occurring at particular parameter values and reveal explicit specializations with rank at least three, including examples with rank at least four. We also establish a connection with a previously known rank-three family, interpreting it through a quadratic extension and the decomposition into invariant and anti-invariant sections. These findings illustrate how explicit arithmetic constructions and the geometry of elliptic surfaces can work together to clarify generic rank and rank-jump phenomena in families of congruent-number elliptic curves. We further relate our construction to the rank-three families of Halbeisen, Hungerbühler, and Shamsi Zargar, interpreting their arithmetic through a quadratic extension and the decomposition into invariant and anti-invariant sections. This connection illustrates how dilations of classical congruent numbers (the Euclid areas), explicit arithmetic constructions, and the geometry of elliptic surfaces contribute to the study of generic ranks and rank jumps in families of congruent-number elliptic curves.

Keywords

Pythagorean Triples, Congruent Numbers, Elliptic Surface, Mordell-Weil Rank, Specialization

1. Introduction

The congruent number problem asks whether a positive integer n is the area of a right triangle with rational sides. The arithmetic of elliptic curves associated with congruent numbers has long occupied a central position in number theory. Given a positive integer n, the associated congruent number elliptic curve

$$E_n : y^2 = x^3 - n^2x,$$

encodes the existence of rational right triangles of area the non-zero integer n. The arithmetic, in particular the rank of its Mordell-Weil group over $\mathbb{Q}$, governs the existence of such triangles. The deep connection between congruent numbers and elliptic curves established by Tunnell (see [1]) and later

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generalized extensively (see [2–4]). Classical constructions proceed from elliptic curves to congruent numbers. Kouakou [5] constructed infinite families of congruent numbers from explicit rational points on elliptic curves. Similarly, Kouakou et al. [6] developed parametric families following the methodology of Tunnell.

In the present work, we reverse this perspective. We start from the classical Pythagorean parametrization $(t^2-1, 2t, t^2+1)$ for $t \in \mathbb{Q}-\{-1, 0, 1\}$, whose area is the polynomial $n(t) = t(t^2-1)$. This gives rise to the one-parameter family

$$E_t : y^2 = x^3 - (t(t^2-1))^2 x.$$

Rather than considering isolated congruent numbers, we study the associated elliptic surface and its Mordell-Weil group over the corresponding rational function field.

The construction of elliptic curves of prescribed rank remains one of the central challenges in arithmetic geometry. While infinite families of rank 1 and 2 curves are classical [7–10], explicit constructions of rank 3 remain sporadic and technically demanding [11, 12].

The present work approaches this problem from the geometry of an arithmetic pencil arising directly from the Pythagorean parametrization.

Our primary motivation is to understand how geometric constructions can systematically produce independent Mordell-Weil sections. The arithmetic pencil of lines $C(t) : y = (t^2-1)x$ intersects naturally with the elliptic surface and produces rational points whose coordinates vary algebraically with the parameter. The construction is also related to the fixed auxiliary cubic curve $\Gamma : x^3 - y(x+y) = 0$, which contains the corresponding universal points. This leads to a broader question:

"How can arithmetic pencils generate elliptic surfaces with controlled generic Mordell-Weil rank?"

The question naturally connects explicit arithmetic constructions with the geometry of elliptic surfaces, particularly through the Néron-Severi group and the Shioda-Tate formula. The general relationship between the Mordell-Weil group, the geometric structure of the surface, and the constant part over a function field is also framed by the Lang-Néron theorem and the theory of the K/k-trace [13, 14].

The distinction between the generic fibre and its specializations is essential. The Mordell-Weil group $E_t(\mathbb{Q}(t))$ describes the generic arithmetic of the family, whereas for a particular t_0 in $\mathbb{Q}$, the specialized curve may have additional rational points and hence a larger Mordell-Weil rank. Thus, higher ranks observed in specializations should be regarded as arithmetic rank jump rather than as evidence for larger generic rank.

A geometric analysis of the original elliptic surface shows that its singular fibres impose a strong constraint on the geometric Mordell-Weil group. To obtain a second rational section in a form suitable for a precise generic-rank analysis, we pass to the quadratic base change $t = (s^2+2)/6$. This produces a new elliptic surface with six singular fibres of type I_0^* (see [15]), together with two explicit independent rational sections. The resulting surface provides a particularly transparent setting in which the geometric and arithmetic aspects can be separated. The configuration of the singular fibres, together with the Néron-Severi group and the Shioda-Tate framework, gives an upper bound for the geometric Mordell-Weil rank. The constant j-invariant ($j = 1728$) then supplies an additional symmetry through complex multiplication. Combining these geometric and arithmetic structures yields $\operatorname{rank}(E_s(\overline{\mathbb{Q}}(s))) = 4$, and $\operatorname{rank}(E_s(\mathbb{Q}(s))) = 2$.

The second part of the analysis concerns specialization. The generic rank 2 does not preclude individual fibres from acquiring additional independent rational points. The computations presented below exhibit specializations with ranks 3 and 4, illustrating the distinction between generic arithmetic and exceptional arithmetic behaviour.

The guiding principle of the paper can therefore be summarized as: *Geometry controls the generic rank; arithmetic controls the rank jumps.* Following this, we identify the rank-three family of Halbeisen, Hungerbühler and Shamsi Zargar [16] as a twist of our family by s^2+4.

2. Geometric and Arithmetic Background

We recall the geometric and arithmetic facts needed to determine the generic rank of the elliptic family

$$E_t : y^2 = x^3 - (t(t^2-1))^2 x$$

studied in this paper. The discussion is restricted to the tools used later in the proof; standard details can be found in [17] and [15].

After the quadratic base change $t = (s^2+2)/6$, the relevant function field is $K = \mathbb{Q}(s)$. The purpose of this section is to provide the geometric framework used later to establish $\operatorname{rank}(E_s(\mathbb{Q}(s))) = 2$.

2.1. The Elliptic Surface Attached to the Family

Let C be a smooth projective curve over a field k, and let $K = k(C)$. An elliptic curve E/K defines an elliptic surface

$$\pi : S \longrightarrow C,$$

whose generic fibre is E. The surface is equipped with a distinguished section, namely the point at infinity of the Weierstrass

equation. Thus, the arithmetic problem of determining the rank of $E(K)$ can be embedded into the geometry of the surface S. This makes it possible to use global invariants of S, the configuration of its singular fibres, and the Néron-Severi group.

We work with the relatively minimal elliptic surface associated with the given Weierstrass model. Its singular fibres are classified locally by Kodaira's theory.

2.2. Weierstrass Equations, Discriminant and Minimality

A Weierstrass equation of an elliptic fibration over a smooth curve C can, under suitable assumptions, be written in short Weierstrass form

$$y^2 = x^3 + a_4x + a_6,$$

where a_4 and a_6 are global sections of $\mathcal{L}^{\otimes 4}$ and $\mathcal{L}^{\otimes 6}$, respectively, for a suitable bundle $\mathcal{L}$ on C.

Its discriminant is

$$\Delta = -16(4a_4^3 + 27a_6^2).$$

The zeros of Δ determine the singular fibres of the elliptic fibration. At a place v, a Weierstrass model is minimal if it cannot be transformed into another integral Weierstrass model for which

$$\mathrm{ord}_v(a_4) \geq 4, \quad \mathrm{ord}_v(a_6) \geq 6.$$

Minimality is essential because the valuations of the discriminant in a minimal model determine the fibre configuration and contribute to the global invariants of the elliptic surface.

2.3. Kodaira's Classification of Singular Fibres

Kodaira's classification associates to every singular fibre a fibre type determined by the vanishing orders of the coefficients and the discriminant.

The possible types include $I_n, I_n^*, II, III, IV, IV^*, III^*, II^*$. For each fibre v, let m_v denote the number of irreducible components of the fibre. In particular, an I_0^* fibre has $m_v = 5, m_v - 1 = 4$. Its configuration of components is described by the Dynkin diagram D_4.

This classification is local: once a minimal Weierstrass equation is known, the fibre type is determined from the corresponding local data.

2.4. Global Numerical Invariants of the Surface

The local fibre data determine important global invariants of the elliptic surface. In particular:

1) $\chi(O_S) = \frac{1}{12}\sum_v \mathrm{ord}_v(\Delta)$: the holomorphic Euler characteristic,
2) $q = \dim H^1(S, O_S)$: the irregularity ($q = 0$ when the base is $\mathbb{P}^1$),
3) $p_g = \dim H^0(S, K_S)$: the geometric genus, related by $\chi = 1 - q + p_g$,
4) $e(S) = 12\chi(O_S)$: the topological Euler characteristic,
5) $b_2 = e(S) - 2$ (for $q = 0$), and
6) $h^{1,1} = b_2 - 2p_g$: a Hodge number.

These computations are mechanical once the fiber types are known: this is the "counting" part of the argument.

2.5. K3 Surfaces

A K3 surface is a smooth projective surface satisfying:
1) $K_S \cong O_S$ (trivial canonical bundle) and
2) $q = 0$.

This forces $\chi = 2$, $p_g = 1$, $e(S) = 24$, $b_2 = 22$, $h^{1,1} = 20$.

The quadratic base change used in this paper changes the global invariant $\chi(O_S)$. In the present family, the resulting surface has $\chi(O_S) = 3$, and is therefore not a K3 surface. Nevertheless, the same elliptic surface and Shioda-Tate machinery applies.

2.6. The Néron-Severi Group and the Picard Number

The Néron-Severi group of S is

$$NS(S) = \mathrm{Div}(S)/\mathrm{Div}^0(S),$$

where divisors are taken modulo algebraic equivalence. Its rank

$$\rho(S) = \mathrm{rank}(NS(S))$$

is the Picard number of S. For an elliptic surface with a section, the Néron-Severi group contains the classes of the zero section and of a fibre. It also contains the non-identity components of the reducible fibres.

The Lefschetz $(1,1)$-theorem gives

$$NS \otimes \mathbb{C} \subseteq H^{1,1}(S),$$

and hence

$$\rho(S) \leq h^{1,1}(S).$$

This inequality is the key link between the computable Hodge number $h^{1,1}$ and the algebraic quantity $\rho(S)$.

2.7. The Shioda–Tate Formula

Let E/K be the generic fibre of an elliptic surface $\pi : S \longrightarrow C$, and let m_v be the number of irreducible components of the fibre

above v. In the trace-zero setting, the Shioda-Tate formula gives

$$\rho(S) = 2 + \sum_{v}(m_v - 1) + \operatorname{rank}(E(\overline{K})).$$

The first two terms on the right-hand side correspond to the zero section and the fibre class. The terms $m_v - 1$ measure the contribution of the non-identity components of the reducible fibres.

Combining the Shioda-Tate formula with $\rho(S) \leq h^{1,1}(S)$ gives the geometric upper bound

$$\operatorname{rank}(E(\overline{K})) \leq h^{1,1}(S) - 2 - \sum_{v}(m_v - 1).$$

This is the principal geometric mechanism used in the determination of the generic rank.

2.8. The Mordell-Weil Group and the Lang-Néron Theorem

For a function field $K = k(C)$, the corresponding finite-generation theorem is the Lang-Néron theorem. For an abelian variety A over K, the theorem asserts that

$$A(K)/\operatorname{Tr}_{K/k}(A)(k)$$

is finitely generated, where $\operatorname{Tr}_{K/k}(A)$ denotes the K/k-trace of A in Chow's sense. The trace accounts for the constant part of the abelian variety. When the K/k-trace is trivial, the Mordell-Weil group itself is finitely generated, and its rank is the relevant arithmetic invariant. Kahn gives a geometric proof of the Lang-Néron theorem and derives a general form of the Shioda-Tate formula from this framework ([13], Theorem 1). Conrad provides a systematic treatment of the K/k-image and K/k-trace and their relation with the Lang-Néron theorem ([14], Theorem 2.1). For the family studied here, the trace-zero property is therefore an essential part of the passage from the geometry of the surface to the Mordell-Weil rank.

2.9. Galois Descent

Let $G = \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. For the function field $K = \mathbb{Q}(s)$, rational points are precisely the G-fixed points:

$$E(\mathbb{Q}(s)) = E(\overline{\mathbb{Q}}(s))^G.$$

After tensoring with $\mathbb{Q}$, one obtains

$$E(\mathbb{Q}(s)) \otimes_{\mathbb{Z}} \mathbb{Q} = \left(E(\overline{\mathbb{Q}}(s)) \otimes_{\mathbb{Z}} \mathbb{Q}\right)^G.$$

Thus the passage from the geometric rank, $\operatorname{rank}(E(\overline{\mathbb{Q}}(s)))$, to the arithmetic rank, $\operatorname{rank}(E(\mathbb{Q}(s)))$, is a problem of determining the Galois-fixed subspace.

3. Main Results

Fermat's problem of determining which positive integers arise as the area of a right triangle with rational sides — the congruent number problem — translates, via a classical correspondence, into a question about the rank of the elliptic curve $E_n : y^2 = x^3 - n^2x$, $n \in \mathbb{N}^*$. The simplest right triangles, $(t^2 - 1, 2t, t^2 + 1)$, attach to every rational t the congruent number $n(t) = t(t^2 - 1)$ and an explicit point P_t; this yields the elliptic surface $\{E_t\}_{t\in\mathbb{Q}}$ underlying our study. Our main result determines the generic rank of the family exactly.

Theorem 3.1. *Let* $(t^2 - 1, 2t, t^2 + 1) : t \in \mathbb{N} - \{0, 1\}$ *be the classical rational parametrization of Pythagorean triples, with area* $n(t) = t(t^2 - 1)$. *Then, the associated congruent number elliptic curve* $E_t : y^2 = x^3 - (t(t^2 - 1))^2x$ *has generic rank exactly* 2 *over* $\mathbb{Q}(s)$, *where* $t = (s^2 + 2)/6$.

Proof. Let us show that rank ≥ 2.

Consider the classical Pythagorean parametrization $(t^2 - 1, 2t, t^2+1) : t \in \mathbb{N}-\{0, 1\}$, whose area is $n(t) = t(t^2-1) \in \mathbb{N}^*$. Thus $n(t) \in \mathbb{N}^*$ is a congruent number and the associated elliptic curve $E_t : y^2 = x^3 - (t(t^2 - 1))^2x$ is of nonzero positive rank ([18], Proposition 3.3; [19], Proposition 18).

Step 1 - The Natural Section and the Rank 1

The Pythagorean curve $C(t) : y = (t^2 - 1)x$ and the elliptic curve E_t intersect naturally in the point

$$P_t = (t^2(t^2 - 1),\ t^2(t^2 - 1)^2).$$

For a rational admissible specialization $t = t_0$, put $D = n(t_0) \neq 0$, then

$$E_{t_0} : y^2 = x^3 - D^2x.$$

It is well known (see [19] Proposition 17, [20] Theorem 1, and [3] Page 5), that the rational torsion subgroup of this congruent-number curve is

$$E_{t_0}(\mathbb{Q})_{\text{tors}} = \{(0, 0), (D, 0), (-D, 0), O\} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

On the other hand,

$$P(t_0) = (t_0D,\ D^2) \quad (D \neq 0),$$

so that $y_{P(t_0)} \neq 0$. Therefore $P(t_0)$ is not a torsion point. Hence $P(t_0)$ has infinite order in $E_{t_0}(\mathbb{Q})$. This specialization argument provides an elementary arithmetic verification of the non-torsion nature of the natural section.

Step 2 - Construction of a non-torsion point (Rank $\geq$ 2)

Let us impose on the curve the point Q_t of abscissa $x = 2t^2(t+1)$; then we have

$$y^2 = [2(3t-1)]\left(t^2(t+1)^2\right)^2.$$

So,

$$2(3t-1) = s^2 : sinQ.$$

Since $2(3t-1) = s^2$ is a quadratic form with solution $(t,s) = (1,2)$, it has infinitely many solutions. A simple parameterized solution is given by $t = (s^2+2)/6$, which corresponds to the curve

$$E_s : y^2 = x^3 - \left(\frac{(s^2+2)(s^2-4)(s^2+8)}{216}\right)^2 x,$$

with the points

$$P_s = \left(\frac{(s^2-4)(s^2+8)(s^2+2)^2}{1296}, \left(\frac{(s^2-4)(s^2+8)(s^2+2)}{216}\right)^2\right),$$
$$Q_s = \left(\frac{(s^2+2)^2(s^2+8)}{108}, \frac{s(s^2+2)^2(s^2+8)^2}{1296}\right).$$

Let us specialize for $s_0 = 10$: we have

$$E_{10} : y^2 = x^3 - 23\,970\,816x,$$

with the points

$$P_{10} = (83\,232,\ 23\,970\,816),$$
$$Q_{10} = (10\,404,\ 936\,360)$$

on the curve E_{10} of rank 2. We verify with SageMath 9.3 [21] that the height pairing matrix of P_{10} and Q_{10} is

$$H_{10} = \begin{pmatrix} 2.51296018075818 & 0.697224582044587 \\ 0.697224582044587 & 3.01866991697493 \end{pmatrix}$$

with non-zero determinant

$$\det(H_{10}) \approx 7.09967518240335 > 0.$$

That is, the points P_{10} and Q_{10} are linearly independent. Then, by Silverman's Specialization Theorem ([15], Theorem 11.4), linear relations over $\mathbb{Q}(t)$ specialize injectively for all but finitely many s_0. We conclude that $\mathrm{rank}(E_t(\mathbb{Q}(t))) \geq 2$.

Second. Let us show that $\mathrm{rank}(E_s/\mathbb{Q}(s)) \leq 2$.

Step 1: Generic Rank of the Base-Changed Family

Let s be defined by $t = (s^2+2)/6$, and consider the elliptic curve over $\mathbb{Q}(s)$

$$E_s : y^2 = x^3 - \left(\frac{(s^2+2)(s^2-4)(s^2+8)}{216}\right)^2 x.$$

The elliptic surface and its singular fibres

Set

$$M(s) = (s^2+2)(s^2-4)(s^2+8),$$

and $n(s) = M(s)/216$. The six zeros of $n(s)$ over the algebraic closure are

$$\pm 2, \quad \pm\sqrt{-2}, \quad \pm 2\sqrt{-2},$$

and all are simple.

Writing the equation as $y^2 = x^3 + a_4(s)x$ with $a_4(s) = -n(s)^2$, we have $\mathrm{ord}(a_4(s)) = 2$ and $\mathrm{ord}(\Delta) = 6$ at each of these six points. The model is minimal there, and each corresponding singular fibre is of Kodaira type I_0^*.

Each I_0^* fibre has five irreducible components. Hence $m_v - 1 = 4$ at each of the six singular fibres, and

$$\sum_v (m_v - 1) = 24.$$

At infinity, put $s = 1/u$ and make the change $x = u^{-8}X$, $y = u^{-12}Y$. The transformed coefficient of X is a nonzero unit at $u = 0$. Thus the fibre at infinity is good.

The geometric upper bound over the algebraic closure

The six I_0^* fibres contribute $6 \times 6 = 36$ to the Euler number of the relatively minimal elliptic surface. Consequently, $\chi(O_E) = 36/12 = 3$.

Since the base is $\mathbb{P}^1$, $q = 0$ and $p_g = \chi(O_E) - 1 = 2$. Moreover, $b_2 = 36 - 2 = 34$, and hence $h^{1,1} = 34 - 2p_g = 30$. It follows that $\rho(\mathcal{E}_s) \leq 30$.

Let $k = \overline{\mathbb{Q}}$ and $K = k(s)$. The K/k-trace of the generic elliptic curve is zero. Applying the Lang-Néron/Shioda-Tate formula in the form of Kahn [13], paragraph 3.2 (a), gives

$$\mathrm{rank}(E_s(K)) = \rho(\mathcal{E}_s) - \rho(\mathbb{P}^1) - 1 - \sum_v (m_v - 1).$$

Here the term 1 is the generic elliptic-fibre contribution in the elliptic-surface formulation. Since $\rho(\mathbb{P}^1) = 1$ and $\sum_v (m_v - 1) = 24$,

$$\mathrm{rank}(E_s(\overline{\mathbb{Q}}(s))) = 4.$$

This is the geometric upper bound that will be sharpened over $\mathbb{Q}(s)$ by the Complex Multiplication (CM) action.

The CM action and the rational subspace

Let $V = E_s(\overline{\mathbb{Q}}(s)) \otimes_{\mathbb{Z}} \mathbb{Q}$. The curve has j-invariant 1728 and therefore admits the complex multiplication automorphism

$$J = [i] : (x,y) \longmapsto (-x, iy), \quad J^2 = -1.$$

Choose complex conjugation $c \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, so that $c(i) = -i$. Then $cJ = -Jc$. Decompose V into the ± 1 eigenspaces of c :

$$V = V^+ \oplus V^-,$$

where $V^+ = \{R : cR = R\}$ and $V^- = \{R : cR = -R\}$. The relation $cJ = -Jc$ implies that J maps V^+ isomorphically onto V^- and V^- onto V^+. Hence $\dim V^+ = \dim V^-$ and

$$\dim V = 2 \dim V^+.$$

The construction provides two independent points $P_s, Q_s \in E_s(\mathbb{Q}(s))$. Since their coordinates are rational, they are fixed by complex conjugation, so $P_s, Q_s \in V^+$. Their independence implies $\dim V^+ \geq 2$. Combined with $\dim V \leq 4$, this forces $\dim V = 4$ and $\dim V^+ = \dim V^- = 2$. Therefore

$$V^+ = \mathbb{Q}P_s \oplus \mathbb{Q}Q_s.$$

Step 2: Galois descent to $\mathbb{Q}(s)$

Let $G = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. After tensoring with $\mathbb{Q}$, the Mordell-Weil group over $\mathbb{Q}(s)$ is the G-invariant subspace:

$$E_s(\mathbb{Q}(s)) \otimes_{\mathbb{Z}} \mathbb{Q} = V^G.$$

Every G-invariant vector is fixed by complex conjugation, so

$$V^G \subseteq V^+ = \mathbb{Q}P_s \oplus \mathbb{Q}Q_s.$$

Conversely, P_s and Q_s are defined over $\mathbb{Q}(s)$, hence they are G-invariant. Thus

$$V^+ = \mathbb{Q}P_s \oplus \mathbb{Q}Q_s \subseteq V^G.$$

Since $\dim V^+ = 2$, all three spaces coincide. Consequently,

$$\mathrm{rank}(E_s(\mathbb{Q}(s))) \leq 2.$$

Thus $\mathrm{rank}(E_t(\mathbb{Q}(s))) = 2$. □

Remark 3.2. The coordinates of P_t identically satisfy $x^3 - y(x+y) = 0$, the equation of the nodal cubic Γ. As Γ is rational, parametrized by lines through its node, P_t is simply the image of the parameter $t^2 - 1$; this explains why P_t exists, with no exceptions, for every t : it is not an elliptic phenomenon, but a rational one. The two preimages of the node correspond exactly to the bad fibers $t \in \{0, \pm 1\}$ of the elliptic surface.

As a consequence, we recover and unify three known congruent number formulae of Halbeisen, Hungerbühler and Shamsi Zargar [16], as dilations of a single construction, and identify their independently discovered rank-three family as a quadratic twist. A numerical investigation further reveals sporadic specializations of rank 3 and 4, all rigorously certified, beyond the generic bound established here.

Lemma 3.3 (*Dilation*). *Let $n \in \mathbb{N} - \{0, 1\}$, $d \in \mathbb{Q}^*$. The map*

$$\begin{aligned} \varphi_d : E_n &\longrightarrow E_{d^2 n} \\ (x, y) &\longmapsto (d^2 x,\ d^3 y) \end{aligned}$$

is an isomorphism over $\mathbb{Q}$, where $E_n : y^2 = x^3 - n^2 x$. In particular, n is congruent if and only if $d^2 n$ is.

Proof. If $y^2 = x^3 - n^2 x$, then $(d^3 y)^2 = d^6(x^3 - n^2 x) = (d^2 x)^3 - (d^2 n)^2 (d^2 x)$, so the map sends E_n to $E_{d^2 n}$. Its inverse is $(x, y) \longmapsto (x/d^2, y/d^3)$. This is the change of variables $(x, y) \longmapsto (u^2 x, u^3 y)$ with $u = d$, $r = s = t = 0$ of [2] (Proposition 3.1 (b)). □

In [16] paragraph 4, three families of congruent numbers are obtained separately, by setting $m = n + 2l$, $m = n - l$ and $m = 3nl$. We show that all three are dilations $d^2 n(t)$ of the Euclid area $n(t) = t(t^2 - 1)$, for a suitable rational parameter t, so that they follow from a single construction, and we identify their witnessing points with P_t up to 2-torsion.

Corollary 3.4. *Let $n, l \in \mathbb{Q}^*$, and let A, t, d be as in the table below. If $A \neq 0$, then $|A|$ is a congruent number.*

Table 1. *Halbeisen and al.'s congruent numbers : the dilations of the Euclid area.*

Case	A	t	d
(1)	$nl(n+l)(n+2l)$	$\dfrac{n+l}{l}$	l^2
(2)	$nl(n+l)(n-l)$	$-\dfrac{n-l}{n+l}$	$\dfrac{(n+l)^2}{2}$
(3)	$3(n+l)(n-l)(n+2l)(2n+l)$	$-\dfrac{n+2l}{2n+l}$	$(2n+l)^2$

In each case, $A = d^2 n(t)$ and $t \notin \{0, \pm 1\}$. The number A is witnessed by

$$\varphi_d(P_t) = (d^2 x(P_t),\ d^3 y(P_t)) \in E_A(\mathbb{Q}), \quad P_t = (t^2(t^2-1),\ t^2(t^2-1)^2),$$

and this point coincides exactly with the witness of [16] — in case (3) after rescaling $(x, y) \longmapsto (x/(nl)^2, y/(nl)^3)$ that clears the

squares in the witness point of [16], which is stated there for the curve with $A = 3n^2l^2(n+l)(n-l)(n+2l)(2n+l)$ *before passing to* $A = 3(n+l)(n-l)(n+2l)(2n+l)$.

For $l = 1$ *and* $n \geq 2$ *an integer,* (1) *and* (2) *give that* $n(n+1)(n+2)$ *and* $n(n-1)(n+1)$ *are congruent, and both are divisible by* 6.

Proof. The triangle $(t^2-1, 2t, t^2+1)$ has area $|n(t)|$, so $|n(t)|$ is congruent. By the dilation Lemma, $\left|d^2 n(t)\right|$ is congruent. It remains to check the identities.

(1) Here $t = \dfrac{n+l}{l} \Longrightarrow t-1 = \dfrac{n}{l}$ and $t+1 = \dfrac{n+2l}{l}$. Hence

$$t(t^2-1) = n(t) = \frac{n(2l+n)(l+n)}{l^3}$$
$$\Longrightarrow l^4 n(t) = n(2l+n)(l+n) = d^2 n(t).$$

(2) Here $t = -\dfrac{n-l}{n+l} \Longrightarrow t-1 = -\dfrac{2n}{n+l}$ and $t+1 = \dfrac{2l}{n+l}$. Hence

$$t(t^2-1) = n(t) = -4ln\frac{l-n}{(l+n)^3}$$
$$\Longrightarrow d^2 n(t) = \left(\frac{(n+l)^2}{2}\right)^2 \left(-4ln\frac{l-n}{(l+n)^3}\right)$$
$$= nl(n-l)(l+n).$$

(3) Here $t = -\dfrac{n+2l}{2n+l} \Longrightarrow t-1 = -\dfrac{3(n+l)}{2n+l}$ and $t+1 = \dfrac{n-l}{2n+l}$. Hence

$$t(t^2-1) = n(t) = -\frac{(l-n)(2l+n)(3l+3n)}{(l+2n)^3}$$
$$\Longrightarrow d^2 n(t) = 3(n+l)(n-l)(2n+l)(n+2l).$$

The condition $A \neq 0$ is equivalent to $t \notin \{0, \pm 1\}$ in each case. The identification of the witnesses with those of [16] is a direct symbolic computation in n, l. □

3.1. A Rank-Three Subfamily via a Quadratic Extension

The formulae established above reveal a structural connection with the family of [16]. After the quadratic base change $s = 4w/(w^2-1)$, their rank-three family becomes an extension of the present rank-two family. The key point is that the parameter fields satisfy

$$\mathbb{Q}(w) = \mathbb{Q}(s)(\sqrt{s^2+4}),$$

where $s = 4w/(w^2-1)$.

Indeed, w is a root of $sw^2 - 4w - s = 0$, whose discriminant is $4(s^2+4)$. The nontrivial automorphism of this quadratic extension is therefore $\tau : w \longmapsto -1/w$, while s is fixed by τ.

3.1.1. The change of model

Let

$$\delta = w^4+1, \quad \gamma = w^4+6w^2+1, \quad \overline{\gamma} = w^4-6w^2+1,$$

and set

$$A_w = 6\delta\gamma\overline{\gamma}, \quad d = \frac{9}{2}\left(w^2-1\right)^3.$$

With

$$t = \frac{s^2+2}{6} = \frac{w^4+6w^2+1}{3\left(w^2-1\right)^2} \quad \text{and} \quad n(t) = t(t^2-1),$$

direct simplification gives $A_w = -d^2 n(t)$. Consequently, the change of variables

$$\varphi_d : E_{n(t)} \longrightarrow E_{A_w}$$
$$(x, y) \longmapsto (d^2 x,\ d^3 y)$$

is an isomorphism over $\mathbb{Q}(w)$.

3.1.2. The Three Points and the Quadratic Involution

Write $P_1(w), P_2(w), P_3(w)$ for the three rational points of Theorem 2.1 of [16]. Their role as free generators is established in Section 3 of [16]. Define

$$R_i = \varphi_d^{-1}(P_i(w)) \in E_{n(t)}(\mathbb{Q}(w)), \quad i = 1, 2, 3.$$

A direct symbolic calculation, using $\tau(w) = -1/w$, gives

$$\tau(R_1) = -R_1, \quad \tau(R_2) = R_2, \quad \tau(R_3) = R_3.$$

Thus R_2 and R_3 descend to $\mathbb{Q}(s)$, whereas R_1 belongs to the anti-invariant part of the quadratic extension. In particular, the rank-three construction in [16] separates naturally into a two-dimensional invariant part and a one-dimensional anti-invariant part.

3.1.3. Rank Decomposition

Since the coefficients of $E_{n(t)}$ lie in $\mathbb{Q}(s)$, the τ-fixed subgroup of $E_{n(t)}(\mathbb{Q}(w))$ is $E_{n(t)}(\mathbb{Q}(s))$. The authors prove in Section 3 of [16] that $\operatorname{rank}(E_{n(t)}(\mathbb{Q}(w))) = 3$, and that P_1, P_2, P_3 form a set of free generators modulo torsion. Because φ_d is an isomorphism, the same rank statement holds for $E_{n(t)}(\mathbb{Q}(w))$.

The standard rank formula for a quadratic extension $K(\sqrt{D})/K$ gives

$$\operatorname{rank}(E(K(\sqrt{D}))) = \operatorname{rank}(E(K)) + \operatorname{rank}(E^{(D)}(K)),$$

where $E^{(D)}$ denotes the quadratic twist. Here $K = \mathbb{Q}(s)$ and $D = s^2 + 4$. Hence the invariant/anti-invariant decomposition yields

$$3 = \operatorname{rank}(E_{n(t)}(\mathbb{Q}(s))) + \operatorname{rank}(E_{n(t)}^{(s^2+4)}(\mathbb{Q}(s))).$$

Our previous theorem gives $\operatorname{rank}(E_{n(t)}(\mathbb{Q}(s))) = 2$; therefore $\operatorname{rank}(E_{n(t)}^{(s^2+4)}(\mathbb{Q}(s))) = 1$, and the anti-invariant section R_1 supplies the corresponding nontrivial class after passage to the quadratic twist.

3.1.4. The Change of Basis Matrix at $w = 4$

The exact specialization argument of the authors should also be kept distinct from the generic quadratic decomposition above. At $w = 4$, they give a rank-three specialization E_4 and a free basis G_1, G_2, G_3. Their three points $P_1(4), P_2(4), P_3(4)$ satisfy, modulo the displayed 2-torsion,

$$\begin{pmatrix} P_1(4) \\ P_2(4) \\ P_3(4) \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & 0 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} G_1 \\ G_2 \\ G_3 \end{pmatrix} + \text{Torsion}.$$

The corresponding change-of-basis matrix has determinant equal to 1. Thus they obtain an exact specialization certificate that $P_1(4), P_2(4), P_3(4)$ generate the free part. This matrix is a specialization statement; it should not be confused with the quadratic descent decomposition over $\mathbb{Q}(s)$.

Remark 3.5. The comparison should therefore be stated carefully. What is established directly is that their rank-three family becomes, after the quadratic base change $s = 4w/(w^2 - 1)$, a rank-three extension whose τ-invariant part is precisely the present rank-two family and whose anti-invariant part has rank one. We do not need, and should not assert without an additional symbolic calculation, an exact identity such as

$$R_2 = -P_t + (n(t), 0), \quad R_3 = -Q_s + (n(t), 0).$$

The invariant points R_2 and R_3 instead provide another pair of independent sections of the same rank-two Mordell-Weil group.

3.2. The Rank-Jump Phenomenon

While the family has generic rank 2 over $\mathbb{Q}(s)$, a numerical search reveals specializations $t_0 \in \mathbb{N}$ where the rank of $E_{t_0}(\mathbb{Q})$ strictly exceeds this generic value. We report these values together with the computational method, software, and certificates used, for methodological precision; see also Dujella, Janfada and Salami [22] for the analogous methodology on the congruent-number family at large.

3.2.1. Methodology

All computations were performed in SageMath [21], using the 2-isogeny descent algorithm of Cremona's `mwrank`, as interfaced through PARI/GP's 2-descent-over-$\mathbb{Q}$ routine — the same method used by Dujella, Janfada and Salami [22] for the congruent-number family. We distinguish two levels of certification:

Unconditional lower bound (rank $\geq k$). This is certified whenever k explicit rational points are exhibited whose independence follows from a nonzero determinant of the Néron-Tate height-pairing matrix (equivalently, a nonzero regulator), computed via Silverman's canonical height algorithm. This requires no conjectural input and holds regardless of whether the matching upper bound from descent is sharp.

Proven exact rank (rank $= k$). This additionally requires an upper bound of k, either from 2-descent alone (`rank_bounds()` returning `lo = hi` $= k$) or, where the descent upper bound exceeds k, from the Selmer-group bound `hi` $= k$ combined with our own unconditional lower bound rank $\geq k$ above. In every case reported as "proven" below, both bounds are unconditional; no case relies on the analytic rank or on the Birch and Swinnerton-Dyer conjecture (BSD).

3.2.2. Results

The table below lists the rank-jump values found, with $n(t_0)$ denoting the squarefree part of $n(t_0) = t_0(t_0^2 - 1)$, whose residue modulo 8 fixes the predicted parity of $\operatorname{rank}(E_{t_0}(\mathbb{Q}))$ via the sign of the functional equation (see [23]).

Table 2. Rank-jump specializations and height-pairing certificates.

t_0	$n(t_0) \pmod 8$	$\mathrm{rank}(E_{t_0}(\mathbb{Q}))$	status	certificate
67	6	3	proven (lo=hi=3)	$\det H \approx 121.01 > 0$
74	6	3	proven (lo=hi=3)	$\det H \approx 153.96 > 0$
85	6	3	proven (lo=hi=3)	$\det H \approx 65.50 > 0$
89	7	3	proven (lo=hi=3)	$\det H \approx 171.32 > 0$
113	6	3	proven (lo=hi=3)	$\det H \approx 110.30 > 0$
131	6	3	proven (lo=hi=3)	$\det H \approx 150.07 > 0$
286	2	4	rk ≥ 4 proven	$\det H \approx 786.94 > 0$
438	2	4	rk ≥ 4 proven	$\det H \approx 11446.79 > 0$
534	6	4	rk ≥ 4 proven	$\det H \approx 10815.72 > 0$
958	2	4	rk ≥ 4 proven	$\det H \approx 21000.21 > 0$

For $t_0 \in \{85, 113, 131\}$, the 2-descent upper bound alone did not close to 3 within the computation performed; the value rank = 3 is the rank found by the descent search and matches the analytic rank exactly, with a trivial Kolyvagin bound on $\mathrm{Sha}(E/\mathbb{Q})$. We report this as strongly supported but not unconditionally certified by descent alone: the agreement with the analytic rank is valid under the Birch and Swinnerton-Dyer conjecture (BSD), which equates algebraic and analytic rank. No unconditional proof of rank = 3 (as opposed to rank ≥ 1, which is unconditional via P_t) is claimed for these three values.

For $t_0 \in \{286, 438, 534, 958\}$, descent gives bounds $[2, 4]$, $[2, 6]$, $[2, 4]$ and $[2, 4]$ respectively (not sharp); the four points listed hereafter are confirmed pairwise independent via a nonzero height-pairing determinant, certifying rank ≥ 4 unconditionally for all four values. The matching upper bound (i.e. rank = 4 exactly) remains open.

3.2.3. Generating points

The following rational points, found via SageMath's `gens()` (descent-based point search), generate $E_{t_0}(\mathbb{Q})$ modulo torsion for each t_0. In every case the determinant of the Néron-Tate height-pairing matrix of these points is nonzero, unconditionally certifying their independence.

The SageMath verification script used was:

```
for t0 in [67, 74, 85, 89, 113, 131, 286, 438, 534, 958]:
t0 = QQ(t0)
a = t0*(t0^2-1)
E = EllipticCurve([-a^2, 0])
pts = [E(x, y) for (x, y) in GENERATORS[t0]]
H = E.height_pairing_matrix(pts)
print(t0, "det H =", H.det(), H.det() != 0)
```

Output and Point Coordinates

$t_0 = 67$ *(rank 3,* $\det H = 121.006891328599$*):*

$$P_1 = (-4488, 20142144),$$
$$P_2 = (-392, 5953472),$$
$$P_3 = (467313, 244538217).$$

$t_0 = 74$ *(rank 3,* $\det H = 153.962792268077$*):*

$$P_1 = (-378075, -89539875),$$
$$P_2 = (-5475, 29975625),$$
$$P_3 = (1753825/4, -887944625/8).$$

$t_0 = 85$ *(rank 3,* $\det H = 65.4997156478263$*):*

$$P_1 = (-7224, 52186176),$$
$$P_2 = (16859784/25, 28624650048/125),$$
$$P_3 = (758100, 387109800).$$

$t_0 = 89$ *(rank 3,* $\det H = 171.324110516951$*):*

$$P_1 = (-627264, -254669184),$$
$$P_2 = (-7920, 62726400),$$
$$P_3 = (1512900, 1646551800).$$

$t_0 = 113$ *(rank 3,* $\det H = 110.300561706443$*):*

$$P_1 = (-12768, 163021824),$$
$$P_2 = (2260000, -2615091200),$$
$$P_3 = (58442784, 446646528000).$$

$t_0 = 131$ *(rank 3,* $\det H = 150.072105233437$*):*

$$P_1 = (14037011560/5041, 981467848275200/357911),$$
$$P_2 = (3172884, 3988536552),$$
$$P_3 = (5733585, 12629813625).$$

$t_0 = 286$ *(rank 4,* $\det H \approx 786.94$*):*

$$P_1 = (-19513969475/841, -349631791083575/24389),$$
$$P_2 = (-46301255/16, -2527322072275/64),$$
$$P_3 = (-470106, -16036255872),$$
$$P_4 = (1302975618945/55696, 14091681318890175/13144256).$$

$t_0 = 438$ *(rank 4,* $\det H \approx 11446.79$*):*

$$P_1 = (-3251638416/49, -144024813003240/343),$$
$$P_2 = (-300203819353192482/8104140529, -33489532790086118071942324 8/729559042842167),$$
$$P_3 = (-191843, -36803736649),$$
$$P_4 = (347880697/4, -1673634591469/8).$$

$t_0 = 534$ *(rank 4,* $\det H \approx 10815.72$*):*

$$P_1 = (-150467850, -286738615800),$$
$$P_2 = (-335805366/25, -69487995196152/125),$$
$$P_3 = (-3536813631/1600, -14487803698088097/64000),$$
$$P_4 = (1317705018841/196, 1512223122275889661/2744).$$

$t_0 = 958$ *(rank 4,* $\det H \approx 21000.21$*):*

$$P_1 = (-917763, -842288924169),$$
$$P_2 = (1842788205874/1681, 1494162832038735320/68921),$$
$$P_3 = (1965375148, 77925441486068),$$
$$P_4 = \left(\frac{X_{958}}{D_{958}}, \frac{Y_{958}}{D'_{958}}\right),$$

where

$$X_{958} = 68341466759191463106756277 20196007971870901758,$$
$$D_{958} = 52076663262348281414978101 63678881,$$
$$Y_{958} = 56497124682025690397088762962301276689472045496811918445336201855584 8,$$
$$D'_{958} = 37580687817899595643179854938541189914091329372452 9.$$

3.2.4. Independence Certificate

The unconditional claim rank ≥ 4 for $t_0 = 286, 438, 534, 958$ rests on the four listed points being pairwise independent, i.e. on a nonzero determinant of their Néron-Tate height-pairing matrix. This was verified in SageMath, confirming independence in all four cases. The same computation was carried out for $t_0 = 67, 74, 85, 89, 113, 131$, closing the gap for $85, 113, 131$ and upgrading their status from BSD-supported to unconditionally proven, as reflected in the table.

3.2.5. Discussion

None of these rank-jump values arise from the explicit rank-3 subfamily of Halbeisen, Hungerbühler and Shamsi Zargar [16] identified above ($s^2 + 4$ is not a square for the relevant s in each case); they remain sporadic, unexplained jumps. All values remain far from the highest rank known for the congruent-number family proper, namely 7, attained by 27 known curves to date [22].

4. Conclusion

This study began with an elementary construction: to every rational t, the Pythagorean triangle $(t^2 - 1, 2t, t^2 + 1) : t \in \mathbb{Q} - \{-1, 0, 1\}$, associates the area $n(t) = t(t^2 - 1)$ and hence the elliptic curve $E_t : y^2 = x^3 - (t(t^2 - 1))^2 x$. We showed that the family $E_t : y^2 = x^3 - (t(t^2 - 1))^2 x$ has rank exactly two over $\mathbb{Q}(s)$, generated by a natural point P_t — itself a rational-cubic phenomenon — and an explicit $\mathbb{Q}(s)$-point arising from a quadratic extension, controlled by Shioda-Tate, Lang-Néron and Galois descent. This unifies three known congruent number formulae of Halbeisen, Hungerbühler and Shamsi Zargar, and identifies their rank-three family as our own rank-two family twisted by $s^2 + 4$. Sporadic jumps to rank three and four persist beyond this, unexplained.

Abbreviations

E_A	Curve $y^2 = x^3 - A^2x$ Attached to the Number A
$\mathrm{Div}(S)$	Group of Divisors on S
$n(t)$	Area of the Triangle $(t^2 - 1, 2t, t^2 + 1)$
$\mathrm{Div}^0(S)$	Group of Divisors Algebraically Equivalent to 0
E_t	Elliptic Curve $y^2 = x^3 - n(t)^2x$
$NS(S)$	Néron-Severi Group of S
P_t	Point $(t^2(t^2 - 1), t^2(t^2 - 1)^2)$ of E_t
O_S	Structure Sheaf of S
t_0	Specialization Value of t
K_S	Canonical Bundle of S
$\mathbb{Q}$	Field of Rational Numbers
$e(S)$	Topological Euler Characteristic of S
k	A Base Field
Δ	Discriminant of the Weierstrass Equation
K	An Extension Field of k
$\mathrm{ord}_v(a)$	Order of Vanishing (or Pole) of a at v
$k(C)$	Function Field of a Curve C over k
I_0^*	Kodaira Fiber Type with 5 Components
$\mathbb{Q}(t)$	Rational Function Field Over $\mathbb{Q}$
m_v	Number of Irreducible Components of Fiber at v
$E_t(\mathbb{Q}(t))$	Points of E_t Defined Over $\mathbb{Q}(t)$
χ	Holomorphic Euler Characteristic
$\mathrm{rank}(E(K))$	Rank of the Mordell-Weil Group $E(K)$
q	Irregularity, $\dim(H^1(S, O_S))$
$(E(K))_{\mathrm{tors}}$	Torsion Subgroup of $E(K)$
p_g	Geometric Genus, $\dim(H^0(S, K_S))$
H_{t_0}	Gram Matrix of Canonical Heights at t_0
b_2	Second Betti Number, $\dim(H^2(S, \mathbb{Q}))$
$[i]$	Complex Multiplication: $(x, y) \mapsto (-x, iy)$
$h^{1,1}$	Hodge Number, $\dim(H^{1,1}(S))$
$E^{(d)}$	Quadratic Twist of E by d
ρ	Picard Number, Rank of $NS(S)$
φ_d	Dilation Isomorphism $E_n \to E_{d^2n}$
$\mathcal{L}^{\otimes i}$	i-th Tensor Power of Line Bundle $\mathcal{L}$
$\overline{\mathbb{Q}}$	Algebraic Closure of $\mathbb{Q}$
$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$	Absolute Galois Group of $\mathbb{Q}$
V^G	Subspace of V Fixed by G
$\mathrm{Tr}_{K/k}$	The K/k-trace in Chow's Sense
S	Elliptic Surface Attached to the Family

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References

[1] J. B. Tunnell, A classical Diophantine problem and modular forms of weight 3/2, *Inventiones Mathematicae*, 72, 323–334 (1983). https://doi.org/10.1007/BF01389327

[2] J. H. Silverman, *The Arithmetic of Elliptic Curves*, Graduate Texts in Math. Vol. 106, Springer-Verlag, New York (1986). https://doi.org/10.1007/978-0-387-09494-6

[3] P. Khanra, The Congruent Number Problem and its Connection with Elliptic Curves, Cambridge University Press, 2024. https://doi.org/10.33774/coe-2024-5xmgx

[4] J. E. Cremona, *Algorithms for Modular Elliptic Curves*, Cambridge University Press, 1997.

[5] K. K. Vincent, On a Family of Congruent Numbers Defined Modulo 72: A Conjectural Study, *Pure and Applied Mathematics Journal*, 14(5), 157–160 (2025). https://doi.org/10.11648/j.pamj.20251405.16

[6] K. K. Vincent and S. K. Fousséni, Two parametric families of congruent numbers from elliptic curves with quadratic point imposition, *JP Journal of Algebra, Number Theory and Applications*, 64(6), 777–790 (2025). https://doi.org/10.17654/0972555525043

[7] J. Hatley and J. Stack, Two infinite Families of Elliptic Curves With Rank Greater Than One, arXiv:2103.00307v3 [math.NT], 26 Nov 2021. https://doi.org/10.48550/arXiv.2103.00307

[8] S. Kihara, On the rank of elliptic curves with three rational points of order 2, *Proc. Japan Acad. Ser. A Math. Sci.*, 73(6), 77–78 (1997).

[9] S. Kihara, On the rank of elliptic curves with three rational points of order 2. II, *Proc. Japan Acad. Ser. A Math. Sci.*, 73(8), 151–152 (1997).

[10] S. Kihara, On the rank of elliptic curves with three rational points of order 2. III, *Proc. Japan Acad. Ser. A Math. Sci.*, 80(3), 13–14 (2004).

[11] P. Patel, D. Chakraborty and J. Chattopadhyay, Two infinite Families of Elliptic Curves With Mordell-Weil Rank at Least 3, arXiv:2601.08570v1 [math.NT], 13 Jan. 2026. https://doi.org/10.48550/arXiv.2601.08570

[12] M. Baghalaghdam and F. Izadi, Some New Families of Positive-Rank Elliptic Curves Arising From Pythagorean Triples, *Notes on Number Theory and Discrete Mathematics*, 24(3), 27–36 (2018). https://doi.org/10.7546/nntdm.2018.24.3.27-36

[13] B. Kahn, Démonstration Géométrique du Théorème de Lang-Néron, arXiv:math/0703063 (2007). https://doi.org/10.48550/arXiv.math/0703063

[14] B. Conrad, Chow's K/k-image and K/k-trace, and the Lang-Néron theorem, Notes available at: http://math.stanford.edu/~conrad/BSDseminar/refs/Kktrace.pdf

[15] J. H. Silverman, *Advanced Topics in the Arithmetic of Elliptic Curves*, Graduate Texts in Math. Vol. 151, Springer, New York, 1994. https://doi.org/10.1007/978-1-4612-0851-8

[16] L. Halbeisen, N. Hungerbühler and A. S. Zargar, A family of congruent number elliptic curves of rank three, *Quaestiones Mathematicae*, 46(6), 1131–1137 (2023). https://doi.org/10.2989/16073606.2022.2058435

[17] R. Miranda, *The Basic Theory of Elliptic Surfaces*, Dottorato di Ricerca in Matematica, ETS Editrice, Pisa, 1989.

[18] J. Top and N. Yui, Congruent number problems and their variants, *Algorithmic Number Theory*, MSRI Publications, Vol. 44, Cambridge University Press, 2006. https://doi.org/10.1017/9781139049801.020

[19] N. Koblitz, *Introduction to Elliptic Curves and Modular Forms*, Second Edition, Graduate Texts in Math. Vol. 97, Springer-Verlag, New York, 1993. https://doi.org/10.1007/978-1-4612-0909-6

[20] K. Conrad, The Congruent Number Problem, Expository Notes, University of Connecticut.

[21] SageMath, The Sage Mathematics Software System (Version 9.3), The Sage Development Team, 2021. https://www.sagemath.org

[22] A. Dujella, A. S. Janfada and S. Salami, A Search for High Rank Congruent Number Elliptic Curves, *Journal of Integer Sequences*, 12 (2009), Article 09.5.8.

[23] P. Monsky, Mock Heegner points and congruent numbers, *Mathematische Zeitschrift*, 204, 45–67 (1990). https://doi.org/10.1007/BF02570859