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Conventional Finite Volumes for Flow Problems in Porous Media Involving Discontinuous Permeability: Convergence Rate Analysis of Cellwise-constant and Linear-spline Solutions

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Abstract: This work develops mathematical aspects of Conventional Finite Volume schemes for flow problems in porous media governed by discontinuous absolute permeability. Focusing on incompressible one-phase flow problems in heterogeneous porous media, a particular attention is put on the homogenized absolute permeability involved in the discrete Darcy velocity over the "interaction zone" between two adjacent control volumes. The first key-step of our presentation consists in putting in place a discrete-function-space frame-work endowed with inner products and their associated norms. Then after adequate mathematical tools are deployed as projection and interpolation operators with their fundamental properties. A discrete version of the Poincaré-Friedrichs inequality is also established and used to get equivalent discrete norms. Interpolation Operators are used to define cellwise-constant and linear-spline approximate solutions. A discrete variational formulation of the finite volume problem is stated and the Lax-Milgram theorem applies (upon projection operator continuity) to show the well posedness of the discrete variational problem. A first order convergence in L^2-norm and in some discrete energy norm has been shown. Sufficient conditions to get higher order convergence rate in L^2-norm and in H_0^1-norm have been stated for linear-spline solutions.

Keywords: Conventional Finite Volumes, Incompressible Flows, Discrete Function Space Frame-Work, Cellwise-constant and Linear-spline Solutions, Rate Convergence

1. Introduction

Let Ω be a bounded open subset of $\mathbb{R}^d$, $d \in \{1, 2, 3\}$ being the space dimension. Note that Ω is not necessary convex (but connected and not empty) with a polygonal boundary denoted by Γ. Let us start with recalling that $H^1(\Omega)$ denotes the usual Sobolev space made up of $L^2(\Omega)-$functions with distributional partial derivatives in $L^2(\Omega)$ while $H_0^1(\Omega)$ is a closed subspace of $H^1(\Omega)$ gathering the functions v such that $\gamma_0(v) = 0$, with $\gamma_0(.)$ denoting the usual Trace operator (over the boundary Γ) defined from $H^1(\Omega)$ to $L^2(\Gamma)$. As usual $H^{\frac{1}{2}}(\Gamma)$ denotes $\gamma_0(H^1(\Omega))$ which is a dense subspace of $L^2(\Gamma)$.

We are interested in the finite volume approximation of the solution u to the following system:

$$- \, div \, [\lambda(x) grad \, u] = f \;\text{ in } \Omega \qquad (1)$$

$$u = 0 \qquad \text{on } \mathit{\Gamma} \qquad (2)$$

where u is the unknown function while λ and f are given functions such that:

$$f \in L^2(\Omega) \qquad (3)$$

and $\lambda(.)$ is a *piecewise-constant function* over Ω, so it verifies what follows:

$$\exists\lambda_{min},\ \lambda_{max} \in \mathbb{R}_+^* \text{ such that,}$$
$$\lambda_{min} \leq \lambda(x) \leq \lambda_{max} \text{ a.e. in } \Omega. \tag{4}$$

Before going into the finite volume approximation of the solution u to the system of equations (1)-(2) we should start with ensuring on existence and uniqueness of this solution thanks to the well-known Lax-Milgram theorem. The Lax-Milgram theory and more can be found for instance in [2, 3, 18]. It is a tool adapted for the analysis of weak (or variational) formulations of 2nd order elliptic problems. This is the way we can see that the system (1)-(2) possesses a unique weak (or variational) solution u in the sense that:

$$\begin{cases} u \in H_0^1(\Omega) \text{ such that :} \\ \mathcal{B}(u,v) = L(v) \qquad \forall v \in H_0^1(\Omega) \end{cases} \tag{5}$$

where we have set:

$$\begin{cases} \mathcal{B}(u,v) = \int_\Omega \lambda(x) grad\, u.grad\, v\, dx \\ L(v) = \int_\Omega f(x)v(x)\, dx \end{cases} \tag{6}$$

Terminology: The above system is named Weak (or Variational) Formulation of the elliptic problem (1)-(2).

Remark 1.1. Under conditions (3)-(4) above one can show that there is equivalence between (1)-(2) and (5)-(6).

2. Conventional Finite Volume Formulation of (1)-(2) in One-dimensional Space

Let us concentrate on the one-dimensional version of the model problem (1)-(2).

2.1. The Model Diffusion Problem and Its Weak Formulation

We are interested here in the following one-dimensional 2nd order elliptic problem :

$$\begin{cases} \text{Find a function } \varphi \text{ such that :} \\ -[\lambda(x)\varphi']'(x) = f(x) \quad in \quad I =]a\,,\, b[, \\ \qquad with \quad \varphi(a) = \varphi(b) = 0\,. \end{cases} \tag{7}$$

Note that the weak (or variational) formulation of the system (7) is the problem that consists to (see for instance [1, 2, 4, 5, 8] concerning this topic):

$$\begin{cases} \text{Find } u \in H^1(I), \text{ with } u(a) = u(b) = 0 \text{ such that :} \\ \int_I \lambda(x)u'(x)v'(x)dx \quad = \quad \int_I f(x)v(x)dx \quad \forall\, v \in H_0^1(I). \end{cases} \tag{8}$$

where $H^1(I)$ is the well-known Sobolev space made up of functions v from $L^2(I)$, with distributional first derivative of v, denoted here by v', belonging to $L^2(I)$. Since I is a bounded interval of $\mathbb{R}$, it is known that $H^1(I) \subset C^0(\overline{I})$, where $\overline{I} = [a, b]$ (see [2] for instance). So the relations $u(a) = u(b) = 0$ get sense. On the other hand, $H_0^1(I)$ denotes a subspace of $H^1(I)$, made up of v such that $v(a) = v(b) = 0$.
The variational formulation (8) is relevant for the discrete formulation of the model problem in terms of Finite Volume scheme, as will be seen later, (or in terms of Finite Element scheme: see for instance [12]).

2.2. Finite Volume Meshes and Relevant Discrete Function Spaces

Let us start with the presentation of gridding procedure which is the starting point of most numerical methods for Boundary- and Initial-Value Problems (including Generalized Equations of Finite Difference Methods introduced by [6] and developed for Computational Structural Mechanics models in [7, 20, 21] for instance).

2.2.1. Finite Volume Meshes

Let us consider a finite increasing sequence of points $\{a = x_0 < x_1 < x_2 < ... < x_N < x_{N+1} = b\}$ from $\overline{I} = [a, b]$. In one-dimensional space we have the following conventional Finite Volume Mesh.

Definition 2.1. (conventional finite volume mesh)

1) Let us set by definition:

$$\begin{cases} K_i = [x_i - h_i^-, x_i + h_i^+] \\ with\ h_i^- = x_i - x_{i-\frac{1}{2}}\ and\ h_i^+ = x_{i+\frac{1}{2}} - x_i \quad \forall 1 \leq i \leq N \end{cases} \tag{9}$$

where the real numbers $\{x_i\}_{i=0}^{N+1}$ and $\{x_{i+\frac{1}{2}}\}_{i=0}^{N}$ are given and are such that:

$$a = x_0 < x_{\frac{1}{2}} < x_1 < x_{\frac{3}{2}} < x_2 < ... <$$
$$... < x_{i-1} < x_{i-\frac{1}{2}} < x_i < x_{i+\frac{1}{2}} < x_{i+1} < ... <$$
$$... < x_{N-1} < x_{N-\frac{1}{2}} < x_N < x_{N+\frac{1}{2}} < x_{N+1} = b.$$

2) Let us set by definition:

$$\begin{cases} K_0 = [x_0 - h_0^-, x_0 + h_0^+] \\ K_{N+1} = [x_{N+1} - h_{N+1}^-, x_{N+1} + h_{N+1}^+] \end{cases} \tag{10}$$

with the convention:

$$h_0^- = h_{N+1}^+ = 0. \tag{11}$$

3) Let us set: $\mathcal{P} = \{K_i\}_{i=0}^{N+1}$, $\mathcal{C} = \{x_i\}_{i=0}^{N+1}$ and $\mathcal{N} = \{x_{i+\frac{1}{2}}\}_{i=0}^{N}$.

The family $\{\mathcal{P}, \mathcal{C}, \mathcal{N}\}$ defines on $\overline{I}$ what we name a finite volume mesh denoted by $\mathfrak{M}$. The elements of $\mathcal{P}$ are called the mesh *control-volumes* (or simply *control-volumes*), those of $\mathcal{C}$ are called the mesh (element) centroids and those of $\mathcal{N}$ are called the control-volume boundaries or mesh interfaces.

Note that each control-volume K_i is attached to one centroid x_i and vice-versa. In what follows h_i denotes the diameter of the control-volume K_i. In other words:

$$h_i = h_i^+ + h_i^-.$$

On the other hand let us define the size h of the mesh $\mathfrak{M}$ as:

Definition 2.2. We set:

$$h = \max_{0 \le i \le N+1} h_i.$$

The control-volumes K_0 and K_{N+1} have their "centroids" on their borders. For this reason these control-volumes are named "degenerate control-volumes".

Definition 2.3. (Cell-Centered and Vertex-Centered Finite Volumes)

4) The Cell-Centered Finite Volume is a Finite Volume Scheme for which the discrete unknowns are all located at $\{x_i\}_{i=1}^{N}$ i.e. at the "centroids".

5) The Vertex-Centered Finite Volume is a Finite Volume Scheme for which the discrete unknowns are all located at the boundaries of control-volumes i.e. at $\{x_{i+\frac{1}{2}}\}_{i=0}^{N}$

Remark 2.1. Note that depending on the boundary conditions discrete unknowns are to be also considered at x_0 or at x_{N+1}. More precisely, with Dirichlet boundary conditions there is no discrete unknown to be considered neither at x_0 nor at x_{N+1}. For Neumann boundary conditions i.e. only the fluxes $-[\lambda\varphi'](x_0)$ and $-[\lambda\varphi'](x_{N+1})$ at end-points are given, so discrete unknowns are to be considered at x_0 and at x_{N+1}. This is the reason why degenerate control-volumes are not involved in the local discrete balance equation for Dirichlet boundary conditions, but are involved for the Neumann boundary conditions. Note also that it is possible to solve a flow (or diffusion) problem with hybrid (or mixed) boundary conditions, that is, Dirichlet condition at one end-point and Neumann condition at the other.

2.2.2. Discrete Function Spaces for Finite Volume Reconstruction of Exact Solutions

Let us start with the following definition.

Definition 2.4. We set:

$$D_{i+\frac{1}{2}} = [x_i, x_{i+1}], \quad \text{for } i = 0, 1, 2, ..., N$$

Let us introduce a family of subintervals of $\overline{I}$, called "diamond mesh" and denoted by $\mathfrak{D}$. This is needed in what follows for defining the discrete derivative of discrete functions associated with the mesh $\mathfrak{M}$. So the mesh $\mathfrak{D}$ depends somewhat on the mesh $\mathfrak{M}$.

Definition 2.5. We set :

$$\mathfrak{D} = \{D_{i+\frac{1}{2}}\}_{i=0}^{N}.$$

Definition 2.6. For a given bounded interval T of $\mathbb{R}$ we set:

$$\mathbb{P}_0(T) = \{v : T \longrightarrow \mathbb{R} \;/\; \exists\, v_T \in \mathbb{R}\; s.t.\; v(x) = v_T \; in\; T\}. \tag{12}$$

where "s.t." stands for such that.

In our frame-work it is interesting to view the elements v of $\mathbb{P}_0(T)$ as functions defined in $T \backslash \{x_g^T, x_d^T\}$, where x_g^T and x_d^T are respectively left and right end-points of the interval T.

Now all the ingredients are gathered to define the finite-dimensional function spaces for the finite volume reconstruction of the exact solution to general 2nd order elliptic problems. In the following definition CFV means "Conventional Finite Volume(s)".

Definition 2.7. (Discrete function spaces for CFV approximations)

6)The discrete function space $\mathbb{S}^{\mathfrak{M},0}$ for the CFV approximation of the exact solution to general 2nd order elliptic problems is defined as:

$$\mathbb{S}^{\mathfrak{M},0} = \prod_{i=0}^{N+1} \mathbb{P}_0(K_i)$$

7) The discrete function set $\mathbb{V}^{\mathfrak{M},0}$ for the CFV approximation of the exact solution to general 2nd order elliptic problems with prescribed *Dirichlet boundary conditions* is the subset of $\mathbb{S}^{\mathfrak{M},0}$ defined as:

$$\mathbb{V}^{\mathfrak{M},0} = \left\{ v \in \mathbb{S}^{\mathfrak{M},0} \;\;/\;\; \begin{cases} v(x) = u_a & in\; K_0 \\ v(x) = u_b & in\; K_{N+1} \end{cases} \right\}$$

where u_a and u_b are prescribed values of v at the domain end-points. In the particular case where $u_a = 0$ and $u_b = 0$ the set $\mathbb{V}^{\mathfrak{M},0}$ is nothing than a subspace of $\mathbb{S}^{\mathfrak{M},0}$, denoted by $\mathbb{S}_0^{\mathfrak{M},0}$ in what follows.

Remark 2.2. Note (or recall) that :

1) The function space $\mathbb{S}^{\mathfrak{M},0}$ is a finite-dimensional space, with dimension equal to $(N + 2)$.

2) The function space $\mathbb{S}_0^{\mathfrak{M},0}$ is a finite-dimensional space, with dimension equal to N.

3) Let us denote by 1_K (with K generic name of control-volumes from the mesh $\mathfrak{M}$) the pseudo-characteristic function

defined *almost everywhere* in I as :

$$1_K(x) \;=\; \begin{cases} 1 & if \quad x \in Int(K) \\ 0 & if \quad x \in Ext(K) \end{cases} \tag{13}$$

where Int(K) and Ext(K) stand for Interior and Exterior of K respectively. The sets of pseudo-characteristic functions $\{1_{K_i}\}_{i=0}^{N+1}$ and $\{1_{K_i}\}_{i=1}^{N}$ are naturally canonical basis of $\mathbb{S}^{\mathfrak{M},0}$ and $\mathbb{S}_0^{\mathfrak{M},0}$ respectively.

In the sequel we will need to use some scalar products defined on the spaces $\mathbb{S}^{\mathfrak{M},0}$. Before defining these scalar products we should introduce the function space $\mathbb{S}^{\mathfrak{D},0}$ defined as :

Definition 2.8. (Space where are lying discrete gradients of discrete functions.)

Let us set: $\mathbb{S}^{\mathfrak{D},0} \;=\; \prod_{i=0}^{N} \mathbb{P}_0(D_{i+\frac{1}{2}})$ where we have set $D_{i+\frac{1}{2}} \;=\; [x_i\,,\,x_{i+1}]$. Define on $\mathbb{S}^{\mathfrak{D},0}$ the following scalar product:

$$(\xi^{\mathfrak{D}}, \zeta^{\mathfrak{D}})_{L^2(I),\mathfrak{D}} \;=\; \sum_{i=0}^{N} h_{i+\frac{1}{2}} \xi_{i+\frac{1}{2}} \zeta_{i+\frac{1}{2}} \tag{14}$$

where we have set

$$h_{i+\frac{1}{2}} \;=\; h_i^+ + h_{i+1}^- \equiv \text{ length of } D_{i+\frac{1}{2}}. \tag{15}$$

A discrete derivation operator $\nabla^{\mathfrak{D}}$ is defined from the space $\mathbb{S}^{\mathfrak{M},0}$ to the space $\mathbb{S}^{\mathfrak{D},0}$ as follows:

Definition 2.9. (Discrete derivation operator)

$$\mathbb{S}^{\mathfrak{M},0} \;\ni\; v_h \longmapsto \nabla^{\mathfrak{D}} v_h = \sum_{i=0}^{N} [\nabla^{\mathfrak{D}} v_h]_{i+\frac{1}{2}} 1_{D_{i+\frac{1}{2}}} \;\in\; \mathbb{S}^{\mathfrak{D},0} \tag{16}$$

where we have set

$$[\nabla^{\mathfrak{D}} v_h]_{i+\frac{1}{2}} \;=\; \frac{1}{h_{i+\frac{1}{2}}} [v_{i+1} - v_i]. \tag{17}$$

It is high time to equip the space $\mathbb{S}^{\mathfrak{M},0}$ with the following scalar products. Indeed they play a key role in the sequel.

First scalar product:

$$\forall\, v_h, w_h \in \mathbb{S}^{\mathfrak{M},0} \qquad (v_h, w_h)_{L^2(I),\mathfrak{M}} \;=\; \sum_{i=0}^{N+1} h_i v_i w_i. \tag{18}$$

Recall that h_i is the one-dimensional Lebesgue measure of the control-volume K_i. This scalar product is associated with the following norm:

$$\forall\, v_h \in \mathbb{S}^{\mathfrak{M},0} \qquad \|\; v_h \;\|_{L^2(I),\mathfrak{M}} = [\sum_{i=0}^{N+1} h_i v_i^2]^{\frac{1}{2}}. \tag{19}$$

Whenever there is no risk of confusion, we will be using the following simplified notations, namely $(v_h, w_h)_{L^2,\mathfrak{M}}$ and $\|\; v_h \;\|_{L^2,\mathfrak{M}}$.

Second scalar product:

$$\forall\, v_h, w_h \in \mathbb{S}^{\mathfrak{M},0} \quad (v_h, w_h)_{H^1(I),\mathfrak{M}} = (v_h, w_h)_{L^2(I),\mathfrak{M}} \;+\; (\nabla^{\mathfrak{D}} v_h\,,\, \nabla^{\mathfrak{D}} w_h)_{L^2(I),\mathfrak{D}} \tag{20}$$

The associated norm denoted by $\|\;.\;\|_{H^1(I),\mathfrak{M}}$ and called discrete $H^1 - norm$ is defined as :

$$\forall\, v_h \in \mathbb{S}^{\mathfrak{M},0} \quad \|\; v_h \;\|^2_{H^1(I),\mathfrak{M}} = \|\; v_h \;\|^2_{L^2(I),\mathfrak{M}} \;+\; \|\; \nabla^{\mathfrak{D}} v_h \;\|^2_{L^2(I),\mathfrak{D}}\;. \tag{21}$$

We have the following important result.

Proposition 2.1. (Discrete version of Poincaré-Friedrichs inequality)

There exists a mesh independent nonnegative constant κ such that

$$\forall\, v_h \in \mathbb{S}_0^{\mathfrak{M},0} \qquad \|\; v_h \;\|_{L^2,\mathfrak{M}} \leq \;\kappa\; \|\; \nabla^{\mathfrak{D}} v_h \;\|_{L^2,\mathfrak{D}}$$

where

$$\|\; \nabla^{\mathfrak{D}} v_h \;\|^2_{L^2,\mathfrak{D}} = \sum_{i=0}^{N} \frac{1}{h_{i+\frac{1}{2}}} [v_{i+1} - v_i]^2.$$

Proof Let v_h be a function from $\mathbb{S}_0^{\mathfrak{M},0}$, that is, $v_h \in \prod_{i=0}^{N+1} \mathbb{P}_0(K_i)$, with (excluding discontinuous points of v_h, that is, precisely speaking, boundary-points of $K_i]_{i=0}^{N+1}$):

$$v_h(x) = v_i \quad in \quad Int[K_i] \qquad \forall\, 0 \leq i \leq N+1$$

where $Int[K_i]$ stands for interior of K_i and where $v_0 = v_{N+1} = 0$ since $v_h \in \mathbb{S}_0^{\mathfrak{M},0}$. So for any arbitrarily fixed $i \in \{0, 1, ..., N, N+1\}$, we have a.e. in K_i:

$$\mid v_h(x) \mid^2 = v_i^2 \leq (\sum_{k=0}^{i-1} \mid v_{k+1} - v_k \mid)^2 \leq (\sum_{k=0}^{N} \sqrt{h_{k+\frac{1}{2}}} \frac{\mid v_{k+1} - v_k \mid}{\sqrt{h_{k+\frac{1}{2}}}})^2$$

By Cauchy-Schawrz inequality we deduce that for all $i \in \{0, 1, ..., N, N+1\}$

$$\mid v_h(x) \mid^2 = v_i^2 \leq [b-a] \parallel v_h \parallel^2_{H_0^1,\mathfrak{M}} \quad a.e. \quad in \quad K_i$$

Integrating the two sides of the previous inequality in K_i and summing over $i \in \{0, 1, ..., N, N+1\}$ lead to the inequality we are looking for, with $\kappa = b - a$.

An immediate consequence of the previous result is what follows.

Proposition 2.2. (Discrete $H_0^1 - norm$)

(i) The following mapping

$$\mathbb{S}_0^{\mathfrak{M},0} \ni v_h \longmapsto \parallel v_h \parallel_{H_0^1,\mathfrak{M}} = [\sum_{i=0}^{N} \frac{1}{h_{i+\frac{1}{2}}} [v_{i+1} - v_i]^2]^{\frac{1}{2}} \equiv \parallel \nabla^{\mathfrak{D}} v_h \parallel_{L^2,\mathfrak{D}} \tag{22}$$

defines a norm over the space $\mathbb{S}_0^{\mathfrak{M},0}$. This norm is called below the discrete $H_0^1 - norm$.

(ii) Moreover on the space $\mathbb{S}_0^{\mathfrak{M},0}$ the discrete $H_0^1 - norm$ is equivalent to the discrete $H^1 - norm$ (defined above on the larger space $\mathbb{S}^{\mathfrak{M},0}$).

Remark 2.3. Note that the discrete $H_0^1 - norm$ is associated with the following inner product:

$$(v_h, w_h)_{\mathbb{S}_0^{\mathfrak{M},0}} = (\nabla^{\mathfrak{D}} v_h \,,\, \nabla^{\mathfrak{D}} w_h)_{L^2(I),\mathfrak{D}}. \tag{23}$$

The function space $\mathbb{S}_0^{\mathfrak{M},0}$ equipped with this inner product is a Hilbert space. Note also that in the sequel the bilinear form $(v_h, w_h)_{\mathbb{S}_0^{\mathfrak{M},0}}$ is denoted also by $(v_h, w_h)_{H_0^1,\mathfrak{M}}$.

Projection operator

Let us introduce the following projection operator denoted by $\Pi^{\mathfrak{M}}$, operating from the well-known Lebesgue space $L^2(I)$ to the discrete function space $S^{\mathfrak{M},0}$.

Definition 2.10. (Projection operator) Define the projection operator $\Pi^{\mathfrak{M}}$ as follows:

$$L^2(I) \ni v \longmapsto \Pi^{\mathfrak{M}} v \in \mathbb{S}^{\mathfrak{M},0} \;\text{ with }\; \Pi^{\mathfrak{M}} v = \sum_{i=0}^{N+1} [\Pi^{\mathfrak{M}} v]_i 1_{K_i}$$

where we have set : $[\Pi^{\mathfrak{M}} v]_i = \frac{1}{h_i} \int_{K_i} v(x)\, dx \;\; \forall\, 0 \leq i \leq N+1$.

Proposition 2.3. The projection operator $\Pi^{\mathfrak{M}}$ is a continuous linear mapping from the function space $L^2(I)$ equipped with its well-known standard-norm (defined by: $\parallel v \parallel_{L^2(I)} = [\int_I \mid v(x) \mid^2 dx]^{\frac{1}{2}}$) to the discrete function space $\mathbb{S}^{\mathfrak{M},0}$ endowed with the norm $\parallel \,.\, \parallel_{L^2(I),\mathfrak{M}}$ defined above by the relation (19).

Proof Let v be an arbitrarily chosen function from the space $L^2(I)$ equipped with its well-known standard-norm. The operator $\Pi^{\mathfrak{M}}$ transforms v into $\Pi^{\mathfrak{M}} v$ lying in $\mathbb{S}^{\mathfrak{M},0}$. By definition, we have

$$\parallel \Pi^{\mathfrak{M}} v \parallel^2_{L^2(I),\mathfrak{M}} \stackrel{def}{=} \sum_{i=0}^{N+1} h_i [\Pi^{\mathfrak{M}} v]_i^2 = \sum_{i=0}^{N+1} h_i (\frac{1}{h_i} \int_{K_i} v(x)\, dx)^2$$

Thanks to Cauchy-Schwarz's inequality we get:
$\parallel \Pi^{\mathfrak{M}} v \parallel^2_{L^2(I),\mathfrak{M}} \leq \parallel v \parallel^2_{L^2(I)}$. This inequality ensures that the projection operator $\Pi^{\mathfrak{M}}$ is continuous since it is obviously linear.

2.3. Conventional Finite Volume Formulation of the Problem (7)

Let us start with describing the general frame-work and introducing some useful definitions and notations.

2.3.1. General Frame-work

In most engineering problems (fluid flows in subsurface and heat conduction in multilayered materials for instance), the diffusion operator coefficient $\lambda(.)$ is supposed to be a piece-wise constant function for geophysical/physical reasons. We adopt that point of view in this work. Putting in place a relevant finite volume mesh is required as the first step towards any finite volume analysis for such mathematical models. In the preceding subsection we have exposed in details the procedure for obtaining such a finite volume mesh $\mathfrak{M} = \{\mathcal{P}, \mathcal{C}, \mathcal{N}\}$. Recall that $\mathcal{P}$ is the family of control-volumes $K_i = [x_i - h_i^-, x_i + h_i^+]$, $\mathcal{C}$ is the family of mesh element centroids $x_i]_{i=0}^{N+1}$ and $\mathcal{N}$ is the family of control-volume boundary points $x_{i+\frac{1}{2}}]_{i=0}^{N}$. Note that centroids $x_i]_{i=1}^{N}$ are the points where approximate values $\overline{\varphi}_i]_{i=0}^{N+1}$ of the exact solution φ are computed by means of a Finite Volume algorithm. In what follows we are going to expose the *main steps* for putting in place a Finite Volume scheme in view to address the problem (7).

General assumptions:

Let $\{I_s\}_{s\in S}$ be the subdivision of $\overline{I}$ associated with the set (assumed finite) of discontinuity points of the diffusion coefficient $\lambda(.)$. Let us assume that:

$$\begin{cases} 1)\text{The discontinuity points of } \lambda(.) \text{ are part of the set } \{x_{\frac{1}{2}}, x_{\frac{3}{2}}, x_{\frac{5}{2}}, ..., x_{N+\frac{1}{2}}\} \\ 2) \quad \forall s \in S, \quad \varphi\mid_{\overline{I}_s} \in C^2(\overline{I}_s) \text{ and we set: } \gamma = \max_{s\in S}[\max_{\overline{I}_s} \mid \varphi'' \mid]. \end{cases} \tag{24}$$

In the sequel we denote by λ_{i_s} the constant value of $\lambda(x)$ in the control-volume K_i included in the portion I_s from the subdivision of $\overline{I}$ due to the diffusion coefficient discontinuity. Whenever there is no risk of confusion λ_{i_s} is simply denoted by λ_i.

Definitions and notations useful for the sequel

Let us define the flow velocity (vector function with d components in d-dimensional space; recall that in this subsection $d = 1$) and the flux (scalar function by definition) at the point $x_{i+\frac{1}{2}}$ represented with $x^-_{i+\frac{1}{2}}$ or $x^+_{i+\frac{1}{2}}$ whether it is seen by an observer lying in $[x_i, x_{i+\frac{1}{2}}]$ or in $[x_{i+\frac{1}{2}}, x_{i+1}]$ respectively:

$$q(x^{\pm}_{i+\frac{1}{2}}, \varphi') = -[\lambda\varphi'](x^{\pm}_{i+\frac{1}{2}}) \tag{25}$$

is the flow velocity at the points $x^{\pm}_{i+\frac{1}{2}}$ and

$$F(x^{\pm}_{i+\frac{1}{2}}, \varphi') = q(x^{\pm}_{i+\frac{1}{2}}, \varphi') \cdot \nu(x^{\pm}_{i+\frac{1}{2}}) \tag{26}$$

is the flux at the points $x^{\pm}_{i+\frac{1}{2}}$, where $\nu(x^+_{i+\frac{1}{2}}) = -1$ and $\nu(x^-_{i+\frac{1}{2}}) = +1$ are the outward unit normal vector in one-dimensional space. The dot "$\cdot$" in the relation (26) represents the standard scalar product in $\mathbb{R}^d$; but here $d = 1$, so this standard scalar product is reduced to the simple multiplication operation in $\mathbb{R}$.

2.3.2. The Discrete Flux Function at Control-volume Boundaries

Step one: Pressure-Velocity reformulation) Rewrite the global flow equations (7) in the following equivalent form :

$$\begin{cases} q_x(x, \varphi') = f(x) \quad \text{(Balance equation in control-volumes) for } x \in K_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}], \quad \forall 1 \le i \le N \\ q(x, \varphi') = -[\lambda_i\, \varphi'(x)] \\ \text{(Diffusion law: Darcy's law in our model problem)} \end{cases} \tag{27}$$

$$\begin{cases} \varphi(x^+_{i+\frac{1}{2}}) = \varphi(x^-_{i+\frac{1}{2}}) \qquad \text{(Continuity principle for } \varphi) \ \forall 0 \le i \le N \\ F(x^+_{i+\frac{1}{2}}, \varphi') + F(x^-_{i+\frac{1}{2}}, \varphi') = 0 \\ \text{(Continuity principle of the flux)}, \end{cases} \tag{28}$$

with the following prescribed Dirichlet boundary conditions (see equation (7)):

$$\varphi(x_0) = \varphi(x_{N+1}) = 0. \tag{29}$$

where q_x is the derivative of the velocity with respect to the space variable.

Step two: Approximation of the fluxes involved in the balance equation over control-volumes.

Integrating in the control-volume K_i the two sides of the balance equation (see the system (27) above) leads to

$$F(x^-_{i+\frac{1}{2}}, \varphi') + F(x^+_{i-\frac{1}{2}}, \varphi') = h_i[\Pi^{\mathfrak{M}} f]_i \qquad \forall\, 1 \le i \le N. \tag{30}$$

where (according to Definition 2.10)

$$[\Pi^{\mathfrak{M}} f]_i = \frac{1}{h_i}\int_{K_i} f(x)\, dx \qquad \forall\, 1 \le i \le N. \tag{31}$$

At this stage it is easily seen that an accurate approximation of the fluxes is required for getting a Finite Volume scheme of higher quality. However never forget that this approximation accuracy is dependent on the mesh structure and the regularity of the solution.

Under the assumptions (24) the Taylor-Lagrange Theorem applies to φ restricted to $[x_i, x_{i+\frac{1}{2}}]$ (which is included in some $\overline{\Omega}_{\widehat{s}}$, with $\widehat{s} \in S$) and leads to

$$\varphi(x_i) = \varphi(x^-_{i+\frac{1}{2}}) - \frac{h_i^+}{1!}\varphi'(x^-_{i+\frac{1}{2}}) + \frac{(h_i^+)^2}{2!}\varphi''(\xi^-_{i+\frac{1}{2}}),$$

$$\text{with} \qquad x_i < \xi^-_{i+\frac{1}{2}} < x_{i+\frac{1}{2}}.$$

Then we deduce

$$F(x^-_{i+\frac{1}{2}}, \varphi') = \frac{\lambda_i}{h_i^+}[\varphi(x_i) - \varphi(x^-_{i+\frac{1}{2}})] + T(x^-_{i+\frac{1}{2}}, h, \varphi'') \quad (32)$$

where, in virtue of (24), we have (with h standing for the mesh size, denoted also by $size(\mathfrak{M})$ and defined in Definition 2.2 above):

$$|\, T(x^-_{i+\frac{1}{2}}, h, \varphi'') \,| \leq \gamma\frac{h}{2}max\{\lambda_s;\ s \in S\} \qquad \forall 0 \leq i \leq N. \quad (33)$$

Applying again Taylor-Lagrange Theorem to φ restricted to $[x_{i+\frac{1}{2}}, x_{i+1}]$ leads similarly to

$$\varphi(x_{i+1}) = \varphi(x^+_{i+\frac{1}{2}}) + \frac{h^-_{i+1}}{1!}\varphi'(x^+_{i+\frac{1}{2}}) + \frac{(h^-_{i+1})^2}{2!}\varphi''(\xi^+_{i+\frac{1}{2}}), \text{ with } x_{i+\frac{1}{2}} < \xi^+_{i+\frac{1}{2}} < x_{i+1}.$$

By analogy we deduce that

$$F(x^+_{i+\frac{1}{2}}, \varphi') = \frac{\lambda_{i+1}}{h^-_{i+1}}[\varphi(x_{i+1}) - \varphi(x^+_{i+\frac{1}{2}})] + T(x^+_{i+\frac{1}{2}}, h, \varphi'') \quad (34)$$

where, in virtue of General Assumptions (24), we have

$$|\, T(x^+_{i+\frac{1}{2}}, h, \varphi'') \,| \leq \gamma\frac{h}{2}max\{\lambda_s;\ s \in S\} \qquad \forall 0 \leq i \leq N. \quad (35)$$

Step three : What expression for the discrete flux function in view to get a conservative scheme ?

This is a decisive step. Our *objective in this step* is to find out a discrete flux function defined at the control-volume boundaries in such a way that the Flux Continuity Principle is respected. This is so far the only way to get a conservative scheme.

Summing side by side the flux relations (32) and (33), and accounting with the exact (or physical) flux continuity at the interface $x_{i+\frac{1}{2}}$ i.e. $[F(x^-_{i+\frac{1}{2}}, \varphi') + F(x^+_{i+\frac{1}{2}}, \varphi')] = 0$, in addition with $\varphi \in H^1_0(I) \subset C^0(\overline{I})$, lead to the following relation:

$$\frac{\lambda_{i+1}}{h^-_{i+1}}[\varphi(x_{i+1}) - \varphi(x_{i+\frac{1}{2}})] + \frac{\lambda_i}{h_i^+}[\varphi(x_i) - \varphi(x_{i+\frac{1}{2}})] + T(x_{i+\frac{1}{2}}, h, \varphi'') = 0 \quad (36)$$

where we have set: $T(x_{i+\frac{1}{2}}, h, \varphi'') = T(x^+_{i+\frac{1}{2}}, h, \varphi'') + T(x^-_{i+\frac{1}{2}}, h, \varphi'')$.

Note that discrete pressure continuity is imposed by setting: $\varphi_r \equiv \varphi(x_r^+) \overset{continuity}{=} \varphi(x_r^-)$, for $r \in \{\frac{1}{2}, 1, \frac{3}{2}, 2, ..., N+\frac{1}{2}, N+1\}$. Since the quantities of the form $\varphi_{i+\frac{1}{2}}$ are viewed as discrete unknowns of second class we look for expressing them as a linear combination of the quantities $\varphi_k]_{k=0}^{N+1}$ plus a term of the form $\mathcal{O}(h^2)$. Thanks to (36) we immediately obtain the following result:

Proposition 2.4. Assume that the conditions for the definition (24) are put in place and that there exist mesh independent nonnegative numbers ω_g, ω_d such that

$$\omega_g h \leq h_i^+, h_i^- \leq \omega_d h \qquad \forall\, 0 \leq i \leq N+1.$$

Then we have for all $i \in \{0, 1, ..., N\}$

$$\varphi_{i+\frac{1}{2}} = \frac{(\lambda_{i+1}/h^-_{i+1})\varphi_{i+1} + (\lambda_i/h_i^+)\varphi_i}{(\lambda_{i+1}/h^-_{i+1}) + (\lambda_i/h_i^+)} + \frac{T(x_{i+\frac{1}{2}}, h, \varphi'')}{(\lambda_{i+1}/h^-_{i+1}) + (\lambda_i/h_i^+)}. \quad (37)$$

with

$$| \frac{T(x_{i+\frac{1}{2}}, h, \varphi'')}{(\lambda_{i+1}/h^-_{i+1}) + (\lambda_i/h^+_i)} | \leq Ch^2 \tag{38}$$

where C is a mesh independent nonnegative number.

Let us go back to the flux expressions $F(x^-_{i+\frac{1}{2}}, \varphi')$ and $F(x^+_{i+\frac{1}{2}}, \varphi')$ (see equations (32) and (34)) with the intention of replacing there $\varphi_{i+\frac{1}{2}}$ with the expression in the RHS (Right-Hand Side) of equation (37). Setting

$$\alpha^+_i = \frac{\lambda_i}{h^+_i} \qquad and \qquad \alpha^-_{i+1} = \frac{\lambda_{i+1}}{h^-_{i+1}}$$

We easily obtain that

$$F(x^-_{i+\frac{1}{2}}, \varphi') = \frac{\alpha^+_i \alpha^-_{i+1}}{\alpha^+_i + \alpha^-_{i+1}}[\varphi_i - \varphi_{i+1}] + E(\varphi", x^-_{i+\frac{1}{2}}) \tag{39}$$

and

$$F(x^+_{i+\frac{1}{2}}, \varphi') = \frac{\alpha^+_i \alpha^-_{i+1}}{\alpha^+_i + \alpha^-_{i+1}}[\varphi_{i+1} - \varphi_i] + E(\varphi", x^+_{i+\frac{1}{2}}) \tag{40}$$

where

$$| E(\varphi", x^+_{i+\frac{1}{2}}) | \leq Ch \; and \; | E(\varphi", x^-_{i+\frac{1}{2}}) | \leq Ch \tag{41}$$

with C standing for diverse mesh-independent nonnegative numbers.

Remark 2.4. (Very important for error estimates) It follows from (39) and (40) that:

$$E(\varphi", x^+_{i+\frac{1}{2}}) + E(\varphi", x^-_{i+\frac{1}{2}}) = 0.$$

Our objective was to find out a discrete flux function (defined at the control-volume boundaries) that satisfies the Flux Continuity Principle. This purpose is achieved with the following definition of the discrete flux function inspired from relations (39) and (40).

Definition 2.11. Let us assume that the true solution φ to the diffusion problem (7) satisfies the required conditions (24). Let φ_h be the function from $\mathbb{S}^{\mathfrak{M},0}$ defined by its components $(\varphi(x_i) \equiv \varphi_i)_{i=0}^{N+1}$ in the canonical basis of $\mathbb{S}^{\mathfrak{M},0}$. Denote by $F_{D_{i+\frac{1}{2}}}(., \nabla^{\mathfrak{D}}\varphi_h)$ the vector made up of discrete fluxes defined on the two sides of the point $x_{i+\frac{1}{2}}$ as follows. For $i = 0, ..., N$ we set:

$$F_{D_{i+\frac{1}{2}}}(x^-_{i+\frac{1}{2}}, \nabla^{\mathfrak{D}}\varphi_h) = \frac{\lambda_i \lambda_{i+1}}{\lambda_i h^-_{i+1} + \lambda_{i+1} h^+_i}[\varphi_i - \varphi_{i+1}],$$

and

$$F_{D_{i+\frac{1}{2}}}(x^+_{i+\frac{1}{2}}, \nabla^{\mathfrak{D}}\varphi_h) = \frac{\lambda_i \lambda_{i+1}}{\lambda_i h^-_{i+1} + \lambda_{i+1} h^+_i}[\varphi_{i+1} - \varphi_i].$$

.

The above definition of the discrete flux function can be expressed in terms of discrete gradient $[\nabla^{\mathfrak{D}}\varphi_h]_{i+\frac{1}{2}}$ as follows:

Definition 2.12. The conditions required in the previous definition are conserved here. We could define the discrete flux function as follows. For $i = 0, ..., N$ we set:

$$F_{D_{i+\frac{1}{2}}}(x^-_{i+\frac{1}{2}}, \nabla^{\mathfrak{D}}\varphi_h) = -\lambda^\star_{D_{i+\frac{1}{2}}}[\nabla^{\mathfrak{D}}\varphi_h]_{i+\frac{1}{2}} \tag{42}$$

and

$$F_{D_{i+\frac{1}{2}}}(x^+_{i+\frac{1}{2}}, \nabla^{\mathfrak{D}}\varphi_h) = \lambda^\star_{D_{i+\frac{1}{2}}}[\nabla^{\mathfrak{D}}\varphi_h]_{i+\frac{1}{2}} \tag{43}$$

where we have set:

$$\lambda^\star_{D_{i+\frac{1}{2}}} = \frac{h_{i+\frac{1}{2}}\lambda_i\lambda_{i+1}}{\lambda_i h^-_{i+1} + \lambda_{i+1}h^+_i} \text{ and } [\nabla^{\mathfrak{D}}\varphi_h]_{i+\frac{1}{2}} = \frac{[\varphi_{i+1} - \varphi_i]}{h_{i+\frac{1}{2}}} \tag{44}$$

with

$$h_{i+\frac{1}{2}} = h^+_i + h^-_{i+1}.$$

Remark 2.5. The coefficient $\lambda^\star_{D_{i+\frac{1}{2}}}$ involved in the definition of the discrete flux function and given by the relation (44) is the harmonic mean value of the following set of diffusion coefficients $\{\lambda_i, \lambda_{i+1}\}$. This harmonic mean value, called "homogenized" or "equivalent" diffusion coefficient of the exchange zone $D_{i+\frac{1}{2}}$, is well-known by Reservoir Engineers. Indeed the determination of homogenized parameters for numerical simulation of large scale multiphase flows in fractured and/or heterogenous petroleum reservoirs is a challenging issue so far (see for instance: [8–11, 13, 14, 16, 17, 19]). We will be coming back on this topic for 2-D elliptic problems in a up-coming work.

2.3.3. Equations of the Conventional Finite Volume Scheme

Here our objective is to put in place the discrete balance equation in each control-volume and the finite volume formulation of (7).

It follows from relations (39)-(40) and Definition 2.11 that

$$F(x^{\varepsilon}_{i+\frac{1}{2}}, \varphi') \;=\; F_{D_{i+\frac{1}{2}}}(x^{\varepsilon}_{i+\frac{1}{2}}, \nabla^{\mathfrak{D}}\varphi_h) \;+\; E(\varphi", x^{\varepsilon}_{i+\frac{1}{2}}), \tag{45}$$

with $\varepsilon \in \{-;+\}$. Using (45) in the following equation of the local mass balance (i.e. mass balance in the control-volume K_i):

$$F(x^{-}_{i+\frac{1}{2}}, \varphi') + F(x^{+}_{i-\frac{1}{2}}, \varphi') = h_i[\Pi^{\mathfrak{M}} f]_i \; \forall 1 \leq i \leq N$$

yields a system of relations (linear with respect to $\varphi_k]_{k=0}^{N+1}$) that reads as:

$$\begin{aligned} F_{D_{i+\frac{1}{2}}}(x^{-}_{i+\frac{1}{2}}, \nabla^{\mathfrak{D}}\varphi_h) \;+\; F_{D_{i+\frac{1}{2}}}(x^{+}_{i-\frac{1}{2}}, \nabla^{\mathfrak{D}}\varphi_h) \quad &= \\ = h_i[\Pi^{\mathfrak{M}} f]_i - E(\varphi", x^{-}_{i+\frac{1}{2}}) - E(\varphi", x^{+}_{i-\frac{1}{2}}) \quad &\forall 1 \leq i \leq N \end{aligned} \tag{46}$$

where we have set:

$$\forall\, 0 \leq j \leq N$$

$$\begin{cases} F_{D_{j+\frac{1}{2}}}(x^{-}_{j+\frac{1}{2}}, \nabla^{\mathfrak{D}}\varphi_h) = \frac{\lambda_j \lambda_{j+1}}{\lambda_j h^{-}_{j+1} + \lambda_{j+1} h^{+}_{j}} [\varphi_j - \varphi_{j+1}] \\ F_{D_{j+\frac{1}{2}}}(x^{+}_{j+\frac{1}{2}}, \nabla^{\mathfrak{D}}\varphi_h) = \frac{\lambda_j \lambda_{j+1}}{\lambda_j h^{-}_{j+1} + \lambda_{j+1} h^{+}_{j}} [\varphi_{j+1} - \varphi_j]. \end{cases} \tag{47}$$

Definition 2.13. (Cellwise-constant approximation of φ)

Recall that φ denotes the exact solution to the problem (7). Let $(\overline{\varphi}_i)_{i=0}^{N+1}$ be the components (in the canonical basis) of a function $\overline{\varphi}_h$ from the space $\mathbb{S}^{\mathfrak{M},0}$. The function $\overline{\varphi}_h$ is called a cellwise-constant approximation of φ if (and only if) the set of its components $\{\overline{\varphi}_i\}_{i=0}^{N+1}$ is solution to the following system of linear equations:

$$\begin{cases} \forall\, 1 \leq i \leq N \\ F_{D_{i+\frac{1}{2}}}(x^{-}_{i+\frac{1}{2}}, \nabla^{\mathfrak{D}}\overline{\varphi}_h) \;+\; F_{D_{i+\frac{1}{2}}}(x^{+}_{i-\frac{1}{2}}, \nabla^{\mathfrak{D}}\overline{\varphi}_h) \;=\; h_i[\Pi^{\mathfrak{M}} f]_i \end{cases} \tag{48}$$

where, for $j = 0, ..., N$, we have set:

$$\begin{cases} F_{D_{j+\frac{1}{2}}}(x^{-}_{j+\frac{1}{2}}, \nabla^{\mathfrak{D}}\overline{\varphi}_h) = \frac{\lambda_j \lambda_{j+1}}{\lambda_j h^{-}_{j+1} + \lambda_{j+1} h^{+}_{j}} [\overline{\varphi}_j - \overline{\varphi}_{j+1}] \\ F_{D_{j+\frac{1}{2}}}(x^{+}_{j+\frac{1}{2}}, \nabla^{\mathfrak{D}}\overline{\varphi}_h) \;=\; \frac{\lambda_j \lambda_{j+1}}{\lambda_j h^{-}_{j+1} + \lambda_{j+1} h^{+}_{j}} [\overline{\varphi}_{j+1} - \overline{\varphi}_j], \end{cases} \tag{49}$$

accounting with the following boundary conditions:

$$\overline{\varphi}_0 \;=\; \overline{\varphi}_{N+1} \;=\; 0. \tag{50}$$

Recall that $[\Pi^{\mathfrak{M}} f]_i$ is defined by the relation (31).

Definition 2.14. (Finite Volume scheme) The system of equations (48)-(50) with discrete unknowns $\{\overline{\varphi}_i\}_{i=1}^{N}$ defines what is called a Finite Volume Scheme or Finite Volume algorithm for numerically solving the problem (7).

Rewriting the LHS (Left-Hand Side) of (48) in terms of components of $\overline{\varphi}_h$ by exploiting the equations (49) leads to the following discrete problem :

$$\begin{cases} \text{Find a discrete function } \overline{\varphi}_h \in \mathbb{S}_0^{\mathfrak{M},0}, \text{ with components } (\overline{\varphi}_1, \overline{\varphi}_2, ..., \overline{\varphi}_N) \\ \text{in the canonical basis of } \mathbb{S}_0^{\mathfrak{M},0}, \textit{ such that } : \\ \frac{\lambda_i \lambda_{i+1}}{\lambda_i h^{-}_{i+1} + \lambda_{i+1} h^{+}_{i}} [\overline{\varphi}_i - \overline{\varphi}_{i+1}] + \frac{\lambda_i \lambda_{i-1}}{\lambda_i h^{+}_{i-1} + \lambda_{i-1} h^{-}_{i}} [\overline{\varphi}_i - \overline{\varphi}_{i-1}] \qquad = h_i[\Pi^{\mathfrak{M}} f]_i \; \forall 1 \leq i \leq N. \end{cases} \tag{51}$$

Notice that this discrete problem is equivalent to the system (48)-(50). The system of equations involved in the previous discrete problem is what is called the conventional (or classical) *Finite Volume Scheme* for the continuous problem (7): see for instance the work of Eymard, Gallouet and Herbin in [5] considered as one of the main references on this topic.

Concluding remarks: Notice that the homogeneous Dirichlet boundary conditions of the continuous problem (7) are involved in the Finite Volume scheme (51) through the fact that we are seeking $\overline{\varphi}_h$ in the space $\mathbb{S}_0^{\mathfrak{M},0}$ of discrete functions. There are some natural questions concerning the Finite Volume scheme given by (51): (i) Does the discrete problem (51) get a solution ? (ii) If the response is affirmative, is there a unique solution ? (iii) If there is a unique solution to (51), is that solution stable with respect to the norms defined in $\mathbb{S}^{\mathfrak{M},0}$ and does the Finite Volume solution $\overline{\varphi}_h$ (in case it exists) converge to the exact solution φ as the mesh size h goes to 0 ? we shall be answering to all those questions in the next subsection.

2.4. Theoretical Analysis of the Finite Volume Scheme Designed for the Problem (7)

Let us start with giving the matrix properties of the discrete problem (51).

2.4.1. Matrix Analysis of the Discrete Problem (51)

Let $A_h = \{A_h^{ij}\}$, with $1 \le i, j \le N$, be the $N \times N$ matrix associated with the finite volume scheme under consideration. We have the following result.

Proposition 2.5. A_h is a symmetric positive definite matrix i.e.

$$\begin{cases} (i) \quad A_h^{ij} \,=\, A_h^{ji} \qquad \forall\, 1 \le i, j \le N \\ \text{and} \\ (ii) \quad (\xi)^t A_h \xi \,>\, 0 \qquad \forall\, \xi \in \mathbb{R}^N \setminus \{0_{\mathbb{R}^N}\} \end{cases} \tag{52}$$

where $(.)^t$ is the transposition operator.

Proof Let ξ be a non zero-vector from $\mathbb{R}^N$, with components $(\xi_i)_{i=1}^N$. We can see from the left-hand side of the discrete balance equation from (51) that for $i = 1, ..., N$:

$$[A_h\xi]_i = \frac{\lambda_i \lambda_{i+1}}{\lambda_i h_{i+1}^- + \lambda_{i+1} h_i^+}[\xi_i - \xi_{i+1}] +$$

$$+ \frac{\lambda_i \lambda_{i-1}}{\lambda_i h_{i-1}^+ \,+\, \lambda_{i-1} h_i^-}[\xi_i - \xi_{i-1}]$$

where we have set $\xi_0 = 0$ and $\xi_{N+1} = 0$. Therefore,

$$(\xi)^t A_h \xi = \sum_{i=0}^{N} \frac{\lambda_i \lambda_{i+1}}{\lambda_i h_{i+1}^- + \lambda_{i+1} h_i^+}[\xi_{i+1} - \xi_i]^2$$

It is then clear that

$$(\xi)^t A_h \xi \,\ge\, 0 \,\forall \xi \in \mathbb{R}^N.$$

Remarking that $(\xi)^t A_h \xi = 0$ implies $\xi = 0$, the positive definiteness of the matrix A_h is proven.

Definition 2.15. Let a and b be two vectors from $\mathbb{R}^n$, with components $\{a_i\}$ and $\{b_i\}$ in the canonical basis of $\mathbb{R}^n$. We define a partial order over $\mathbb{R}^n$ as follows.

$$a \le b \quad \Longleftrightarrow\; 0 \le b_i - a_i \;\forall 1 \le i \le n. \tag{53}$$

In particular a vector from $\mathbb{R}^n$, with positive components, is said greater than or equal to zero-vector of $\mathbb{R}^n$, denoted $0_{\mathbb{R}^n}$ or simply 0 if there is no risk of confusion. Notice that when $n = 1$ we recover the well-known total order relation over $\mathbb{R}$. Remark that the partial order relation given by definition 2.15 easily extends to space of matrices.

Definition 2.16. (Monotone matrix)

Let M be an $n \times n$ real matrix. M is monotone if : (i) M is invertible, and (ii) The inverse M^{-1} of M is such that $0 \le M^{-1}$ i.e.

$$0 \le [M^{-1}]_{ij},\; \forall 1 \le i, j \le n.$$

A monotone matrix gets the following characterization (see for instance [4]).

Proposition 2.6. Let M be an $n \times n$ real matrix. M is monotone if and only if :

$$\forall\, \xi \in \mathbb{R}^n \; 0 \,\le\, M\xi \;\Longrightarrow\; 0 \,\le\, \xi.$$

The monotonicity of a matrix associated with a discrete formulation of a linear 2nd order elliptic problem ensures that the involved numerical scheme meets the *positivity-preserving property* (often named *discrete maximum principle*). Before checking that the finite volume scheme (51) meets that property let us state it for the continuous problem (7): see [2, 3] for instance.

Proposition 2.7. Denote by $\mathbb{R}_+^*$ the set of nonnegative real numbers and assume that:

(i) The diffusion coefficient $\lambda(.)$ lies in $C^1(\overline{I})$ and satisfies the following (strict ellipticity) condition:

$$\exists\, \lambda^{min} \in \mathbb{R}_+^* \; s.t. \; \lambda^{min} \le \lambda(x) \quad \forall\, x \in \overline{I};$$

(ii) The function f lies in $C^0(\overline{I})$ and satisfies the following positivity condition:

$$f(x) \,\ge\, 0 \;\text{ in }\; \overline{I}.$$

Then the unique solution φ in $C^2(I) \cap C^0(\overline{I})$ for the diffusion problem (7) satisfies the following property:

$$\varphi(x) \,\ge\, 0 \; in \; \overline{I} \;\; \text{(Positivity Preserving Property)}.$$

Proposition 2.8. (Monotonicity property) The matrix A_h associated with the discrete problem (51) is monotone.

Proof Following the classical technique exposed in the literature (see for instance [5]), let $\omega = \{\omega_i\}_{i=1}^N$ be a vector from $\mathbb{R}^N$ such that

$$0 \,\le\, A_h \omega \tag{54}$$

We should prove that all the components of ω are positive. The previous inequality is equivalent to what follows : for $i = 1, ..., N$,

$$0 \leq \frac{\lambda_i \lambda_{i+1}}{\lambda_i h_{i+1}^- + \lambda_{i+1} h_i^+}[\omega_i - \omega_{i+1}] + \frac{\lambda_i \lambda_{i-1}}{\lambda_i h_{i-1}^+ \; + \; \lambda_{i-1} h_i^-}[\omega_i - \omega_{i-1}] \tag{55}$$

with

$$\omega_0 \; = \; \omega_{N+1} \; = \; 0. \tag{56}$$

Let us set

$$\omega_{min} = \min\{\omega_i \; ; \; i = 0, 1, 2, ..., N\} \tag{57}$$

and

$$\alpha \overset{def}{=} \min\{i \in \{0, 1, ..., N\} \text{ such that } \omega_i = \omega_{min}\}. \tag{58}$$

Suppose that

$$1 \; \leq \; \alpha \; \leq \; N. \tag{59}$$

Therefore (55) is satisfied for $i = \alpha$, that is,

$$0 \; \leq \; \frac{\lambda_\alpha \lambda_{\alpha+1}}{\lambda_\alpha h_{\alpha+1}^- + \lambda_{\alpha+1} h_\alpha^+}[\omega_\alpha - \omega_{\alpha+1}] + + \frac{\lambda_\alpha \lambda_{\alpha-1}}{\lambda_\alpha h_{\alpha-1}^+ \; + \; \lambda_{\alpha-1} h_\alpha^-}[\omega_\alpha - \omega_{\alpha-1}],$$

It follows from the definition of α (see relation (58)) that

$$0 \; \leq \; \frac{\lambda_\alpha \lambda_{\alpha+1}}{\lambda_\alpha h_{\alpha+1}^- + \lambda_{\alpha+1} h_\alpha^+}[\omega_\alpha - \omega_{\alpha+1}] + + \frac{\lambda_\alpha \lambda_{\alpha-1}}{\lambda_\alpha h_{\alpha-1}^+ \; + \; \lambda_{\alpha-1} h_\alpha^-}[\omega_\alpha - \omega_{\alpha-1}] \; < \; 0$$

Since $0 < 0$ is not possible, we have necessarily that $\alpha = 0$ and thus $\omega_{min} = \omega_0 = 0$. The positivity of all the components of the vector ω is proven.

Remark For proving that A_h is monotone a novel and elementary technique has been exposed in [15]. It is based on geometric arguments in any space dimension. Let us proceed with that technique for our one-dimensional situation. Consider $\omega = \{\omega_i\}_{i=1}^N$ any vector from $\mathbb{R}^N$ such that

$$0 \; \leq \; A_h \omega \, . \tag{60}$$

We should prove that all the components of ω are positive. The previous inequality is equivalent to what follows.

$$0 \; \leq \; \frac{\lambda_i \lambda_{i+1}}{\lambda_i h_{i+1}^- + \lambda_{i+1} h_i^+}[\omega_i - \omega_{i+1}] + \frac{\lambda_i \lambda_{i-1}}{\lambda_i h_{i-1}^+ \; + \; \lambda_{i-1} h_i^-}[\omega_i - \omega_{i-1}] \qquad \forall \, 1 \leq i \leq N, \tag{61}$$

with

$$\omega_0 \; = \; \omega_{N+1} \; = \; 0. \tag{62}$$

Let us suppose that the least component of ω is ω_σ, with $1 \; \leq \; \sigma \; \leq \; N$. So for $i = \sigma$, we get from (61) that

$$0 \; \leq \; \frac{\lambda_\sigma \lambda_{\sigma+1}}{\lambda_\sigma h_{\sigma+1}^- + \lambda_{\sigma+1} h_\sigma^+}[\omega_\sigma - \omega_{\sigma+1}] + \frac{\lambda_\sigma \lambda_{\sigma-1}}{\lambda_\sigma h_{\sigma-1}^+ \; + \; \lambda_{\sigma-1} h_\sigma^-}[\omega_\sigma - \omega_{\sigma-1}] \; \leq \; 0. \tag{63}$$

Therefore

$$\omega_{\sigma-1} = \omega_\sigma = \omega_{\sigma+1}. \tag{64}$$

(i) If $\sigma - 1 = 0$ or $\sigma + 1 = N + 1$ then the positivity of all the components of ω is proven since $\omega_0 = \omega_{N+1} = 0$. It is the end of the proof.

(ii) Otherwise, consider the semi-line L_σ with origin x_σ and passing trough $x_{\sigma-1}$. It is clear that the relations (64) hold for all the interior centroids x_k belonging to the semi-line L_σ. The concerned centroids are $x_1 < x_2 < x_3 < ... < x_{\sigma-1} < x_\sigma$. So we can assert that:

$$\omega_{k-1} = \omega_k = \omega_{k+1} \qquad \forall \, 1 \leq k \leq \sigma. \tag{65}$$

Taking $k = 1$ in (65) leads straightly to the positivity of all the components of ω. Note that the second technique for proving the positivity preserving property of the matrix A_h may be seen as less elegant than the classical one. However it is easily extensible to higher space dimension (in particular two- or three-dimensional space as we will see when dealing with 2-D elliptic problems).

Remark Notice that the sparse structure of the matrix A_h together with its symmetry and positivity definiteness give rise to the use of a large game of powerful linear solvers for addressing the linear square system (51). Let us list for instance Jacobi's method, Gauss-Seidel's method, Conjugate-gradient method, etc.

2.4.2. Variational Formulation of the Discrete Problem (51)

Recall that $\overline{\varphi}_h$ is a cellwise-constant function, with components $(\overline{\varphi}_1, \overline{\varphi}_2, ..., \overline{\varphi}_N)$ in the canonical basis of $\mathbb{S}_0^{\mathfrak{M},0}$, defined as solution to the discrete problem (51). Let $v_h \in \mathbb{S}_0^{\mathfrak{M},0}$, with components $(v_1, v_2, ..., v_N)$ in the canonical basis of $\mathbb{S}_0^{\mathfrak{M},0}$. Multiplying the two sides of the discrete balance equation from the discrete problem (51) by v_i and summing on $i \in \{1, 2, ..., N\}$ lead to what follows :

$$\sum_{i=0}^{N} \frac{\lambda_i \lambda_{i+1}}{\lambda_i h_{i+1}^- + \lambda_{i+1} h_i^+} [\overline{\varphi}_{i+1} - \overline{\varphi}_i][v_{i+1} - v_i] = (\Pi^{\mathfrak{M}} f, v_h)_{L^2,\mathfrak{M}} \qquad \forall v_h \in \mathbb{S}_0^{\mathfrak{M},0} \tag{66}$$

Let us name $\mathfrak{B}(.,.)$ the bilinear form defined on the discrete function space $\mathbb{S}_0^{\mathfrak{M},0}$ by

$$\mathfrak{B}(v_h, w_h) \overset{def}{=} \sum_{i=0}^{N} \frac{\lambda_i \lambda_{i+1}}{\lambda_i h_{i+1}^- + \lambda_{i+1} h_i^+} [v_{i+1} - v_i][w_{i+1} - w_i].$$

Any discrete function solving the discrete problem (51) is necessarily a solution of the following variational problem.

Definition 2.17. (Variational problem)

The variational formulation of the discrete problem (51) consists to

$$\begin{cases} \text{Find a function } \overline{\varphi}_h \in \mathbb{S}_0^{\mathfrak{M},0} \quad \text{ such that} \\ \mathfrak{B}(\overline{\varphi}_h, v_h) \;=\; (\Pi^{\mathfrak{M}} f, v_h)_{L^2,\mathfrak{M}} \qquad \forall\, v_h \in \mathbb{S}_0^{\mathfrak{M},0}. \end{cases} \tag{67}$$

We have the following obvious result.

Proposition 2.9. (Equivalence result)

The discrete problem (51) is equivalent to the variational problem (67).

Proof It is done in two steps.

1) *Step 1*: It is well understood (from the way the variational problem is introduced) that any solution of the discrete problem (51) is a solution of the variational problem (67).

2) *Step 2*: Let $\overline{\varphi}_h \in \mathbb{S}_0^{\mathfrak{M},0}$ be a solution to (67), and let $(\overline{\varphi}_1, \overline{\varphi}_2, ..., \overline{\varphi}_N)$ be its components in the canonical basis of $\mathbb{S}_0^{\mathfrak{M},0}$. Therefore those components satisfy the variational equation involved in the variational problem (67). That variational equation is equivalent to the following system of equations:

$$\mathfrak{B}(\overline{\varphi}_h, 1_{K_j}) \;=\; (\Pi^{\mathfrak{M}} f, 1_{K_j})_{L^2,\mathfrak{M}} \qquad \forall\, j \in \{1, 2, 3, ..., N\} \tag{68}$$

where 1_{K_j} is the j^{th} element of the canonical basis of the space $\mathbb{S}_0^{\mathfrak{M},0}$. Recall that

$$1_{K_j}(x) \;=\; \begin{cases} 1 & if \quad x \in Int(K_j) \\ \\ 0 & if \quad x \in Ext(K_j) \end{cases} \tag{69}$$

where $Int(K_j)$ and $Ext(K_j)$ are respectively the interior and the exterior of the control-volume K_j. According to the definition of 1_{K_j}, the right-hand side of the equation (68) is equal to $\int_{K_j} f(x)dx$ while its left-hand side is equal to $\frac{\lambda_j \lambda_{j+1}}{\lambda_j h_{j+1}^- + \lambda_{j+1} h_j^+}[\overline{\varphi}_j - \overline{\varphi}_{j+1}] + \frac{\lambda_{j-1}\lambda_j}{\lambda_{j-1} h_j^- + \lambda_j h_{j-1}^+}[\overline{\varphi}_j - \overline{\varphi}_{j-1}]$. So we recover the well-known discrete balance equation in the control-volume K_j, governing the discrete problem (51).

We have the following result.

Proposition 2.10. (Existence, Uniqueness and Stability)

Recall that h is the mesh size (with vocation for going to zero) and assume that:

1) There exist mesh independent nonnegative numbers ω_g and ω_d such that:

$$\omega_g h \;\le\; h_i^-, h_i^+ \;\le\; \omega_d h \qquad \forall\, 0 \le i \le N+1;$$

2) The diffusion coefficient $\lambda(.)$ is a piece-wise constant function, with a finite number of discontinuity points, and is strictly positive almost everywhere in I;

3) The sink/source term $f(.)$ is lying in $L^2(I)$.

Then the variational problem (67) gets a unique solution.

Moreover that solution, denoted by $\overline{\varphi}_h$, satisfies the following stability inequality:

$$\| \,\overline{\varphi}_h\, \|_{H_0^1,\mathfrak{M}} \le \; C \,\|\, f \,\|_{L^2(I)} \tag{70}$$

where C is a mesh independent nonnegative number.

The proof of the above Proposition is a straightforward application of Lax-Milgram theorem which is a powerful tool for the variational analysis of linear elliptic equations of order $2m$, with m given nonnegative integer. The Lax-Milgram theorem is stated as follows.

Theorem 2.1. (Lax-Milgram)

Let H be a Hilbert space and $\| \;.\; \|$ the norm associated with the scalar product defined on H. Let L be a linear form and Ψ a bilinear form both of them defined on H and satisfying the following properties:

1) Ψ is continuous i.e. there exists a nonnegative real number γ such that for all $v, w \in H$ we have:

$$| \Psi(v,w) | \leq \gamma \parallel v \parallel \parallel w \parallel .$$

2) Ψ is coercive or H-elliptic i.e. there is α a nonnegative real number such that for all $v \in H$ we have:

$$\alpha \parallel v \parallel^2 \leq \Phi(v,v).$$

3) L is continuous, that is, L lies in the topological dual H' of H. Let $\langle .,. \rangle$ denotes the duality pairing between H' and H.

Then the variational problem that consists to:

$$\begin{cases} \text{Find } \phi \in H \text{ such that} \\ \Psi(\phi, v) \;=\; \langle L, v \rangle \qquad \forall\, v \in H \end{cases} \tag{71}$$

possesses a unique solution.

The proof of Lax-Milgram's theorem can be found in many classical books of Functional Analysis. See for instance [1–3].

Let us give now the proof of Proposition 2.10 asserting existence, uniqueness and stability of the solution $\overline{\varphi}_h$ to the variational problem (67).

Proof We proceed in two steps.

1) Step 1: Existence and Uniqueness. It suffices to check that the conditions of Lax-Milgram are fulfilled. We start with noticing that $\mathbb{S}_0^{\mathfrak{M},0}$ is a Hilbert space with respect to the scalar product (23) associated with the discrete $H_0^1 - norm$ given by (2.2).

(i) Let us prove Continuity and Coercivity of the bilinear form $\mathfrak{B}(.,.)$. For all $v_h,\, w_h \in \mathbb{S}_0^{\mathfrak{M},0}$, we have :

$$| \mathfrak{B}(v_h, w_h) |^2 \overset{def}{=} | \sum_{i=0}^{N} \frac{\lambda_i \lambda_{i+1}}{\lambda_i h_{i+1}^- + \lambda_{i+1} h_i^+} [v_{i+1} - v_i] \; [w_{i+1} - w_i] \; |^2 .$$

According to assumption 2) the first observation is that there exist two nonnegative numbers λ_{min} and λ_{max} such that

$$\lambda_{min} \leq \lambda(x) \leq \lambda_{max} \qquad a.e. \quad in \quad I.$$

According to assumption 1) and the previous inequalities, the second observation is that the effective diffusion coefficient (see Remark 2.5) $\lambda^\star_{D_{i+\frac{1}{2}}}$ is bounded as follows:

$$\frac{\omega_g \lambda_{min}^2}{2\omega_d \lambda_{max}} \;\leq\; \lambda^\star_{D_{i+\frac{1}{2}}} \;\overset{def}{=}\; \frac{h_{i+\frac{1}{2}} \lambda_i \lambda_{i+1}}{\lambda_i h_{i+1}^- + \lambda_{i+1} h_i^+} \;\leq\; \frac{\omega_d \lambda_{max}^2}{2\omega_g \lambda_{min}} \;(*).$$

These two observations put together with the Cauchy-Schwarz inequality lead to the continuity of the bilinear form $\mathfrak{B}(.,.)$. On the other hand, the coercivity of $\mathfrak{B}(.,.)$ straightly follows from the first inequality of $(*)$.

(ii) Continuity of the linear form $(\Pi^{\mathfrak{M}} f, .)_{L^2,\mathfrak{M}}$: For all $v_h \in \mathbb{S}_0^{\mathfrak{M},0}$, the Cauchy-Schwarz inequality applies and gives

$$| \, (\Pi^{\mathfrak{M}} f, v_h)_{L^2,\mathfrak{M}} \, | \leq \parallel \Pi^{\mathfrak{M}} f \parallel_{L^2,\mathfrak{M}} \parallel v_h \parallel_{L^2,\mathfrak{M}}$$

The continuity of $(\Pi^{\mathfrak{M}} f, .)_{L^2,\mathfrak{M}}$ follows from application of the discrete Poincaré-Friedrichs inequality (Proposition 2.1) and Proposition 2.10.

2) Step 2: Stability of the finite volume solution $\overline{\varphi}_h$. Take $v_h = \overline{\varphi}_h$ in the variational equation (67) and use the coercivity of the bilinear form $\mathfrak{B}(.,.)$ and the continuity of the linear form $(\Pi^{\mathfrak{M}} f, .)_{L^2,\mathfrak{M}}$.

2.4.3. Convergence of $\overline{\varphi}_h$

Note that $\overline{\varphi}_h$, the cellwise-constant function (approximating the exact solution φ) does not belong to the Sobolev space $H_0^1(I)$ as does φ. So we are going to give convergence result of $\overline{\varphi}_h$ to φ only in $L^2 - norm$. Let us establish an important intermediate result. Denote by E_h the error-function from the space $\mathbb{S}_0^{\mathfrak{M},0}$ with components $\{E_i\}_{i=1}^N$ (in the canonical basis of this space) defined by

$$E_i = \varphi_i - \overline{\varphi}_i \qquad \forall 1 \in \{1,2,3,...,N\}. \tag{72}$$

Lemma 2.1. (Error estimates) Under the assumptions of Proposition 2.10, there exists a mesh-independent nonnegative number C such that:

$$\parallel E_h \parallel_{H_0^1,\mathfrak{M}} \leq C\, h, \; \parallel E_h \parallel_{L^2,\mathfrak{M}} \leq \; C\, h, \; \parallel E_h \parallel_{L^\infty,\mathfrak{M}} \overset{def}{=} \max_{1 \leq i \leq N} | \, E_i \, | \leq \; C\, h. \tag{73}$$

Proof Let us start with subtracting side by side equation (48) from (46) and accounting with (47) and (49). Therefore we are led to the following so-called Error system of equations:

$$\begin{cases} \frac{\lambda_i \lambda_{i+1}}{\lambda_i h_{i+1}^- + \lambda_{i+1} h_i^+} [E_i - E_{i+1}] + \frac{\lambda_i \lambda_{i-1}}{\lambda_i h_{i-1}^+ + \lambda_{i-1} h_i^-} [E_i - E_{i-1}] = \\ = \mathcal{E}^-_{i+\frac{1}{2}} + \mathcal{E}^+_{i-\frac{1}{2}} \qquad \forall 1 \leq i \leq N, \end{cases} \tag{74}$$

where $E_0 = E_{N+1} = 0$, and where we have set :

$$\mathcal{E}^-_{i+\frac{1}{2}} = -E(\varphi", x^-_{i+\frac{1}{2}}) \quad \text{ and } \quad \mathcal{E}^+_{i-\frac{1}{2}} = -E(\varphi", x^+_{i-\frac{1}{2}}) \tag{75}$$

with (see Remark 2.4):

$$E(\varphi", x^-_{i+\frac{1}{2}}) + E(\varphi", x^+_{i+\frac{1}{2}}) = 0 \tag{76}$$

and (see relations (41)):

$$\mid E(\varphi", x^-_{i+\frac{1}{2}}) \mid \leq Ch \text{ and } \mid E(\varphi", x^+_{i+\frac{1}{2}}) \mid \leq Ch \quad \forall\, i \in \{1, 2, 3, ..., N\} \tag{77}$$

where C stands for diverse mesh-independent nonnegative numbers.

Multiplying the two sides of equation (74) with E_i and summing on $i \in \{1, 2, ..., N\}$ leads to what follows:

$$\mathfrak{B}(E_h, E_h) = \sum_{i=1}^{N} E_i[\mathcal{E}^+_{i-\frac{1}{2}} + \mathcal{E}^-_{i+\frac{1}{2}}]$$

Thanks to Remark 2.4 and the coercivity of the bilinear form $\mathfrak{B}(.,.)$ we can see that

$$\frac{\omega_g \lambda^2_{min}}{2\omega_d \lambda_{max}} \parallel E_h \parallel^2_{H^1_0,\mathfrak{M}} \leq \sum_{i=0}^{N} \mathcal{E}^+_{i+\frac{1}{2}}[E_{i+1} - E_i] \;\leq\; C\,h \sum_{i=0}^{N} \sqrt{h_{i+\frac{1}{2}}} \frac{\mid E_{i+1} - E_i \mid}{\sqrt{h_{i+\frac{1}{2}}}} \;\leq$$

$$(by\ Cauchy - Schwarz) \;\leq\; C\,h\sqrt{b-a} \parallel E_h \parallel_{H^1_0,\mathfrak{M}} . \tag{78}$$

The second inequality of (73) follows from the discrete version of Poincaré-Friedrichs inequality (see Proposition 2.1). Let us prove the third inequality. For any $i \in \{1, 2, ..., N\}$, we have $\mid E_i \mid^2 \leq [\sum_{i=0}^{N} \mid E_{i+1} - E_i \mid]^2 \leq (b-a) \parallel E_h \parallel^2_{H^1_0,\mathfrak{M}}$ (thanks to Cauchy-Schwarz inequality).

Lemma 2.2. Consider the assumptions of Proposition 2.10 and let $\varphi^{\mathfrak{M}}$ be the unique discrete function from the space $\mathbb{S}^{\mathfrak{M},0}_0$ such that :

$$\varphi^{\mathfrak{M}}_{|K_i} \stackrel{def}{=} \varphi(x_i) \equiv \varphi_i \qquad \forall 0 \leq i \leq N+1.$$

So we have

$$\parallel \varphi - \varphi^{\mathfrak{M}} \parallel_{L^2(I)} \leq C\,h$$

where C is a mesh-independent nonnegative number.

Proof Let us arbitrarily choose $i \in \{0, 1, ..., N, N+1\}$ and $x \in K_i$. From the assumption: $\varphi_{|K_i} \in C^1(\overline{K}_i)$, with $\overline{K}_i$ part of $\overline{I}_s$, for some $s \in S$, we get by Taylor-Lagrange theorem that for all $x \in K_i$ we have

$$[\varphi_{|K_i} - \varphi^{\mathfrak{M}}_{|K_i}](x) = \frac{1}{1!}[\varphi_{|K_i} - \varphi^{\mathfrak{M}}_{|K_i}]'(\xi_x).[x - x_i]$$

with $\min\{x, x_i\} \leq \xi_x \leq \max\{x, x_i\}$. Therefore

$$\int_{K_i} [\varphi_{|K_i} - \varphi^{\mathfrak{M}}_{|K_i}]^2(x)dx \leq h^2 \int_{K_i} \mid [\varphi_{|K_i}]'(\xi_x) \mid^2 dx \leq$$

$$\leq h_i h^2 [\max_{\overline{I}_s} \mid \varphi' \mid]^2 \leq Ch^2 h_i$$

where

$$C = [\max_{\overline{I}_s} \mid \varphi' \mid]^2.$$

Therefore

$$\sum_{i=0}^{N+1} \int_{K_i} [\varphi_{|K_i} - \varphi^{\mathfrak{M}}_{|K_i}]^2(x)dx \;\leq\; \widehat{C}\,h^2$$

i.e.

$$\parallel \varphi - \varphi^{\mathfrak{M}} \parallel^2_{L^2(I)} \leq \widehat{C}\,h^2.$$

This is the end of the proof.

We can state now the $L^2 - convergence$ of $\overline{\varphi}_h$ to φ as follows.

Proposition 2.11. (Convergence of the conventional finite volume solution)
Under the assumptions of Proposition 2.10 there exists a mesh-independent nonnegative number, let us say Υ, such that

$$\parallel \varphi - \overline{\varphi}_h \parallel_{L^2(I)} \leq \Upsilon\,h.$$

.

Proof Thanks to the triangular inequality we have

$$\parallel \varphi - \overline{\varphi}_h \parallel_{L^2(I)} \leq \parallel \varphi - \varphi^{\mathfrak{M}} \parallel_{L^2(I)} + \parallel \varphi^{\mathfrak{M}} - \overline{\varphi}_h \parallel_{L^2(I)}$$

The proof is completed thanks to Lemma 2.2 that applies to the first term and Lemma 2.1 that applies to the second term of the Right-Hand Side of the preceding inequality.

2.5. Linear-spline Approximation of φ

We expose here a finite-volume-based reconstruction of the exact solution φ, requiring that the candidates lie in $\mathbb{L}$inear $\mathbb{S}$pline function space $\mathbb{LS}^{\mathfrak{D}}$ associated with the mesh $\mathfrak{D} = \{D_{i+\frac{1}{2}} = [x_i, x_{i+1}]\}_{i=0}^{N}$ introduced in subsection 2.2. So doing the reconstructed solution can be shown to lie in $H^1_0(I)$, just like the exact solution.

2.5.1. Preliminary Notions and Results

Recall that

$$\mathbb{LS}^{\mathfrak{D}} = \{v_h \in C^0(\overline{I}) \ / \ v_h\,|_{D_{i+\frac{1}{2}}} \in \mathbb{P}_1(D_{i+\frac{1}{2}})\ \forall\, 0 \le i \le N\},$$

where $\mathbb{P}_1(D_{i+\frac{1}{2}})$ is the space of real polynomials of degree ≤ 1, restricted to the segment $D_{i+\frac{1}{2}}$.

Proposition 2.12. The function space $\mathbb{LS}^{\mathfrak{D}}$ is a subspace of the Sobolev space $H^1(I)$.

This result is a straight consequence of the following Lemma.

Lemma 2.3. Let Ω be an open bounded domain included in $\mathbb{R}^d$, with $d \in \{1,2,3\}$. Let $\mathcal{T}$ be a family of sub-domains of $\overline{\Omega}$ defining a partition of Ω in the sense that for all $T, L \in \mathcal{T}$ we have :

$$\begin{cases} T \text{ is nonempty open subset of } \Omega \text{ and } \overline{\Omega} = \cup_{T\in\mathcal{T}}\overline{T} \\ T \ne L \quad \Longrightarrow \quad T\cap L = \emptyset \quad and \quad \overline{T}\cap\overline{L} \,=\, \Gamma_T\cap\Gamma_L \\ \text{ where } \Gamma_\diamond \text{ is the boundary of } \diamond, \text{ for all } \diamond \in \mathcal{T}. \end{cases} \tag{79}$$

Let v be a *continuous* function in $\overline{\Omega}$ and assume that:

$$\forall T \in \mathcal{T} \qquad v_{|_T} \in H^1(T)$$

where $v_{|_T}$ stands for the restriction of v within T.

Then v lies in $H^1(\Omega)$.

Proof. Let v be a function from the space of continuous functions in $\overline{\Omega}$. For the sake of simplification of notations the function $v_{|_T}$ is simply denoted by v inside T and on Γ_T, except when there is risk of confusion.

(i) Since Ω is bounded in $\mathbb{R}^d$, v lies in $L^2(\Omega)$.

(ii) It remains to check that the first order distributional derivatives of v lie also in $L^2(\Omega)$. For this purpose, let us choose arbitrarily a direction, let us say α, in the space $\mathbb{R}^d$. Denote by $\mathbf{D}_\alpha v$ the distributional derivative of v in the α^{th} direction. We should prove that $\mathbf{D}_\alpha v \in L^2(\Omega)$. By definition we have

$$< \mathbf{D}_\alpha v, \phi >= - < v, \frac{\partial\,\phi}{\partial x_\alpha} > \qquad \forall \phi \in \mathcal{D}(\Omega)$$

where $\mathcal{D}(\Omega)$ is the space of functions of class C^∞ in Ω, with a compact support included in Ω. It follows that we get for all $\phi \in \mathcal{D}(\Omega)$:

$$\begin{aligned} < \mathbf{D}_\alpha v, \phi > &= -\int_\Omega v\frac{\partial\,\phi}{\partial x_\alpha}\,dx \\ &= -\sum_{T\in\mathcal{T}}\int_T v\frac{\partial\,\phi}{\partial x_\alpha}\,dx \\ &= -\sum_{T\in\mathcal{T}}[\int_{\Gamma_T} v\phi\nu_\alpha^T d\sigma - \int_T \phi\frac{\partial\,v}{\partial x_\alpha}\,dx] \\ &= \sum_{T\in\mathcal{T}}\int_T \phi\frac{\partial\,v}{\partial x_\alpha}\,dx - \sum_{T,L\in\mathcal{T};\,T\ne L}\int_{\Gamma_T\cap\Gamma_L} v\phi[\nu_\alpha^T+\nu_\alpha^L]\,d\sigma \end{aligned}$$

where $\nu_\alpha^\diamond$ is the α^{th} component of the outward normal unit vector to the boundary of a sub-domain $\diamond$ from $\mathcal{T}$.

Remarking that (due to continuity of v and ϕ over Ω together with the fact that $\nu_\alpha^T + \nu_\alpha^L \,=\, 0$ on $\Gamma_T\cap\Gamma_L$ for all adjacent sub-domains T and L from $\mathcal{T}$) we have

$$\sum_{T,L\in\mathcal{T};\,T\ne L}\int_{\Gamma_T\cap\Gamma_L} v\phi[\nu_\alpha^T+\nu_\alpha^L]\,d\sigma = 0 \qquad \forall \phi \in \mathcal{D}(\Omega)$$

we deduce that

$$\mathbf{D}_\alpha v = \sum_{T\in\mathcal{T}}\frac{\partial\,v_{|_T}}{\partial x_\alpha}\mathbf{1}_T \qquad in \qquad \mathcal{D}'(\Omega).$$

Since the right-hand side of the preceding equality lies in $\prod_{T\in\mathcal{T}} L^2(T) \;\equiv\; L^2(\Omega)$ and the α^{th} direction is arbitrarily chosen the proof is ended.

Remark 2.6. Recall that $I \;=\; [a,b]$. An immediate consequence of the preceding Proposition is that the following function space

$$\begin{aligned} \mathbb{LS}_0^{\mathfrak{D}} = \{v_h \in C^0(\overline{I}) \ / \ v_h\,|_{D_{i+\frac{1}{2}}} &\in \mathbb{P}_1(D_{i+\frac{1}{2}}) \\ \forall\, 0 \le i \le N,\ v_h(a) &= v_h(b) = 0\} \end{aligned}$$

is a subspace of the Sobolev space $H_0^1(I)$.

Proposition 2.13. The space $\mathbb{LS}^{\mathfrak{D}}$ is a finite dimensional space with dimension equal to $(N+2)$. Moreover there exists a (canonical) basis $\{\Phi_i\}_{i=0}^{N+1}$ of $\mathbb{LS}^{\mathfrak{D}}$ such that:

$$\Phi_i(x_j) \,=\, \delta_{ij} \qquad \forall\, 0 \le i,j \le N+1$$

where δ_{ij} is the Kronecker symbol i.e.

$$\delta_{ij} \;=\; \begin{cases} 1 & if \quad i = j \\ 0 & otherwise. \end{cases} \tag{80}$$

Proof It will be done in two steps.

1) *First step :* We should show that the space $\mathbb{LS}^{\mathfrak{D}}$ gets a finite dimension equal to $(N+2)$. It is obvious that $\mathbb{LS}^{\mathfrak{D}}$ gets a finite dimension as it satisfies the following "inequality":

$$\mathbb{LS}^{\mathfrak{D}} \subset \prod_{i=0}^{N} \mathbb{P}_1(D_{i+\frac{1}{2}})$$

and we have

$$dimension[\prod_{i=0}^{N} \mathbb{P}_1(D_{i+\frac{1}{2}})] = 2(N+1).$$

So $dimension[\mathbb{LS}^{\mathfrak{D}}] \leq 2(N+1)$. Let us show now that $dimension[\mathbb{LS}^{\mathfrak{D}}] = N+2$. For that purpose let us introduce the linear mapping Λ from $\mathbb{LS}^{\mathfrak{D}}$ to $\mathbb{R}^{N+2}$ as follows:

$$\Lambda : \quad \mathbb{LS}^{\mathfrak{D}} \quad \longrightarrow \quad \mathbb{R}^{N+2}$$

$$v_h \longmapsto (v(x_0), v(x_1), ..., v(x_N), v(x_{N+1})).$$

We are going to show that Λ is a bijective linear mapping. The linearity of Λ is quite obvious. So we will focus on its bijective status.

(i) Let us start with showing that Λ is a surjective mapping. Given a vector from the space $\mathbb{R}^{N+1}$, with components $(\beta_0, \beta_1, ..., \beta_N, \beta_{N+1})$ in the canonical basis of $\mathbb{R}^{N+1}$, there exists a unique affine function $g^{i+\frac{1}{2}}$ defined in $[x_i, x_{i+1}]$ such that

$$g^{i+\frac{1}{2}}(x) = \begin{cases} \beta_i & \text{if} \quad x = x_i \\ \beta_{i+1} & \text{if} \quad x = x_{i+1}. \end{cases} \tag{81}$$

More precisely we have

$$g^{i+\frac{1}{2}}(x) = \beta_i[\frac{x-x_{i+1}}{x_i-x_{i+1}}]+\beta_{i+1}[\frac{x-x_i}{x_{i+1}-x_i}] \quad \text{in} \quad [x_i, x_{i+1}].$$

Now define a piece-wise affine function g in $[a, b]$ as follows:

$$g\,|_{D_{i+\frac{1}{2}}}(x) = g^{i+\frac{1}{2}}(x) \quad (\text{recall that} : D_{i+\frac{1}{2}} = [x_i, x_{i+1}]).$$

The function g is obviously a continuous function in $[a, b]$ and therefore it lies in the space $\mathbb{LS}^{\mathfrak{D}}$. So the surjective status of Λ is proven.

(ii) Let us end the proof with showing that Λ is injective. We set

$$Ker[\Lambda] = \{v_h \in \mathbb{LS}^{\mathfrak{D}} \,/\, \Lambda(v_h) = 0_{\mathbb{R}^{N+2}}\}.$$

Since Λ is linear to show that this mapping is injective it suffices to show that $Ker[\Lambda]$ is reduced to $\{0_{\mathbb{LS}^{\mathfrak{D}}}\}$. It is clear that $\{0_{\mathbb{LS}^{\mathfrak{D}}}\} \subset Ker[\Lambda]$ since Λ is a linear mapping. Let us concentrate on showing that $Ker[\Lambda] \subset \{0_{\mathbb{LS}^{\mathfrak{D}}}\}$. Consider an arbitrarily chosen function v_h from $Ker[\Lambda]$ such that $\Lambda[v_h] = 0_{\mathbb{R}^{N+2}}$. This means that $v(x_0) = v(x_1) = v(x_2) = ... = v(x_N) = v(x_{N+1}) = 0$. So in any sub-interval $D_{i+\frac{1}{2}} \equiv [x_i, x_{i+1}]$ there are two roots of v_h and $v_h\,|_{D_{i+\frac{1}{2}}}$ is a polynomial of degree ≤ 1. It implies that $v_h\,|_{D_{i+\frac{1}{2}}} \equiv 0$ for any $0 \leq i \leq N$. So $v_h \equiv 0$ in $[a, b]$.

We have shown that Λ is a bijective linear mapping from $\mathbb{LS}^{\mathfrak{D}}$ to $\mathbb{R}^{N+2}$. So $dimension[\mathbb{LS}^{\mathfrak{D}}] = N+2$.

2) *Second step:* The mapping Λ ensures the existence and uniqueness of the family of functions $\{\Phi_i\}_{i=0}^{N+1}$ from $\mathbb{LS}^{\mathfrak{D}}$ such that

$$\Phi_i(x_j) = \delta_{ij} \qquad \forall\, 0 \leq i, j \leq N+1$$

This family is a basis of $\mathbb{LS}^{\mathfrak{D}}$ if and only if it is made up of linearly independent vectors. We are going to prove the linear independency of the family $\{\Phi_i\}_{i=0}^{N+1}$. For that purpose let us introduce the family of linear forms $\{L_i\}_{i=0}^{N+1}$ defined on $\mathbb{LS}^{\mathfrak{D}}$ by

$$\langle L_j, v_h \rangle = v_h(x_j) \qquad \forall\, 0 \leq j \leq N+1.$$

where $\langle\, .\, , . \rangle$ is the duality operator. Consider any family of real numbers $\{\gamma_i\}_{i=0}^{N+1}$ such that

$$\sum_{i=0}^{N+1} \gamma_i \Phi_i = 0_{\mathbb{LS}^{\mathfrak{D}}}.$$

For an arbitrarily chosen j in $\{0, 1, 2, ..., N, N+1\}$ we have

$$0 = \langle L_j, \sum_{i=0}^{N+1} \gamma_i \Phi_i \rangle = \sum_{i=0}^{N+1} \gamma_i \langle L_j, \Phi_i \rangle = \sum_{i=0}^{N+1} \gamma_i \delta_{ij} = \gamma_j.$$

This is the end of the proof.

Proposition 2.14. The basis $\{\Phi_i\}_{i=0}^{N+1}$ of the space $\mathbb{LS}^{\mathfrak{D}}$ possesses the following interesting property:

$$v_h(x) = \sum_{i=0}^{N+1} v_h(x_i)\Phi_i(x) \quad in\ [a, b] \qquad \forall\, v_h \in \mathbb{LS}^{\mathfrak{D}}.$$

Proof Easy exercise.

It follows from the preceding result that:

Proposition 2.15. The family of functions $\{\Phi_i\}_{i=1}^{N}$ is a basis of the space $\mathbb{LS}_0^{\mathfrak{D}}$ and satisfies the following remarkable property:

$$v_h(x) = \sum_{i=1}^{N} v_h(x_i)\Phi_i(x) \quad in \quad [a, b] \qquad \forall\, v_h \in \mathbb{LS}_0^{\mathfrak{D}}.$$

Let us introduce an interpolation operator $\Pi^{\mathfrak{D}}$ from the space $C^0(\overline{I})$ to the space $\mathbb{LS}^{\mathfrak{D}}$ as follows:

$$v \quad \longmapsto \quad \Pi^{\mathfrak{D}} v = \sum_{i=0}^{N+1} v(x_i)\Phi_i.$$

Taking $v = \varphi$ (exact solution to the model problem (7)) we have, by definition,

$$[\Pi^{\mathfrak{D}}(\varphi)](x) = \sum_{i=0}^{N+1} \varphi(x_i)\Phi_i(x).$$

Recall that $\varphi(x_i)$ is also denoted by φ_i, for $i = 0, ..., N+1$.

Proposition 2.16. Assume that the exact solution φ to the problem (7) is lying in $C^2(\overline{I})$, i.e. φ is a classical solution to (7). Then

$$\| \varphi - \Pi^{\mathfrak{D}}(\varphi) \|_{L^2(I)} \leq C\, h^2. \tag{82}$$

and

$$\| \varphi - \Pi^{\mathfrak{D}}(\varphi) \|_{H^1(I)} \leq C\, h. \tag{83}$$

where C denotes here as always diverse mesh-independent nonnegative real numbers.

Proof 1) Let us prove (82).

Fix arbitrarily $i \in \{0, 1, 2, ..., N\}$. Then for all $x \in D_{i+\frac{1}{2}} = [x_i, x_{i+1}]$ we have, according to Lagrange interpolation theory, the existence of some real number $\xi_x^{i+\frac{1}{2}}$, with $x_i < \xi_x^{i+\frac{1}{2}} < x_{i+1}$, such that

$$\varphi_{|D_{i+\frac{1}{2}}}(x) - [\Pi^{\mathfrak{D}}(\varphi)]_{|D_{i+\frac{1}{2}}}(x) \;=\; \frac{\varphi"(\xi_x^{i+\frac{1}{2}})}{2!}[x - x_i][x - x_{i+1}]. \tag{84}$$

Therefore

$$\mid \varphi_{|D_{i+\frac{1}{2}}}(x) - [\Pi^{\mathfrak{D}}(\varphi)]_{|D_{i+\frac{1}{2}}}(x) \mid^2 \leq \left[\frac{\max_{\overline{I}} \mid \varphi" \mid}{2!}\right]^2 [x_{i+1} - x_i]^4 \leq \Lambda [x_{i+1} - x_i]^4$$

with Λ mesh-independent. Integrating in $D_{i+\frac{1}{2}}$ all sides of the previous double inequalities and summing on $i \in \{0, 1, 2, ..., N\}$ yield

$$\| \varphi - \Pi^{\mathfrak{D}}(\varphi) \|^2_{L^2(I)} \leq \Lambda' h^4.$$

2) Let us prove now (83).

Fix again arbitrarily a number x in $[x_i, x_{i+1}]$. According to Taylor-Lagrange theorem we have, with $x_i < \theta_x < x_{i+1}$,

$$[\varphi - \Pi^{\mathfrak{D}}(\varphi)]'(x) = [\varphi - \Pi^{\mathfrak{D}}(\varphi)]'(x_i) + \frac{1}{1!}\varphi"(\theta_x).[x - x_i] \quad \forall x_i \leq x \leq x_{i+1}. \tag{85}$$

On the other hand we have, by application of Taylor-Lagrange theorem to the function $[\varphi - \Pi^{\mathfrak{D}}(\varphi)](x)$ over the interval $[x_i, x]$ we get, for all $x_i \leq x \leq x_{i+1}$,

$$[\varphi - \Pi^{\mathfrak{D}}(\varphi)](x) = [\varphi - \Pi^{\mathfrak{D}}(\varphi)](x_i) + \frac{1}{1!}[\varphi - \Pi^{\mathfrak{D}}(\varphi)]'(x_i).[x - x_i] + \frac{1}{2!}\varphi"(\tau_x).[x - x_i]^2. \tag{86}$$

From (84) and (86) we see that one can rewrite (85) in the following form

$$[\varphi - \Pi^{\mathfrak{D}}(\varphi)]'(x) = \frac{\varphi"(\xi_x)}{2!}[x - x_{i+1}] + [\varphi"(\theta_x) - \frac{1}{2}\varphi"(\tau_x)][x - x_i] \qquad \forall x_i < x \leq x_{i+1} \tag{87}$$

Since $\varphi"$ lies in $C^0(\overline{I})$ it is clear that

$$\mid [\varphi - \Pi^{\mathfrak{D}}(\varphi)]'(x) \mid^2 \leq C\, h^2 \quad a.e. \quad in \quad D_{i+\frac{1}{2}} \tag{88}$$

Integrating the both sides of the previous inequality in $D_{i+\frac{1}{2}}$ and summing over $i \in \{0, 1, 2, ..., N\}$ lead to what follows:

$$\| [\varphi - \Pi^{\mathfrak{D}}(\varphi)]' \|_{L^2(I)} \leq C\, h.$$

This is the end of the proof.

2.5.2. Approximation of φ in $\mathbb{LS}_0^{\mathfrak{D}}$

The exact solution φ to the diffusion problem under consideration (in this part of our work) lies in the Sobolev space $H_0^1(I)$. According to Remark 2.6, the space $\mathbb{LS}_0^{\mathfrak{D}}$ is a subspace of $H_0^1(I)$. The first step of the proof of Proposition 2.13 shows that there exists a bijection between $\mathbb{R}^{N+2}$ and $\mathbb{LS}^{\mathfrak{D}}$. Following the same technique one can easily show that there exists a bijection between $\mathbb{R}^N$ and $\mathbb{LS}_0^{\mathfrak{D}}$. So the vector $(\overline{\varphi}_1, ..., \overline{\varphi}_N)$ is associated with some function $\overline{\varphi}_h^\star$ from $\mathbb{LS}_0^{\mathfrak{D}}$, defined as:

$$\overline{\varphi}_h^\star(x) = \sum_{i=1}^{N} \overline{\varphi}_i \Phi_i(x) \qquad \forall\, x \in \overline{I}.$$

It is reasonable to think that this function is a nice candidate for a finite volume approximation of the exact solution φ. We mean by "nice candidate" a function lying in $H_0^1(I)$ and converging faster to φ than does $\overline{\varphi}_h$ (see Proposition 2.11). Let us investigate this allegation. For that purpose let us look for the estimates of $\| \varphi - \overline{\varphi}_h^\star \|_{L_2(I)}$ and $\| \varphi - \overline{\varphi}_h^\star \|_{H_0^1(I)}$. Let us start with setting that:

$$\overline{\varphi}_h^\star(x) = \varphi(x) + \mathcal{E}_h(x) \quad in \quad \overline{I} \tag{89}$$

We know from Lemma 2.1 that

$$| \mathcal{E}_h(x_i) | \leq C\, h \qquad \forall 0 \leq i \leq N+1. \tag{90}$$

Based on this result we make the assumption that there exists some $\alpha \geq 1$ such that

$$| \mathcal{E}_h(x) | \leq C\, h^\alpha \qquad \forall x \in \overline{I}. \tag{91}$$

According to Lagrange interpolation theory, for all $x_i \leq x \leq x_{i+1}$ there exists some real number $\xi_x^{i+\frac{1}{2}}$, with $x_i < \xi_x^{i+\frac{1}{2}} < x_{i+1}$, such that:

$$\varphi_{|_{D_{i+\frac{1}{2}}}}(x) - [\Pi^{\mathfrak{D}}(\varphi)]_{|_{D_{i+\frac{1}{2}}}}(x) \;=\; \frac{\varphi"(\xi_x^{i+\frac{1}{2}})}{2!}[x-x_i][x-x_{i+1}] \tag{92}$$

Thanks to (89) the previous equality can be put in the following form:

$$\overline{\varphi}_h^\star(x) - [\Pi^{\mathfrak{D}}(\varphi)](x) \;=\; \frac{\varphi"(\xi_x^{i+\frac{1}{2}})}{2!}[x-x_i][x-x_{i+1}] + \mathcal{E}_h(x) \quad \forall x_i \leq x \leq x_{i+1} \tag{93}$$

Therefore

$$| \overline{\varphi}_h^\star(x) - [\Pi^{\mathfrak{D}}(\varphi)](x) |^2 \leq C[h^4 + h^{2\alpha}] \quad \forall x_i \leq x \leq x_{i+1} \tag{94}$$

Integrating the two sides of the previous inequality in $D_{i+\frac{1}{2}}$ and summing over $i \in \{0,1,2,...,N\}$ lead to what follows:

$$\| \overline{\varphi}_h^\star - [\Pi^{\mathfrak{D}}(\varphi)] \|_{L^2(I)} \leq C\, h^{\frac{1}{2}\min\{4,2\alpha\}} \;=\; C\, h^{\min\{2,\alpha\}} \tag{95}$$

On the other hand we have:

$$\forall\, x_i \leq x \leq x_{i+1} \begin{cases} [\Pi^{\mathfrak{D}}(\varphi)](x) = \varphi_i \frac{x-x_{i+1}}{x_i-x_{i+1}} + \varphi_{i+1}\frac{x-x_i}{x_{i+1}-x_i} \\ \\ \overline{\varphi}_h^\star(x) = \overline{\varphi}_i \frac{x-x_{i+1}}{x_i-x_{i+1}} + \overline{\varphi}_{i+1}\frac{x-x_i}{x_{i+1}-x_i} \end{cases} \tag{96}$$

We need the assumption consisting in saying that there exist two mesh-independent nonnegative real numbers ϖ_g and ϖ_d such that

$$\varpi_g\, h \;\leq\; h_{i+\frac{1}{2}} \;\leq\; \varpi_d\, h \quad \forall\, 0 \leq i \leq N. \tag{97}$$

Note that an equivalent version of this assumption has been formulated in Proposition 2.10. It follows from (96) that for all $x_i \leq x \leq x_{i+1}$ we have

$$[\Pi^{\mathfrak{D}}(\varphi)]'(x) \;-\; [\overline{\varphi}_h^\star]'(x) = -\frac{\varphi_i - \overline{\varphi}_i}{h_{i+\frac{1}{2}}}[x-x_{i+1}] + \frac{\varphi_{i+1} - \overline{\varphi}_{i+1}}{h_{i+\frac{1}{2}}}[x-x_i] \tag{98}$$

Thanks to assumptions (91) and (97) it is easily seen that

$$| \; [\Pi^{\mathfrak{D}}(\varphi)]'(x) \;-\; [\overline{\varphi}_h^\star]'(x) \; |^2 \leq C\, h^{2(\alpha-1)} \qquad in \quad \overline{I} \tag{99}$$

where $\alpha \geq 1$ is mesh-independent and comes from assumption (91).

Let us summarize the main ideas exposed above concerning the approximation of φ in the space $\mathbb{LS}_0^{\mathfrak{D}}$.

Lemma 2.4. Let us assume that (97) holds. Let us also assume that the model problem (7) possesses a solution φ lying in $C^2(\overline{I})$ and that the real-valued function $\mathcal{E}_h$ defined in $\overline{I}$ as follows:

$$\overline{\varphi}_h^\star \;=\; \varphi \;+\; \mathcal{E}_h$$

is such that there exist mesh-independent nonnegative numbers C and α, with $\alpha \in [1, +\infty[$, such that

$$| \mathcal{E}_h(x) | \leq C\, h^\alpha \qquad in \quad \overline{I}.$$

Then we have what follows:

$$\| [\Pi^{\mathfrak{D}}(\varphi)] - \overline{\varphi}_h^\star \|_{L^2(I)} \leq C\, h^{\min\{2,\alpha\}}. \tag{100}$$

Under the condition $1 < \alpha \leq 2$, we have that:

$$\| \; [\Pi^{\mathfrak{D}}(\varphi)]' \;-\; [\overline{\varphi}_h^\star]' \; \|_{L^2(I)} \leq C\, h^{\alpha-1}. \tag{101}$$

Recall that C stands for diverse mesh-independent nonnegative numbers.

Proposition 2.17. (Error estimates for finite volume solution lying in $\mathbb{LS}_0^{\mathfrak{D}}$)

Let us assume that (97) holds. Let us also assume that the model problem (7) possesses a solution φ lying in $C^2(\overline{I})$ and that the real-valued function $\mathcal{E}_h$ defined in $\overline{I}$ as follows:

$$\overline{\varphi}_h^\star = \varphi + \mathcal{E}_h$$

is such that there exist mesh-independent nonnegative numbers C and α, with $1 \le \alpha \le 2$, such that

$$| \mathcal{E}_h(x) | \le C\, h^\alpha \qquad in \quad \overline{I}.$$

Then we have what follows:

$$\| \varphi - \overline{\varphi}_h^\star \|_{L^2(I)} \le C\, h^\alpha. \tag{102}$$

Moreover, if $1 < \alpha \le 2$ then

$$\| \varphi - \overline{\varphi}_h^\star \|_{H_0^1(I)} \le C\, h^{\alpha-1}. \tag{103}$$

Abbreviations

CFV Conventional Finite Volume(s)
RHS Right-Hand Side

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Conflicts of Interest

The authors declare no conflicts interest.

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