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Some New Results on S-prime Ideals of a Finite Commutative Ring as S-meet Semilattice

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Abstract: Let $\mathfrak{R}$ be the finite commutative ring with unity and I_s be the S-prime ideal of a ring $\mathfrak{R}$. The set L_s forms a partially ordered set (poset) by the subset relation. Initially, the interplay of the semilattice theoretic properties of a poset with the ring theoretic properties are studied with suitable examples. The number of maximal chain of a poset is compared with the number of prime ideals of a ring. It is proved that every maximal element of a poset is the prime ideal of a ring. A ring of prime powers is shown as a lattice. If the order of the ring is the product of two primes, then the trivial ideal is expressed as the meet of every pair of a poset. Further, the cardinality of the poset is determined in terms of the divisors of the order of the ring . A new meet-semilattice called the S-meet semilattice $(L_s, \wedge, \subseteq)$ is defined and the generalized Hasse diagrams of the S-meet semilattice of a ring of prime powers, product of prime powers are drawn in this paper in order to find the properties of S-meet semilattice. Finally, the ideals, the prime ideals and the maximal ideals of the S-meet semilattice are described in terms of the down-sets of S-meet semilattice where the results are listed with an example at the end.

Keywords: Prime Ideal, S-prime Ideal, Nilpotent Ideal, Meet-semilattice

1. Introduction

The ideals of a commutative ring have been analyzed by many authors and are still being studied. Inspired by its interesting properties, many authors have discovered new ideals, namely: S-prime ideal [1], almost S-prime ideal [2], J-ideals [3], $(2, n)$-ideals [4] and so on. These are still being researched by some authors. In the research field, the lattice theoretic aspects of algebraic structure have been examined long and some authors have continued the lattice theory studies beyond the commutative ring theory. In lattice theory, the various ideals of a commutative ring are defined [5–8].

Later, through its development the researchers delved deeper into the subset of a lattice and a semilattice. This semilattice is said to be a meet-semilattice (join-semilattice) in which each pair of elements of semilattice has a meet (join) $\wedge(\vee)$. Over the past few years, the research on meet-semilattice has been carried out by many authors. The category of semilattice by Alfered and Kimura [9] determined injective and projective semilattices which is the starting point of the construction of free meet-semilattice over a poset. Elliott Evans [10] investigated the $B(M)$ in terms of order theory and effects of this on $B(M)$ are characterized. Aiswarya and Afrinayesha [11] expanded the ideal of a lattice to $(L, \wedge)$. Also they [12] discussed 0-distributive meet-semilattice and they studied the characteristics of ideals and filters of meet-semilattice. A non-empty set K which is ordered is called a *meet-semilattice (join-semilattice)* [13], if any two elements of K have a meet(join) in K for all $x, y \in K$. It is denoted by $(L, \wedge)[(L, \vee)]$. A lattice consists of a join-semilattice and a meet-semilattice.

Kalamani and Mythily [14] introduced a new finite graph called S-prime ideal graph of $\mathfrak{R}$ where an edge connects the vertices a, b if sa or sb in an ideal I for some s in S, $a, b \in \mathfrak{R}$ whenever the the product ab in an ideal where $S \subseteq \mathfrak{R}$ which is disjoint from I of $\mathfrak{R}$. An ideal I of $\mathfrak{R}$ is *S-prime* if $\exists\ s \in S \ni$ for all $a, b \in \mathfrak{R}$ with $ab \in I$ then $sa \in$ or $sb \in I$. Some examples [15] show that what are the I_s in a ring. Mythily and Kalamani [15] studied about properties and classifications of

S-prime ideals. Also, some new graphs [16, 17] are defined and studied their algebraic, graph theoretic properties which are from a finite ring and abelian group.

This work is motivated by previous work on I_s of $\mathfrak{R}$; it gives a generalization of S-prime ideals of $\mathfrak{R}$ of some order. This research was carried out by taking the collection of I_s of $\mathfrak{R}$ which gives some results based on semilattice. In section 2, some properties of the $(L, \wedge, \subseteq)$ and $(L, \vee, \subseteq)$ are presented with suitable examples. Section 3 gives generalization of an S-prime ideal. In section 4, the ideals, the prime ideals and the maximal ideals of a S-meet semilattice are generalized by using the down-sets of L_s. In this paper, the ring $\mathfrak{R}$ is considered as a finite commutative ring with unity. Let I_p and I'_p be the prime and non-prime ideals of $\mathfrak{R}$ respectively and the given Hasse diagrams of S-meet semilattice of a ring is given in the order of p^t, p^tq, p^tqr and p^tq^tr $t, s \geq 1$ where $p \neq q \neq r$ and $t, s \in \mathbb{Z}^+$. Also, a poset which is neither an S-meet semilattice nor an S-join semilattice is shown.

2. Properties of an S-meet Semilattice

Let L_s be the collection of all I_s of $\mathfrak{R}$. The set L_s is a poset with the usual $\subseteq$ relation. The poset L_s is called S-meet semilattice (S-join semilattice) if every pair of elements of L_s has a meet (join). It is denoted as $(L_s, \wedge)[L_s, \vee]$ or $(L_s, \wedge, \subseteq)[L_s, \vee, \subseteq]$. It is not necessary that every pair of elements of L_s has meet and join.

The following example shows that the collection of all the I_s of a commutative ring with unity of order p^tq^t does not form a meet-semilattice(join-semilattice).

Example 2.1. Let $\mathfrak{R} = 72$ and their S-prime ideals are $<2>, <3>, <6>, <12>, <18>$ and $<24>$. The set $L_s = \{<2>, <3>, <6>, <12>, <18>, <24>\}$ is a poset which is not a $(L_s, \wedge)$ or $(L_s, \vee)$ whose Hasse diagram is given in Figure 1.

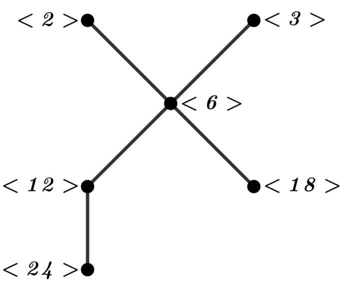

Figure 1. *Hasse diagram of a ring of order* 72.

In this section, some results based on the S-meet semilattice and S-join semilattice are discussed. If $\mathfrak{R}$ is a ring of order p^t, p^tq, p^tqr and p^tq^sr , $t, s \geq 1$ where p, q, r are distinct primes and $t, s \in \mathbb{Z}^+$, then the collection L_s is a S-meet semilattice. Figures 2 - 4 give the generalized Hasse diagram of the S-meet semilattice $(L_s, \wedge)$ of the ring of order p^t, p^tq, p^tqr and p^tq^tr.

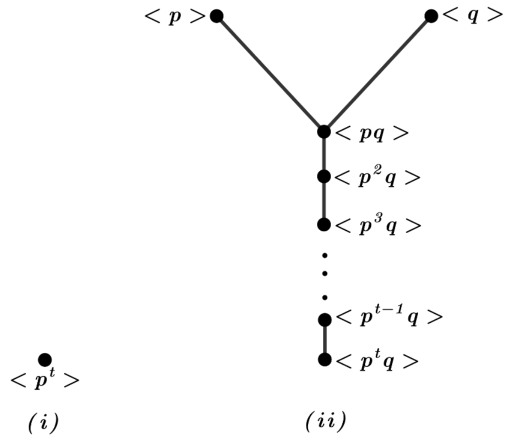

Figure 2. *S-Meet Semilattice of a ring of order* (i) p^t (ii) p^tq.

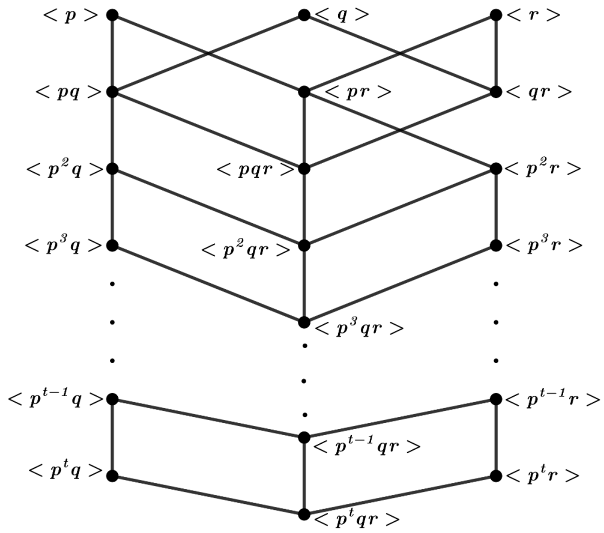

Figure 3. *S-Meet Semilattice of a ring of order* p^tqr.

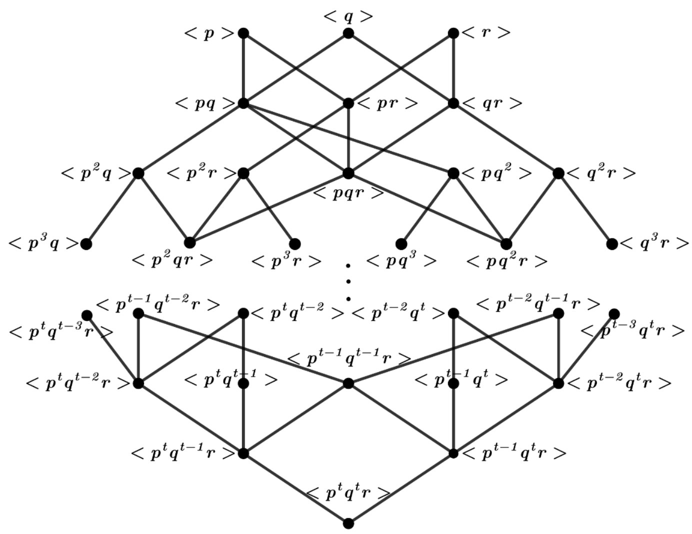

Figure 4. *S-Meet Semilattice of a ring of order* p^tq^tr.

Theorem 2.2. A maximal chain of a poset L_s has exactly one

I_p.

Proof Let $\mathfrak{R}$ be a finite ring and C be the maximal chain of L_s. Every I_s is contained in I_p and hence every maximal chain in L_s has I_p. Let P and Q be the two prime ideals of $\mathfrak{R}$. The ideals P and Q are elements of L_s. If the maximal chain C of L_s has both prime ideals P and Q, then they are comparable by the $\subseteq$ relation of L_s. This means that either P is in Q or Q is in P in a ring $\mathfrak{R}$. This is not possible. Thus the maximal chain has exactly one I_p.

Corollary 2.3. The number of maximal chain of L_s is greater than or equal to the number of I_p and they are equal only if the order of the ring $\mathfrak{R}$ is $p^t q$.

Theorem 2.4. The ideal P is I_p if and only if P is the maximal element of the poset L_s.

Proof Let P be the ideal of $\mathfrak{R}$, Assume that P is I_p. Since every I_p is I_s, P is an element of L_s. Let C be the maximal chain with the prime P. If I is an element of C then $I = I'_p$ by Theorem 2.2. This means that $I \subset P$ and hence $I \wedge P = I\ \forall$ I in C. Therefore, P is the maximal element of C and hence it is the maximal element of L_s.

Conversely assume that P is the maximal element of L_s. Let C be the maximal chain with the maximal element P. If I is any element in the chain C, then $I \wedge P = I$. This implies that the S-prime ideal $I \subseteq P$. Now, $I = I'_p$ which is a subset of P in $\mathfrak{R}$. It gives that P is I_p. Thus the maximal element P of L_s is I_p.

Corollary 2.5. The ideal I is the trivial ideal of a ring $\mathfrak{R}$ if and only if I is the minimal element of L_s.

Theorem 2.6. If $\mathfrak{R}$ is a local ring, then the set $(L_s, \vee, \wedge)$ is a lattice.

Proof Suppose a ring $\mathfrak{R}$ is local. It has a unique S-prime ideal of $\mathfrak{R}$. The poset L_s has a unique element. Obviously, $(L_s, \wedge, \vee, \subseteq)$ is a lattice.

Theorem 2.7. If $\mathfrak{R}$ is a ring of order pq, then $I \wedge J$ is the minimal element of L_s for every $I, J \in L_s$.

Proof Let $\mathfrak{R}$ be a ring of order pq. It has three S-prime ideals I_p, I_q and the trivial ideal. The ideals I_p and I_q are prime ideals generated by p and q respectively. Since the trivial ideal is contained in both I_p and I_q, the poset $(L_s, \wedge)$ is the meet semilattice and the trivial ideal is the minimal element of L_s. If I is in L_s, then I is I_s and I is either I_p or trivial ideal of $\mathfrak{R}$.

Let I and J be in L_s. Then either both are prime or one is prime and the other is trivial. If $I = I_p$ and $J = I_q$, then $I \wedge J$ is the trivial ideal. If I is I_p and J is the trivial ideal, then $I \wedge J$ is the trivial ideal. Since the trivial ideal of $\mathfrak{R}$ is the minimal element of L_s, $I \wedge J$ is the minimal element of L_s for all I and J in L_s.

Theorem 2.8. Let $L_{s'}$ be the proper subset of $(L_s, \wedge, \subseteq)$. If $L_{s'}$ contains the minimal element of L_s, then $(L_{s'}, \wedge, \subseteq)$ is an S'-meet semilattice.

Proof Let $L_{s'}$ be the proper subset of L_s. Every subset of a poset is a poset and hence $(L_{s'}, \subseteq)$ is a partially ordered set. Let I and J be any two elements of $L_{s'}$. Now, I and J are elements of the L_s. Since $(L_s, \wedge, \subseteq)$ is an S-meet semilattice, $\exists K_1 \in L_s \ni I \wedge J = K_1$. If $K_1 \in L_{s'}$, then $(L_{s'}, \wedge, \subseteq)$ is an S'-meet semilattice. If not, let C be the maximal chain in L_s with the element K_1. Remove the element K_1 from C, $\exists K_2 \subseteq K_1$ in $L_s \ni I \wedge J = K_2$.

If $K_2 \in L_{s'}$ then $(L_{s'}, \wedge, \subseteq)$ is an S'-meet semilattice. Otherwise, the same process is repeated until $I \wedge J$ is the minimal element. Since the minimal element is in $L_{s'}$, $(L_{s'}, \wedge, \subseteq)$ is an S'-meet semilattice.

Corollary 2.9. The I_p need not be a maximal element of S'-meet semilattice $(L_{s'}, \wedge)$.

3. Generalization of an S-prime Ideal

Theorem 3.1. Let $\mathfrak{R}$ be a ring of order n. If $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} ... p_k^{\alpha_k}$ where $\alpha_i \geq 1$ and p_i are distinct primes, $i = 1, 2, 3, ..., k$ then the cardinality of the partially ordered set $(L_s, \subseteq)$ is $\tau(n) - \prod_{i=1}^{k} \alpha_i$, where $\tau(n)$ is the number of divisors of n.

Proof Let I'_s be the non S-prime ideal of $\mathfrak{R}$ which is of order n and $\tau(n)$ be the number of divisors of n. It is to be noted that the number of ideals of a ring of order n is same as the number of divisors of n. Since I_s are proper ideals of $\mathfrak{R}$, $\mid L_s \mid < \tau(n)$. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} ... p_k^{\alpha_k}$ where $p_1, p_2, ..., p_k$ are different primes and α_i, $i = 1, 2, 3, ..., k$ are integers greater than or equal to 1. Let I be an ideal of $\mathfrak{R}$ and $I = < p_1^{\beta_1} p_2^{\beta_2} p_3^{\beta_3} ... p_k^{\beta_k} >$ where $0 \leq \beta_i \leq \alpha_i$, $i = 1, 2, 3, ..., k$. Let $S_1, S_2, S_3, ..., S_k$ be the complements of $< p_1 >, < p_2 >, < p_3 >, ..., < p_k >$ respectively. Then $S_1, S_2, S_3, ..., S_k$ are the multiplicative closed subsets of $\mathfrak{R}$ disjoint from I.

Case (i): $\beta_i \neq 1$ for all i

Now, I is I'_p. Assume that $s : I$ is prime for some $s \in S_i$, $i = 1, 2, 3, ..., k$. Let $s : I = < p_1 >$. Now, $\beta_1 \neq 0$ and $s = p_1^{\beta_1 - 1} p_2^{\beta_2} p_3^{\beta_3} ... p_k^{\beta_k}$. The element s becomes an element of all I_p and hence it is a nilpotent element. This means that s is not an element of S which is a contradiction. Therefore, $s : I$ is I'_p for all $s \in S_i$, $i = 1, 2, 3, ..., k$. That is, $I = I'_s$.

Case (ii): $\beta_i = 1$ for some i

Let $\beta_1 = 1$ and $I = < p_1 p_2^{\beta_2} p_3^{\beta_3} ... p_k^{\beta_k} >$ where $0 \leq \beta_i \leq \alpha_i$, $i = 2, 3, ..., k$. Then $\exists s \in S_1 \ni s : I = < p_1 >$. The ideal I is I_s. The I'_s of $\mathfrak{R}$ are $\mathtt{R}, < p_1^{\beta_1} >, < p_2^{\beta_2} >, < p_3^{\beta_3} >, ..., < p_k^{\beta_k} >, < p_1^{\beta_1} p_2^{\beta_2} >, < p_1^{\beta_1} p_3^{\beta_3} >, ..., < p_1^{\beta_1} p_k^{\beta_k} >, < p_1^{\beta_1} p_2^{\beta_2} p_3^{\beta_3} >, < p_1^{\beta_1} p_2^{\beta_2} p_4^{\beta_4} >, ..., < p_1^{\beta_1} p_2^{\beta_2} p_3^{\beta_3} ... p_k^{\beta_k} >$ where $0 \leq \beta_i \leq \alpha_i$ and $\beta_i \neq 1$. The number of I'_s is $1 + (\alpha_1 - 1) + (\alpha_2 - 1) + (\alpha_3 - 1) + ... + (\alpha_k - 1) + (\alpha_1 - 1)(\alpha_2 - 1) + (\alpha_1 - 1)(\alpha_3 - 1) + ... + (\alpha_1 - 1)(\alpha_k - 1) + (\alpha_1 - 1)(\alpha_2 - 1)(\alpha_3 - 1) + ... + (\alpha_1 - 1)(\alpha_2 - 1)...(\alpha_k - 1)$. Simplifying this, the number of $I'_s = \alpha_1 \alpha_2 \alpha_3 ... \alpha_k$.

$$\begin{aligned} \mid L_s \mid &= \text{Number of ideals of } \mathfrak{R} - \text{ Number of } I'_s \\ &= \tau(n) - \prod_{i=1}^{k} \alpha_i. \end{aligned}$$

Hence the cardinality of the partially ordered set $(L_s, \subseteq)$ is $\tau(n) - \prod_{i=1}^{k} \alpha_i$.

Example 3.2. Let $\mathfrak{R} = \mathbb{Z}_{180}$, $n = 180 = 2^2.3^2.5$ and $\tau(n) = 18$. The 18 ideals are generated by the divisors of 180. The non S-prime ideals are $\mathfrak{R}$, $< 4 >$, $< 9 >$, and $< 36 >$.

The number of I_s' is 4 which is the product $\alpha_1\alpha_2\alpha_3$. There are 14 I_s of $\mathfrak{R}$ and they are $< 2 >, < 3 >, < 5 >, < 6 >, < 10 >, < 12 >, < 15 >, < 18 >, < 20 >, < 30 >, < 45 >, < 60 >, < 90 >$ and $< 180 >$. Let L_s be the collection of I_s. $\mid L_s \mid = \tau(n) - \alpha_1\alpha_2\alpha_3 = 18 - 4 = 14$. The Hasse diagram for a poset $L_s = \{< 2 >, < 3 >, < 5 >, < 6 >, < 10 >, < 12 >, < 15 >, < 18 >, < 20 >, < 30 >, < 45 >, < 60 >, < 90 >, < 180 >\}$ is shown in Figure 5.

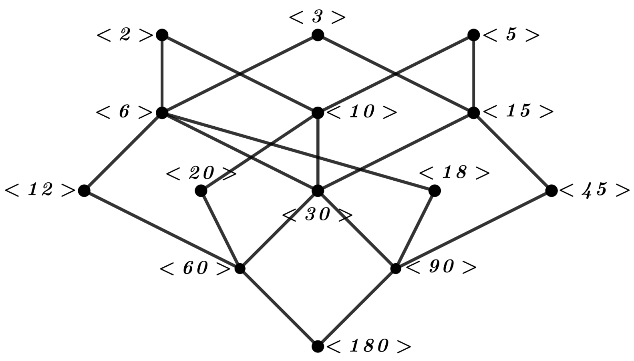

Figure 5. *S-Meet Semilattice of a ring of order* 180.

4. Ideals of an S-meet Semilattice $(L_s, \wedge)$

In this section, the ideals and the down-sets of S-meet semilttice L_s are defined. The subsequent theorems explain how the down-sets of L_s are related to the ideals, the prime ideals and the maximal ideals of L_s. Denote the maximal element of L_s and that of I as $\mathfrak{M}_k$ and $\mathfrak{E}_k$ respectively where $k = 1, 2, 3, ..., r-1$ and the minimal element of L_s as $\mathfrak{m}_1$.

Definition 4.1. Let L_s be an ordered set and M be the subset of a S-meet semilattice. The set M is said to be a *down-set* if a is in M, b is in L_s and $b \subseteq a$ then b is in M. It is denoted by $\downarrow M$.

Definition 4.2. A sub$S$-meet semilattice K is an *ideal* of L_s iff k is in K and a in L_s imply that $k \wedge a$ is in K.

The sub$S$-meet semilattice $\{0\}$ and L_s are the ideals of a S-meet semilattice $(L_s, \wedge)$.

Theorem 4.3. Let I be the subset of S-meet semilattice $(L_s, \wedge)$. Then I is an ideal of L_s iff I is

(i) the down-set of $\mathfrak{m}_1$ if I is trivial

(ii) the union of t down-sets of L_s if I is non-trivial where $t = 1, 2, 3, ..., r-1$ and $|L_s| = r$.

Proof Let a_1, a_2, a_3,... a_r be the elements of L_s. Let D_i be the down-sets of a_i of L_s, where $i = 1, 2, 3, ..., r-1$ and let $a_r = \mathfrak{m}_1$. Let I be an ideal of the S-meet semilattice $(L_s, \wedge)$ and $|L_s| = r$.

Case (i) I is trivial.

The ideal I contains only the $\mathfrak{m}_1$. The meet of every other element of L_s with $\mathfrak{m}_1$ is $\mathfrak{m}_1$ itself. Hence, I is the down-set of $\mathfrak{m}_1$. i.e, $I = D_r$.

Case (ii) I is non-trivial.

The ideal I has a maximal element as it is a sub S-meet semilattice. Let $a_i = \mathfrak{E}_i$ be unique, where $i = 1, 2, 3, ..., r-1$. If $a_j \subseteq a_i, i \neq j,\ i, j = 1, 2, 3, ..., r-1$ then $a_j \in I$. By the definition of down-sets, $I = D_i, i = 1, 2, 3, ..., r-1$. Let $a_i = \mathfrak{E}_i, a_j = \mathfrak{E}_j$ where $i \neq j$ and $i, j = 1, 2, 3, ..., r-1$. If $a_k \subseteq a_i$ or $a_k \subseteq a_j, i \neq j \neq k, i, j, k = 1, 2, 3, ..., r-1$ then $a_k \in I$. By the definition of down-sets, $I = D_i \cup D_j, i \neq j$ and $i, j = 1, 2, 3, ..., r-1$. The ideal may contain at most $(r-1)$ maximal elements. Continuing the above process, $I = \cup_{i=1}^{r-1} D_i$ where D_i is the down-set of a_i. Hence for the non-trivial case if the ideal I contains t maximal elements where $t = 1, 2, 3, ..., r-1$ then I is the union of t down-sets of L_s.

Definition 4.4. An ideal I of a S-meet semilattice is *prime* iff $a, b \in L_s$ and $a \wedge b \in I$ imply that a is in I or b is in I.

Theorem 4.5. Let L_s be the S-meet semilattice with $\mathfrak{M}_k$ where $k \geq 1$ and let I be the subset of $(L_s, \wedge)$. Then I is a prime ideal of L_s iff I is the union of t down-sets of $\mathfrak{M}_k$ where $t = 1, 2, 3, ..., k-1$ and $k > 1$.

Proof Let D_k be the down-sets of $\mathfrak{M}_k$. Let I be the prime ideal of L_s. If $k = 1$, then I is not a proper ideal. So, $k \neq 1$. Let b be any element of L_s such that $b \neq \mathfrak{M}_i$ for any $i = 1, 2, 3, ...k$ and $b = \mathfrak{E}_k$. Then $b = \mathfrak{M}_i \wedge \mathfrak{M}_j$ for $i \neq j$. Since I is prime , $\mathfrak{M}_i$ or $\mathfrak{M}_j$ is in I. Let us assume that $\mathfrak{M}_i \in I$. Since $b = \mathfrak{E}_k, \mathfrak{M}_i \subseteq b$. This is a contradiction. Hence $b = \mathfrak{M}_i$ for some i. For every $i, \mathfrak{E}_i = \mathfrak{M}_k$ for some k. If I has all $\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{M}_3, ..., \mathfrak{M}_k$ then I is not proper. So, the ideal I will have less than k elements. Let t be the number of $\mathfrak{E}_k$'s where $t = 1, 2, 3, ..., k-1$. Then by the proof of the first part of Theorem 4.3, I is the union of t down-sets D_i where $t = 1, 2, 3, ..., k-1$. So, the prime ideal I is the union of t down-sets of $\mathfrak{M}_k$.

Conversely, assume that I is the union of t down-sets of $\mathfrak{M}_k$ where $i = 1, 2, 3, ..., k-1$. Let $k > 1$. By Theorem 4.3, I is an ideal of L_s. Let $t = 1$. Then $I = D_i$ where D_i is the down-set of $\mathfrak{M}_i, i = 1, 2, 3, ..., k$. Let $a, b \in L_s$ such that $a \wedge b \in D_i$. This implies that either $a \subseteq \mathfrak{M}_i$ or $b \subseteq \mathfrak{M}_i$. By the definition of down-set, either $a \in D_i$ or $b \in D_i$ and hence D_i is a prime ideal. i.e., $I = D_i$ is a prime ideal. Let $t = 2$. Then $I = D_i \cup D_j$ where D_i and D_j are the down-sets of $\mathfrak{M}_i$ and $\mathfrak{M}_j$ respectively where $i \neq j$ and $i, j = 1, 2, 3, ..., k$. Let $a, b \in L_s$ such that $a \wedge b \in D_i \cup D_j$. Then $a \wedge b \in D_i$ or $a \wedge b \in D_j$ and $a \subseteq \mathfrak{M}_i$ or $b \subseteq \mathfrak{M}_i$ or $a \subseteq \mathfrak{M}_j$ or $b \subseteq \mathfrak{M}_j$. i.e., $a \subseteq \mathfrak{M}_i$ or $\mathfrak{M}_j$ or $b \subseteq \mathfrak{M}_i$ or $\mathfrak{M}_j$. By the definition of down-sets, $a \in D_i$ or D_j or $b \in D_i$ or D_j. i.e., $a \in D_i \cup D_j$ or $b \in D_i \cup D_j$. Hence, $D_i \cup D_j$ is a prime ideal of L_s. By continuing in the same way, it can be concluded that if I is the union of $(k-1)$ down-sets of $\mathfrak{M}_i, i = 1, 2, 3, ..., k$, then I is the prime ideal of L_s. Thus, if I is the union of t down-sets of $\mathfrak{M}_k$, then I is a prime ideal.

Definition 4.6. An ideal I of a S-meet semilattice is *maximal* if $I \neq L_s$ and the only ideals containing I are I and L_s.

Theorem 4.7. Let L_s be the S-meet semilattice with $\mathfrak{M}_k$ where $k \geq 1$ and I be the subset of $(L_s, \wedge)$. Then I is a maximal ideal of L_s iff it is the union of $(k-1)$ down-sets of $\mathfrak{M}_i, i = 1, 2, 3, ..., k$.

Proof Let D_k be the down-sets of $\mathfrak{M}_k, k = 1, 2, 3, ..., r-1$. Let I be the maximal ideal of L_s. Every maximal ideal is a prime ideal and hence I is prime. By Theorem 4.5, I is the union of t down-sets of $\mathfrak{M}_k$ where $t = 1, 2, 3, ..., k-1$ and $k > 1$. The prime ideal is maximal when $t = k-1$. Therefore, I is the union of $(k-1)$ down-sets of $\mathfrak{M}_i, i = 1, 2, 3, ..., k$.

Conversely, assume that I is the union of $(k-1)$ down-sets of $\mathfrak{M}_i, i = 1, 2, 3, ..., k$. By Theorem 4.3, I is an ideal of L_s. The ideal $I = \cup_{i=1}^{k-1} D_i$ and $L_s = \cup_{i=1}^{k} D_i$. That is I is the proper ideal of L_s. If $a \neq \mathfrak{M}_i$ is in the ideal D_i then $D_a \subseteq D_i$ where D_a is the down-set of a in L_s. Hence, $\cup_{a \in L_s} D_a = \cup_{i=1}^{k} D_i = L_s$ and $I = L_s - \mathfrak{M}_k$. Every proper ideal of L_s is contained in I and there is no proper ideal that contains I. If I is contained in any ideal I' of L_s, then either $I' = I$ or $I' = L_s$. So, I is a maximal ideal of L_s. Thus, the union of $(k-1)$ down-sets of $\mathfrak{M}_i, i = 1, 2, 3, ..., k$ is a maximal ideal.

Corollary 4.8. Every prime ideal of S-meet semilattice L_s is maximal if the ring is of order $p^t q$, where p and q are distinct primes and $t \geq 1$.

Example 4.9. Let $\mathtt{R} = \mathbb{Z}_{30}$ and the poset L_s has the elements are $< 2 >, < 3 >, < 5 >, < 6 >, < 10 >, < 15 >$ and $< 30 >$. Figure 6 shows a Hasse diagram of the S-meet semilattice $(L_s, \wedge)$.

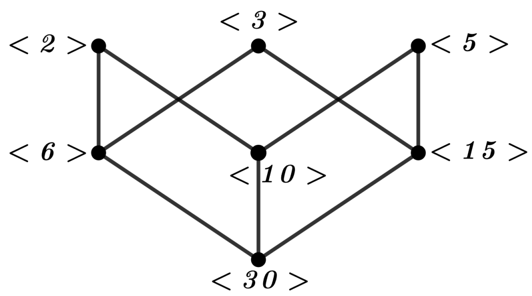

Figure 6. *S-Meet Semilattice of a ring of order* 30.

The ideals in terms of down-sets of S-meet semilattice are listed. $I_1 = \downarrow < 2 >$ (Prime), $I_2 = \downarrow < 3 >$ (Prime), $I_3 = \downarrow < 5 >$ (Prime), $I_4 = \downarrow < 6 >, I_5 = \downarrow < 10 >, I_6 = \downarrow < 15 >, I_7 = \downarrow < 30 >, I_8 = \downarrow < 2 > \cup \downarrow < 3 >$ (Prime and Maximal), $I_9 = \downarrow < 2 > \cup \downarrow < 5 >$ (Prime and Maximal), $I_{10} = \downarrow < 3 > \cup \downarrow < 5 >$ (Prime and Maximal), $I_{11} = \downarrow < 2 > \cup \downarrow < 15 >, I_{12} = \downarrow < 3 > \cup \downarrow < 10 >, I_{13} = \downarrow < 5 > \cup \downarrow < 6 >, I_{14} = \downarrow < 6 > \cup \downarrow < 10 >, I_{15} = \downarrow < 6 > \cup \downarrow < 15 >, I_{16} = \downarrow < 10 > \cup \downarrow < 15 >, I_{17} = \downarrow < 6 > \cup \downarrow < 10 > \cup \downarrow < 15 >, I_{18} = \downarrow < 2 > \cup \downarrow < 3 > \cup \downarrow < 5 >$.

5. Conclusion

In this paper, the number of S-prime ideals in the poset is generalized and the concept of S-meet semilattice is introduced whereas some properties of S-meet semilattice are studied with suitable examples. Finally, the results on ideals, prime ideals and maximal ideals of an S-meet semilattice are described in terms of the down-sets of S-meet semilattice and they are listed.

Abbreviations

$\mathfrak{R}$	Finite Commutative Ring with Unity
S_i	Multiplicative Subset of $\mathfrak{R}$
I_p	Prime Ideal of $\mathfrak{R}$
I'_p	Non-prime Ideal of $\mathfrak{R}$
I_s	S-prime Ideal of $\mathfrak{R}$
I'_s	Non S-prime Ideal of $\mathfrak{R}$
$\tau(n)$	Number of Divisors of n
L_s	Collection of All S-Prime Ideals of $\mathfrak{R}$
C	Maximal Chain of L_s
$\mathfrak{M}_k$	Maximal Element of L_s
$\mathfrak{C}_k$	Maximal Ideal of I of L_s
$\mathfrak{m}_1$	Minimal Element of L_s
D_i	Down-sets of L_s

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Conflicts of Interest

The authors declare no conflicts of interest.

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