



# Eigenvalue Intervals for Fractional Boundary Value Problems with Nonlinear Boundary Conditions

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**Abstract:** In presented paper, we study eigenvalue intervals for fractional boundary value problems with nonlinear boundary conditions. In this case, for the existence of at least one positive solution of the boundary value problem the new sufficient conditions are established. By means of example, the main results is illustrated. Finally, given comparsion obtained results with others.

**Keywords:** Fractional Differential Equation, Boundary Value Problem, Nonlinear Nonlocal Boundary Condition, Positive Solution, Fractional Green's Function, Guo–Krasnosel'skii Fixed Point Theorem

## 1. Introduction

Fractional differential equations have attracted great attention of mathematicians. It is caused both by the intensive development of the theory of fractional calculus itself and by the applications, see [1–4]. Recently, many scholars used techniques of nonlinear analysis (fixed-point theorems, Leray–Schauder theory, Adomian decomposition method, etc.) to deal with the existence of positive solutions of nonlinear boundary value problems of fractional differential equation, see [5–12].

Xu et al. [13] considered the existence of positive solutions for the following problem

$$D_{0+}^{\alpha}(u(t)) = f(u(t)) = 0, \quad 0 < t < 1,$$

$$u(0) = u(1) = u'(0) = u'(1) = 0,$$

where  $3 < \alpha \leq 4$  is a real number,  $D_{0+}^{\alpha}$  is the Riemann–Liouville fractional derivative. By using the properties of the Green function, they gave some multiple positive solutions for singular and nonsingular boundary value problems, and also they gave uniqueness of solution for singular problem by means of Leray–Schauder nonlinear alternative, Guo–Krasnosel'skii fixed point theorem on cones and a mixed monotone method.

Zhao et al. [14] studied the existence of positive solutions for the nonlinear fractional differential equation boundary value problem

$$D_{0+}^{\alpha}(u(t)) = \lambda f(u(t)) = 0, \quad 0 < t < 1,$$

$$u(0) + u'(0) = 0, \quad u(1) + u'(1) = 0,$$

where  $1 < \alpha \leq 2$  is a real number,  $D_{0+}^{\alpha}$  is the Caputo fractional derivative,  $\lambda > 0$  and  $f: [0, +\infty) \rightarrow [0, +\infty)$  is continuous. By the properties of the Green function and Guo–Krasnosel'skii fixed point theorem on cones, the eigenvalue intervals of the nonlinear fractional differential equation boundary value problem are considered, some sufficient conditions for the nonexistence and existence of at least one or two positive solutions for the boundary value problem are established.

Sun et al. [15] studied the existence of positive solutions for the nonlinear fractional differential equation boundary value problem

$$D_{0+}^{\alpha}(u(t)) = \lambda f(u(t)) = 0, \quad 0 < t < 1,$$

$$u(0) = u(1) = u'(0) = u'(1) = 0,$$

where  $3 < \alpha \leq 4$  is a real number,  $D_{0+}^{\alpha}$  is the Riemann–Liouville fractional derivative,  $\lambda > 0$  and  $f: (0, +\infty) \rightarrow (0, +\infty)$

is continuous. By the properties of the Green function and Guo–Krasnosel'skii fixed point theorem on cones, the eigenvalue intervals of the nonlinear fractional differential equation boundary value problem are considered, some sufficient conditions for the nonexistence and existence of at least one or two positive solutions for the boundary value problem are established.

There also results on fractional boundary value problem with integral boundary conditions, such as Vong [16] studied positive solutions of the nonlocal boundary value problem for a class of singular fractional differential equations with an integral boundary condition,

$${}^C D_{0^+}^\alpha(u(t)) + f(t, u(t)) = 0, \quad 0 \leq t \leq 1$$

$$u'(0) = \dots = u^{(n-1)}(0) = 0, \quad u(1) = \int_0^1 u(s) d\mu(s)$$

where  $n \geq 2$ ,  $\alpha \in (n-1, n)$  and  $\mu$  is a function of boundary variation. By the method of upper and lower solutions together with the Schauder fixed point theorem, the existence results for the boundary value problem are obtained.

Feng et al. [18] considered the existence of at least one positive solution of the fractional differential equation

$$D_{0^+}^\alpha(u(t)) + f(u(t)) = 0, \quad 0 < t < 1,$$

$$u(0) = 0, \quad u(1) = H_1(\phi(u)) + \int_E H_2(u(s)) ds,$$

where  $E \subset (0, 1)$  is some measurable set,  $1 < \alpha \leq 2$  is a real number,  $D_{0^+}^\alpha$  is the Riemann–Liouville fractional derivative,  $u(1) = H_1(\phi(u)) + \int_E H_2(u(s)) ds$  represents a nonlinear nonlocal boundary condition. By imposing some relatively mild structural conditions on  $f$ ,  $H_1$ ,  $H_2$  and  $\phi$ , some sufficient conditions for the existence of at least one positive solution for the boundary value problem are established.

Motivated by the above works, we discuss the existence of at least one positive solution of the boundary value problem

$$D_{0^+}^\alpha(u(t)) + \lambda f(u(t)) = 0, \quad 0 < t < 1, \quad (1)$$

$$u(0) = 0, \quad u(1) = H_1(\phi(u)) + \int_E H_2(u(s)) ds \quad (2)$$

where  $E \subset (0, 1)$  is some measurable set,  $1 < \alpha \leq 2$  is a real number,  $D_{0^+}^\alpha$  is the Riemann–Liouville fractional derivative,  $\lambda$  is a positive parameter and  $f: (0, +\infty) \rightarrow (0, +\infty)$  is continuous,  $\phi$  in (2) is a linear functional having the form

$$\phi(u) = \int_0^1 u(t) d\theta(t). \quad (3)$$

The rest of this paper is organized as follows. In section 2, we review some necessary preliminaries and new properties for Green's function. In section 3, we present the main results of this paper; also we give an illustrative example to support our new results.

## 2. Preliminaries

In this section, we give some lemmas to facilitate analysis of problem (1) and (2). And the background materials from fractional calculus theory can be found in [17].

Lemma 1 (following in [18]) Let  $u, h \in C[0, 1]$  and  $1 < \alpha \leq 2$ . The unique solution of problem

$$D_{0^+}^\alpha(u(t)) + h(u(t)) = 0, \quad 0 < t < 1, \quad (4)$$

$$u(0) = 0, \quad u(1) = H_1(\phi(u)) + \int_E H_2(u(s)) ds \quad (5)$$

is

$$u(t) = t^{\alpha-1} [H_1(\phi(u)) + \int_E H_2(u(s)) ds] + \int_0^1 G(t, s) h(s) ds,$$

Where

$$G(t, s) = \begin{cases} \frac{t^{\alpha-1}(1-s)^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 < s \leq t < 1, \\ \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 < t \leq s < 1. \end{cases} \quad (6)$$

Here  $G(t, s)$  is called the Green function of boundary value problem (4) and (5).

The following properties of the Green function different from [10] play important roles in this paper.

Lemma 2 The function  $G(t, s)$  defined by (6) satisfies the following conditions:

- (1)  $G(t, s) > 0$ , for  $t, s \in (0, 1)$ ;
- (2) There exists a positive constant  $\gamma \in (0, \gamma(t, s))$  satisfies  $(\alpha-1)\gamma \leq t^{\alpha-1}$  such that

$$(\alpha-1)\gamma t^{\alpha-1}(1-s)^{\alpha-1} \leq \Gamma(\alpha)G(t, s) \leq t^{\alpha-1}(1-s)^{\alpha-1},$$

where  $t, s \in (0, 1)$  and  $\gamma(t, s) = \min\{s(1-t), t(1-s)\}$ .

Proof. By the expression of  $G(t, s)$ , it is clear that  $G(t, s) > 0$  and  $\Gamma(\alpha)G(t, s) \leq t^{\alpha-1}(1-s)^{\alpha-1}$  for  $t, s \in (0, 1)$ .

When  $s \leq t$ , we have

$$\begin{aligned} \Gamma(\alpha)G(t, s) &= t^{\alpha-1}(1-s)^{\alpha-1} - (t-s)^{\alpha-1} \\ &= (\alpha-1) \int_{t-s}^{t-s} x^{\alpha-2} dx \\ &\geq (\alpha-1) t^{\alpha-2}(1-s)^{\alpha-2} s(1-t) \\ &\geq (\alpha-1) t^{\alpha-2}(1-s)^{\alpha-1} s(1-t) \end{aligned}$$

On the other hand, when  $s \geq t$ , we have

$$\begin{aligned} \Gamma(\alpha)G(t, s) &= t^{\alpha-1}(1-s)^{\alpha-1} \\ &= (\alpha-1) \int_0^{t-s} x^{\alpha-2} dx \end{aligned}$$

$$\begin{aligned} &\geq (\alpha-1) t^{\alpha-2} (1-s)^{\alpha-2} t(1-s) \\ &\geq (\alpha-1) t^{\alpha-1} (1-s)^{\alpha-1} t(1-s). \end{aligned}$$

Let  $\gamma(t, s) = \min\{t(1-s), s(1-t)\}$ , then we can find a constant  $\gamma \in (0, 1)$  satisfies  $(\alpha-1)\gamma \leq t^{\alpha-1}$  such that  $\Gamma(\alpha)G(t, s) \geq (\alpha-1)\gamma t^{\alpha-1} (1-s)^{\alpha-1}$ . The proof is complete.

If we set  $q(t) = t^{\alpha-1}$ ,  $k(s) = (1-s)^{\alpha-1}$ , then  $(\alpha-1)\gamma q(t)k(s) \leq \Gamma(\alpha)G(t, s) \leq q(t)k(s)$ .

Lemma 3 (following in [19]) Let  $X$  be a Banach space, and let  $P \subset X$  be a cone in  $X$ . Assume  $\Omega_1, \Omega_2$  are open subsets of  $X$  with  $0 \in \Omega_1 \subset \Omega_1 \subset \Omega_2$ , and let  $S: P \rightarrow P$  be a completely continuous operator such that, either

(A<sub>1</sub>)  $\|Sw\| \leq \|w\|$ ,  $w \in P \cap \partial\Omega_1$ ,  $\|Sw\| \geq \|w\|$ ,  $w \in P \cap \partial\Omega_2$ , or

(A<sub>2</sub>)  $\|Sw\| \geq \|w\|$ ,  $w \in P \cap \partial\Omega_1$ ,  $\|Sw\| \leq \|w\|$ ,  $w \in P \cap \partial\Omega_2$ .

Then  $S$  has a fixed point in  $P \cap (\bar{\Omega}_2 \setminus \Omega_1)$ .

For convenience, we give some assumptions which will be used in the next section.

(C<sub>1</sub>):  $H_1: [0, +\infty) \rightarrow [0, +\infty)$  and  $H_2: [0, +\infty) \rightarrow [0, +\infty)$  be real-valued, continuous functions.

(C<sub>2</sub>): For  $u \in C[0, 1]$ , there exists a constant  $M_1 \in [0, 1]$  satisfies  $|\phi(u)| \leq M_1 \|u\|$ .

(C<sub>3</sub>): For each given  $\varepsilon > 0$ , there are  $M_\varepsilon > 0$  and a constant  $M_2$  such that  $|H_1(x) - M_2 x| < \varepsilon M_2 x$ , whenever  $x > M_\varepsilon$ .

(C<sub>4</sub>): There exists a function  $F: [0, +\infty) \rightarrow [0, +\infty)$  satisfying  $F(x) \leq M_3 x$ , for some  $M_3 \geq 0$ , having the property that for each  $\varepsilon > 0$  given, there exists  $M_\varepsilon > 0$  such that  $|H_2(x) - F(x)| < \varepsilon F(x)$ , whenever  $x > M_\varepsilon$ .

### 3. Main Results

In this section, we will establish the existence of positive solution for boundary value problem (1) and (2) by fixed point theorem in [19]. As an application, an example is given to illustrate the main results.

Lemma 4. Let Banach space  $E = C[0, 1]$  be endowed with the norm  $\|u\| = \sup |u(t)|$ . Define the cone  $P \subset E$  by  $P = \{u \in E: u(t) \geq \gamma(\alpha-1)\|u\|, t \in [0, 1]\}$  and the operator  $A_\lambda: P \rightarrow E$  by

$$(A_\lambda u)(t) = t^{\alpha-1} [H_1(\phi(u)) + \int_E H_2(u(s))ds] + \lambda \int_0^1 G(t, s) f(u) ds \quad (7)$$

Then,  $A_\lambda: P \rightarrow P$ .

Proof. By lemma 2.4 (see [19]), if  $u \in P$ , we have

$$\begin{aligned} \|(A_\lambda u)(t)\| &= \sup \left| t^{\alpha-1} [H_1(\phi(u)) + \int_E H_2(u(s))ds] + \lambda \int_0^1 G(t, s) f(u) ds \right| \\ &\leq [H_1(\phi(u)) + \int_E H_2(u(s))ds] + \frac{\lambda}{\Gamma(\alpha)} \int_0^1 q(t)k(s)f(u)ds. \end{aligned}$$

On the other hand,

$$\begin{aligned} (A_\lambda u)(t) &= t^{\alpha-1} [H_1(\phi(u)) + \int_E H_2(u(s))ds] + \lambda \int_0^1 G(t, s) f(u) ds \\ &\geq t^{\alpha-1} [H_1(\phi(u)) + \int_E H_2(u(s))ds] + \frac{\lambda(\alpha-1)\gamma}{\Gamma(\alpha)} \int_0^1 q(t)k(s)f(u)ds \\ &\geq \gamma(\alpha-1) \|(A_\lambda u)(t)\|. \end{aligned}$$

Thus,  $A_\lambda: P \rightarrow P$ .

Lemma 5 The operator  $A_\lambda$  in lemma 4 is completely continuous.

The proof of this lemma is similar to lemma 3.1 in [18], we omit it.

For convenience, we denote

$$M_4 = \frac{1}{\Gamma(\alpha)} \int_0^1 k(s)ds, \quad M_5 = \frac{(\alpha-1)^2 \gamma^2}{\Gamma(\alpha)} \int_0^1 k(s)ds,$$

$$F_0 = \limsup_{u \rightarrow 0^+} \frac{f(u)}{u}, \quad f_\infty = \limsup_{u \rightarrow \infty} \frac{f(u)}{u}.$$

Theorem 1 Assume (C<sub>1</sub>) – (C<sub>4</sub>) hold, if there exists  $s \in (0, 1)$  such that

$q(l)f_\infty(\alpha-1)^2 \gamma^2 (1-M_1 M_2 - M_3 m(E)) > F_0$ , then for each

$$\lambda \in \left( \frac{1}{q(l)f_\infty M_5}, \frac{1-M_1 M_2 - M_3 m(E)}{F_0 M_4} \right), \quad (8)$$

the boundary value problem (1) and (2) has at least one positive solution. Here we impose  $(q(l)f_\infty M_5)^{-1} = 0$ , if  $f_\infty = +\infty$  and  $(F_0 M_4)^{-1} = \infty$ , if  $F_0 = 0$ .

Proof. Let  $\lambda$  be in (8), we can find a very small  $\varepsilon$  satisfy

$$\frac{1}{q(l)(f_\infty - \varepsilon)M_5} \leq \lambda \leq \frac{1-M_1 M_2 - M_3 m(E)}{(F_0 + \varepsilon)M_4}. \quad (9)$$

By the definition of  $F_0 = 0$ , there exists  $r_1 > 0$  such that

$$f_u \leq (F_0 + \varepsilon)u, \quad 0 < u < r_1. \quad (10)$$

So, if  $u \in P$  with  $\|u\| = r_1$ , then by (9) and (10), we have

$$\begin{aligned} \|(A_\lambda u)(t)\| &\leq [H_1(\phi(u)) + \int_E H_2(u(s))ds] + \frac{\lambda}{\Gamma(\alpha)} \int_0^1 q(t)k(s)f(u)ds + \\ &\leq |H_1(\phi(u)) - M_2(\phi(u))| + |M_2(\phi(u))| + \int_E |H_2(u) - F(u)|ds + \int_E |F(u)|ds \end{aligned}$$

$$\frac{\lambda}{\Gamma(\alpha)} \int_0^1 k(s)(F_0 + \varepsilon)r_1 ds$$

$$\leq M_2 \varepsilon \phi(u) + M_1 M_2 \|u\| + (1 + \varepsilon) M_3 m(E) \|u\| + \lambda (F_0 + \varepsilon) M_4 \|u\|$$

$$\leq (M_1 M_2 (1 + \varepsilon) + (1 + \varepsilon) M_3 m(E) + \lambda (F_0 + \varepsilon) M_4) \|u\| \leq \|u\|.$$

Hence, if we choose  $\Omega_1 = \{u \in E: \|u\| < r_1\}$ , then

$$\|A_\lambda u\| \leq \|u\|, u \in P \cap \partial\Omega_1. \quad (11)$$

From the definition of  $f_\infty$ , we can find  $r_3 > 0$  such that

$$f_u \geq (f_\infty - \varepsilon)u, \text{ for } u \geq r_3. \quad (12)$$

So, if  $u \in P$  with  $\|u\| = r_2 = \max\{2r_1, r_3\}$ , then by (9) and (12), we have

$$\begin{aligned} \|(A_\lambda u)(t)\| &\geq (A_\lambda u)(l) \geq \lambda \int_0^1 G(l, s) f(u(s)) ds \\ &\geq \frac{\lambda q(l)(\alpha-1)\gamma}{\Gamma(\alpha)} \int_0^1 k(s) f(u(s)) ds \\ &\geq \frac{\lambda q(l)(\alpha-1)^2 \gamma^2}{\Gamma(\alpha)} \int_0^1 k(s) (f_\infty - \varepsilon) \|u\| ds \\ &= \lambda q(l) M_5 (f_\infty - \varepsilon) \|u\| \geq \|u\|. \end{aligned}$$

Thus, if we set  $\Omega_2 = \{u \in E : \|u\| < r_2\}$ , then

$$\|A_\lambda u\| \geq \|u\|, u \in P \cap \partial\Omega_2. \quad (13)$$

Now, from (11), (13) and Lemma 4, we guarantee that  $A_\lambda$  has a fix point  $u \in P \cap (\bar{\Omega}_2 \setminus \Omega_1)$  with  $r_1 \leq \|u\| \leq r_2$ , and clearly  $u$  is a positive solution of (1) and (2). The proof is complete.

Now, we will present an example to illustrate the main result.

Example Let  $f(u) = u^a$ ,  $H_1(x) = xe^x + x$ ,  $H_2(x) = 3u + u^{\frac{1}{2}}$ ,

$\phi(u) = \frac{u}{2}$ , consider the boundary value problem

$$D_{0^+}^{\frac{3}{2}}(u(t)) + \lambda u^a = 0, \quad 0 < t < 1 \quad (14)$$

$$u(0) = 0, u(1) = \frac{u}{3} e^{-\frac{u}{3}} + \frac{u}{3} + \int_{[\frac{1}{10}, \frac{1}{5}]} (3u + u^{\frac{1}{2}}) ds. \quad (15)$$

It is easy to see  $F_0 = 0$ ,  $f_\infty = +\infty$ ,  $m(E) = \frac{1}{10}$  and

$$q(l) f_\infty (\alpha-1)^2 \gamma^2 (1 - M_1 M_2 - M_3 m(E)) > F_0$$

Thus, by Theorem 1, the boundary value problem (14) and (15) has a positive solution for each  $\lambda \in (0, +\infty)$ .

## 4. Conclusion

In this work by means of preliminaries lemma1, 2, 3, 4, 5 proves the Theorem1, which is established the existence of positive solution for boundary value problem (1), (2) with

fixed point theorem in [19]. This is a new sufficient conditions for the existence of at least one positive solution for the boundary value problem of (1), (2). which in case of  $\lambda=1$ , for boundary value problem (1), (2) including the result of [18].

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