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Analysis on the Properties of a Permutation Group

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Abstract: The structures of the subgroups play an important role in the study of the nature of symmetric groups. We calculate the 11300 subgroups of the permutation group S_7 by group-theoretical approach. The analytic expressions for the numbers of subgroups are obtained. The subgroups of the permutation group S_7 are all represented in an alternative way for further analysis and applications.

Keywords: Permutation Group, Subgroup, Lagrange's Theorem, Cayley's Theorem

1. Introduction

The study of permutation groups is of significance for the development of group-theoretical approach [1, 2, 3]. On one hand, each row in the multiplication table of a finite group shows a permutation of group elements such that every finite group is a subgroup of a permutation group. On the other, the analysis of the tensor indices requires the theory of Young's permutation operators. The survey of the structures of the subgroups is meaningful to understand the properties of the permutation groups. Many researches have discussed the computing the subgroups of permutation groups [4]. Since the order of the permutation group S_n, $g = |S_n| = n!$, increases rapidly with the increase of the number n, it is generally considered that it would be quite difficult to calculate the subgroups of a permutation group by group-theoretical method when the number n is getting larger. The subgroups of S_7 mainly come from the computer program [5, 6, 7]. However, we believe that if we can calculate the subgroups of S_n through the group-theoretical method, then we can not only be independent of the computer program, but also we can study the properties of the subgroups of S_n with analytic methods, provide the explanation for the pretty huge numbers of subgroups of different orders. It also might be possible that the research indicates some useful information for the simplification of the computer programs.

In the following section, we will make preliminary sketches of various non-isomorphic groups for a few finite groups. Then, we will analyze the properties of the group S_7 and calculate the subgroups of the permutation group S_7 in section 3. In this section, the analytic expressions for the numbers of subgroups are also presented by the numbers of group elements in the classes. We discuss its possible applications with the results and represent the subgroups in an alternative way in section 4. We conclude in the final section after pointing out various directions for future investigations.

2. Preliminaries

In the finite group, one may try to know how many non-isomorphic groups of a given order of n. Generally, the answer to this question is not yet given. Here, we present all non-isomorphic groups of orders less than 14. That is what we need to calculate the subgroups of the group S_7.

The Lagrange's theorem [8, 9] states that for any finite group G, the order of every subgroup H of G divides the order of G. It implies that every group of prime order is cyclic. If the order of the finite group is a prime number, g = 2, 3, 5, 7, 11, or 13, it can only be the cyclic group, denoted by $C_n = \{E, \mathrm{R}, \cdots, R^{n-1}\}$, where R is a generator, $R^n = E$, and E is the identity.

If the order of a finite group is $g = 2n$ (n = 2, 3, 5, 7), where n is a prime number, it can only be either the cyclic group C_{2n} or the dihedral group D_n.

If the order of the group is 8, there are five non-isomorphic groups [10, 11]. The first is the cyclic group C_8. The second is the dihedral group D_4, where two generators can be denoted by R and S_0, satisfying $R^4 = S_0^2 = E$. The third is an Abelian group, $C_{4h} = C_4 \times C_2$, where the generators satisfy $R^4 = S_0^2 = E$ and $RS_0=S_0R$. The fourth is also a commutative group, $D_{2h} = V_4 \times C_2$, and the generators satisfy $R_1^2 = R_2^2 = S_0^2 = E$. The fifth is a quaternion group Q_8, the generators satisfy $R_1^4 = S_0^4 = E$.

There are two non-isomorphic groups of order 9. One is the cyclic group C_9. The other is a direct product of two cyclic groups, $C_3 \times C_3$, where the generators satisfy $R^3 = S^3 = E$,, and $RS=SR$.

If the order of the group is 12, there are five non-isomorphic groups. The first is a cyclic group C_{12}. The second is the dihedral group D_6 where the generators satisfy $R^6 = S_0^2 = E$. The third group is denoted by T or A_4 where the generators T^2 and R_1 satisfy $(T^2)^2 = R^3 = E$. The fourth is denoted by Q, and the generators is chosen as R and S, satisfying $R^6 = S^4 = E$. The fifth is the group C_{6h} and the generators satisfy $R^6 = S_0^2 = E$ and $RS_0=S_0R$.

3. The Properties of the Subgroups of S_7

Before we get started with the subgroups of S_7, we need to be clear about the number of group elements in S_7, $g = |S_7| = 7! = 5040$, and the number of its conjugate classes, g_C=15. The 15 classes [α], the number of elements $n_{[\alpha]}$ in the classes, the order of the elements and one representative element in each class are presented in Table 1.

Table 1. Group elements and classes of the group S_7.

class [α]	number of elements n [α] in the class [α]	one element in the class	order of elements
[1111111]	1	(1) (2) (3) (4) (5) (6) (7)	1
[211111]	21	(12)	2
[22111]	105	(12) (34)	2
[2221]	105	(12) (34) (56)	2
[31111]	70	(123)	3
[331]	280	(123) (456)	3
[4111]	210	(1234)	4
[421]	630	(1234) (56)	4
[511]	504	(12345)	5
[3211]	420	(123) (45)	6
[322]	210	(123) (45) (67)	6
[61]	840	(123456)	6
[7]	720	(1234567)	7
[52]	504	(12345) (67)	10
[43]	420	(1234) (567)	12

In the following, we will analyze the subgroups of S_7 of different orders for non-isomorphic subgroups. Although there are a considerable number of subgroups, it will be shown that the number can be connected with the numbers of group elements in the classes by analytical expressions.

As can be seen, the subgroups of order 2 will take the form of {E, (12)} or {E, (12) (34)}, or {E, (12) (34) (56)}. Where there is an element of order 2, there is a subgroup of order 2. According to the number of elements in each classes in Table 1, if we denote the total number of cyclic subgroups of order 2 by N (2), then it is equal to the number of all elements of order 2,

$$N(2) = n_{[2]} + n_{[22]} + n_{[222]} = 231. \qquad (1)$$

Therefore, there are 231 cyclic subgroups of order 2 in the permutation group S_7.

The subgroups of order 3 will look like {E, (123), (132)} or {E, (123) (456), (132) (465)}. If the total number of subgroups of order 3 is denoted by N(3), then the connection between N(3) and the number of elements in the classes is

$$N(3) = \frac{1}{2}\{n_{[3]} + n_{[33]}\} = 175. \qquad (2)$$

That is, there are 175 cyclic subgroup of order 3 in the group S_7.

The subgroups of order 4 of the group S_7 need to be analyzed carefully. According to the preliminaries, there are two non-isomorphic groups of order 4, a cyclic group and an inversion group. Notice that the table 1 indicates that elements of order 4 are included in the class [4] and the class [42]. The cyclic subgroups are like {E, (1234), (13) (24), (1432)} or {E, (1234) (56), (13) (24), (1432) (56)}. The number of the cyclic subgroup of order 4 is half of the number of all elements of order 4. The inversion subgroups might take the form of {E, (12), (34), (12) (34)} or {E, (12) (34), (13) (24), (14) (23)}. Since three elements, (56), (57) and (67), are all the elements of order 2 in S_7, there will be other forms of inversion subgroups of order 4, such as {E, (12), (34) (56), (12) (34) (56)}, {E, (12) (34), (12) (56), (34) (56)} or {E, (12) (34), (13) (24) (56), (14) (23) (56)}. The number of inversion subgroup of order 4 is analyzed to be $\left\{n_{[22]} + \frac{1}{3}n_{[22]} + n_{[222]} \times 3 + n_{[22]} + n_{[22]} \times 3\right\}$. Therefore, the analytical expression between the total number of the subgroups of order 4 and the number of elements in the classes is

$$N(4) = \frac{1}{2}\{n_{[4]} + n_{[42]}\} + \left\{n_{[22]} + \frac{1}{3}n_{[22]} + 3n_{[222]} + n_{[22]} + 3n_{[22]}\right\} = 420 + 875 = 1295. \qquad (3)$$

The cyclic subgroups of order 5 are like {E, (12345), (13245), (14235), (15234)} and the total number of subgroups is calculated to be

$$N(5) = \frac{1}{4}n_{[2]} = 126. \qquad (4)$$

There are two non-isomorphic groups of order 6. Through careful analysis, it is found that the generators in the 735 cyclic subgroups can be chosen like {(123456)} or {(123) (45)} or {(123) (45) (67)}. There are 910 dihedral subgroup D_3, the generators can be chosen as {(123), (23)} or {(123), (23) (45)} or {(123), (23) (45) (67)} or {(123) (456), (23) (56)} or {(123) (456), (15) (24) (36)}. Therefore, the total number of the subgroups of order 6 in S_7 is verified to be

$$N(6) = \frac{1}{2}\{n_{[6]} + n_{[322]} + n_{[32]}\} + \left\{\frac{1}{2}n_{[3]} + \frac{1}{2}n_{[3]} \times 6 + \frac{1}{2}n_{[3]} \times 3 + \frac{1}{2}n_{[3]} \times 3 + \frac{1}{2}n_{[33]}\right\} = 735 + 910 = 1645. \qquad (5)$$

The cyclic subgroups of order 7 are like {E, (1234567), (1357246), (1473625), (1526374), (1642753), (1765432)} and the total number of subgroups is

$$N(7) = \frac{1}{6} n_{[7]} = 120. \qquad (6)$$

The preliminaries indicate that there are five non-isomorphic groups of order 8. It can be found that there is no cyclic subgroup of order 8 or subgroup which is isomorphic to the quaternion group. There are 1050 dihedral subgroup D_4, the generators can be chosen like {(1234), (13)} or {(1234), (13) (56)} or {(1234) (56), (12) (34)} or {(1234) (56), (12) (34) (56)}. There are 315 subgroups of order 8 which is isomorphic to C_{4h}, the generators can be chosen like {(1234) (56), (56)}. There are 210 subgroups of order 8 which is isomorphic to D_{2h}, the generators can be chosen like {(12) (34), (12) (56), (12)} or {(12) (34), (13) (24) (56), (56)}. The connection between the total number of subgroups $N(8)$ and the numbers of elements in the classes n_α are found to be

$$\mathrm{N}(8) = \left\{\frac{1}{2} n_{[4]} + \frac{1}{2} n_{[4]} \times 3 + \frac{1}{2} n_{[42]} + \frac{1}{2} n_{[42]}\right\} + \left\{\frac{1}{2} n_{[42]}\right\} + \left\{n_{[222]} + \frac{1}{3} n_{[22]} \times 3\right\} = 1050 + 315 + 210 = 1575. \qquad (7)$$

There is no cyclic subgroup C_9 in the permutation group $\boldsymbol{S}_7$. The subgroups of order 9 in $\boldsymbol{S}_7$ can be expressed as $C_3 \times C_3$, such as {E, (123), (132), (456), (465), (123) (456), (132) (465), (123) (465), (132) (456)}. It is found that there are 70 subgroups of order 9,

$$N(9) = \frac{1}{4} n_{[33]} = 70. \qquad (8)$$

There are 378 subgroups of order 10. The generators in the 126 cyclic groups C_{10} can be taken like {(12345) (67)}. There are 252 dihedral groups D_5, the generators can be like {(12345), (15) (24)} or {(12345), (15) (24) (67)}. The calculate-expression is

$$N(10) = \frac{1}{4} n_{[52]} + \frac{1}{4} n_{[5]} \times 2 = 378. \qquad (9)$$

Using the means of similar analysis, there are 1715 subgroups of order 12. The generators in the 105 cyclic groups C_{12} can be like {(1234) (567)}. There are 1155 dihedral groups D_6, the generators can be chosen as {(123) (45), (12)} or {(123) (45), (12) (67)} or {(123456), (14) (23) (56)} or {(123) (45) (67), (12)} or { (123) (45) (67), (12) (45)} or {(123) (45) (67), (12) (46) (57)}. There are 140 subgroups of order 12 which are isomorphic to C_{6h}, the generators can be like {(123) (45) (67), (67)} or {(123) (45) (67), (46) (57)}. There are 210 subgroups of order 12 which are isomorphic to A_4, the generators can be like {(13) (24), (243)} or {(13) (24), (243) (567)} or {(15) (34), (146) (253)}. There are 105 subgroups of order 12 which are isomorphic to the group Q, the generators can be like {(123) (46) (57), (23) (4567)}.

There are 120 subgroups of order 14, the generators can be chosen as {(1234567), (12) (37) (46)}.

There are no subgroups of order 11 or 13 in the group S_7 according to the Lagrange's theorem. The subgroups, whose order are more than 14, can be deduced from the results what we have gotten and the analysis of the structures of the subgroups. Due to the length limit, we take the subgroup of order 16 as example. Refer to the subgroups of order 8 which are isomorphic to the group D_4, we find that the subgroups of order 16 are isomorphic to D_{4h}, such as {E, (1234), (13) (24), (1432), (13), (14) (23), (24), (12) (34), (56), (1234) (56), (13) (24) (56), (1432) (56), (12) (34) (56), (13) (56), (14) (23) (56), (24) (56)}. The total number of subgroups of order 16 is calculated to be 315,

$$N(16) = \frac{1}{2} n_{[4111]} \times 3 = 315. \qquad (10)$$

Similarly, the subgroups of order 18 can be referred to the subgroups of order 9, the subgroups of order 21 can be referred to the subgroups of order 7, the subgroups of order 144 can be referred to the subgroups of order 72, etc., and the analytical expressions can all be get through careful analysis. The subgroups of orders larger than 240 can be get directly from the analysis of the properties of the permutation group. Calculate the number of 6-combinations of 7, $\mathrm{C}_7^6 = 7$, then we know that there are 7 subgroups of order 720 which are isomorphic to S_6, correspondingly, there are 7 subgroups of order 360 which are isomorphic to A_6. We have 1 subgroup of order 2520, which is denoted by A_7, composed by all even permutations of S_7. On the basis of the calculation, it is found that when we calculate the subgroups of order m of the permutation group S_n, we can make full use of the results about the subgroups of order less than m of S_n and the results of subgroups of order m of $S_{(n-1)}$.

4. Discussions

In general, a group can be described by the generators. Since the choice of the generators is not unique, researchers have different choices. A group can also be described by giving all of the elements, whereas it appears to be redundant, especially when the order of the group is large. These two methods have been used in previous calculations. Here, for the convenience of analysis, we represent the subgroups in an alternative way, as shown in Table 2.

The subgroups of the permutation group S_7 are represented in the form of $[\alpha_1]^{a_1} [\alpha_2]^{a_2} \cdots [\alpha_q]^{a_q}$, where $[\alpha_1]$, $[\alpha_2]$ $\cdots$ and $[\alpha_q]$ are the classes of the group S_7. The expression means that this subgroup is a group of order $\{a_1 + a_2 + \cdots + a_q + 1\}$. Excluding the identity, there are q classes in this group and there are a_q elements in the class $[\alpha_q]$. Another distinct advantage of this expression is the order of the elements can be determined as soon as one sees the class $[\alpha_q]$ in the expression. For example, the expression of $[222]^1$ $[33]^2$ $[6]^2$ represents that in the group of order 6, there is one element of order 2 in the class [222], two elements of order 3 in the class [33] and two elements of order 6 in the class [6].

The presented research would widen the application of the Cayley's theorem [12, 13, 14]. The theorem stated that every finite group of order n is isomorphic to a subgroup of a permutation group S_n. The order of the corresponding permutation subgroup, directly from the Cayley's theorem, is usually the same as the order of S_n, i.e., n!. To study a group of order n! would be more difficult than to study a group of order n. Generally, it is not always an easy task to find the corresponding permutation subgroup with the same order as an arbitrary finite group. Now, we have obtained 11300 subgroups of S_7. The expressions of these subgroups can reveal how many classes are included and how many elements there are in each class. These results would be quite useful in the study of the structures and properties of finite groups.

It is known that there are many non-isomorphic groups for a given order n and there are also several expressions consist of different classes for the isomorphic groups. When we choose the most appropriate form for a finite group, it should be noticed that the group elements of same order in different classes might have different meanings in the applications. Take for instance, the elements of order 2 in the octahedron group O. It can be found that the elements in the class [2] and the class [22] are all the elements of order 2. However, the elements in the class [2] represent the rotation around the 2-fold axes connecting the midpoints of two opposite edges, while the elements in the class [22] represent the rotations around the three coordinate axes through the angle π respectively. The permutation subgroup which is isomorphic to the group O is {E, (123), (132), (234), (243), (124), (142), (134), (143), (12) (34), (13) (24), (14) (23), (12), (13), (14), (23), (24), (34), (1234), (1243), (1324), (1432), (1342), (1423) }. Therefore, among several forms of the subgroups of order 24 which are all isomorphic to the group O in the Table 2, the appropriate form of the corresponding permutation subgroup for the group O should be $[2]^6\,[22]^3\,[3]^8\,[4]^6$.

To summarize, on the basis of theoretical calculation and analysis, there are 11300 subgroups of the permutation group S_7. Besides two trivial subgroups, there are 231 cyclic subgroups of order 2, 175 cyclic subgroups of order 3, 1295 subgroups of order 4, 126 cyclic subgroups of order 5, 1645 subgroups of order 6, 120 cyclic subgroups of order 7, 1575 subgroups of order 8, 70 subgroups of order 9, 378 subgroups of order 10, 1715 subgroups of order 12, 120 subgroups of order 14, 315 subgroups of order 16, 350 subgroups of order 18, 378 subgroups of order 20, 120 subgroups of order 21, 1435 subgroups of order 24, 245 subgroups of order 36, 126 subgroups of order 40, 120 subgroups of order 42, 315 subgroups of order 48, 63 subgroups of order 60, 175 subgroups of order 72, 105 subgroups of order 120, 35 subgroups of order 144, 30 subgroups of order 168, 21 subgroups of order 240, 7 subgroups of order 360, 7 subgroups of order 720, 1 subgroup of order 2520. The subgroups have all been expressed by the classes [α] of the group S_7 in Table 2.

Table 2. *The 11300 subgroups of the group S_7.*

Order m	Total number of the subgroups $N(m)$	subgroups expressed by the classes	number for each form of subgroup
1	1	$[1]^1$	1
		$[2]^1$	21
2	231	$[22]^1$	105
		$[222]^1$	105
3	175	$[3]^2$	35
		$[33]^2$	140
		$[22]^1\,[4]^2$	105
		$[22]^1\,[42]^2$	315
		$[2]^2\,[22]^1$	105
4	1295	$[22]^3$	35
		$[22]^3$	105
		$[2]^1\,[22]^1\,[222]^1$	315
		$[22]^1\,[222]^2$	315
5	126	$[5]^4$	126
		$[2]^1\,[3]^2\,[32]^2$	210
		$[222]^1\,[33]^2\,[6]^2$	420
		$[22]^1\,[3]^2\,[322]^2$	105
6	1645	$[222]^3\,[3]^2$	105
		$[2]^3\,[3]^2$	35
		$[22]^3\,[3]^2$	210
		$[22]^3\,[33]^2$	420
		$[222]^3\,[33]^2$	140
7	120	$[7]^6$	120
		$[2]^2\,[22]^3\,[4]^2$	105
		$[22]^3\,[222]^2\,[4]^2$	315
		$[2]^2\,[22]^1\,[222]^2\,[42]^2$	315
8	1575	$[22]^5\,[42]^2$	315
		$[2]^3\,[22]^3\,[222]^1$	105
		$[2]^1\,[22]^3\,[222]^3$	105
		$[2]^1\,[22]^1\,[222]^1\,[4]^2\,[42]^2$	315
9	70	$[3]^4\,[33]^4$	70
		$[22]^5\,[5]^4$	126
10	378	$[222]^5\,[5]^4$	126
		$[2]^1\,[5]^4\,[52]^4$	126

Order *m*	**Total number of the subgroups *N*(*m*)**	**subgroups expressed by the classes**	**number for each form of subgroup**
		$[22]^{1}\ [3]^{2}\ [322]^{2}\ [4]^{2}\ [43]^{4}$	105
		$[2]^{4}\ [22]^{3}\ [3]^{2}\ [32]^{2}$	210
		$[2]^{1}\ [22]^{3}\ [222]^{3}\ [3]^{2}\ [32]^{2}$	210
		$[22]^{3}\ [222]^{4}\ [33]^{2}\ [6]^{2}$	420
		$[2]^{3}\ [22]^{1}[222]^{3}\ [3]^{2}\ [322]^{2}$	105
		$[22]^{7}\ [3]^{2}\ [322]^{2}$	105
12	1715	$[22]^{1}\ [222]^{6}\ [3]^{2}\ [322]^{2}$	105
		$[22]^{3}\ [3]^{2}\ [322]^{6}$	35
		$[2]^{2}\ [22]^{1}\ [3]^{2}\ [32]^{4}\ [322]^{2}$	105
		$[3]^{8}\ [22]^{3}$	35
		$[33]^{8}\ [22]^{3}$	70
		$[33]^{8}\ [22]^{3}$	105
		$[22]^{1}\ [3]^{2}\ [322]^{2}\ [42]^{6}$	105
14	120	$[7]^{6}\ [222]^{7}$	120
16	315	$[2]^{3}\ [22]^{5}\ [222]^{3}\ [4]^{2}\ [42]^{2}$	315
		$[2]^{3}\ [3]^{4}\ [32]^{6}\ [33]^{4}$	140
18	350	$[22]^{9}\ [3]^{4}\ [33]^{4}$	70
		$[222]^{3}\ [3]^{4}\ [33]^{4}\ [6]^{6}$	140
		$[22]^{5}\ [4]^{10}\ [5]^{4}$	126
20	378	$[22]^{5}\ [42]^{10}\ [5]^{4}$	126
		$[2]^{1}\ [22]^{5}\ [222]^{5}\ [5]^{4}\ [52]^{4}$	126
21	120	$[7]^{6}\ [33]^{14}$	120
		$[2]^{6}\ [22]^{3}\ [3]^{8}\ [4]^{6}$	35
		$[22]^{9}\ [3]^{8}\ [42]^{6}$	105
		$[2]^{1}\ [22]^{3}\ [222]^{3}\ [3]^{8}\ [32]^{8}$	105
		$[22]^{3}\ [222]^{6}\ [33]^{8}\ [4]^{6}$	105
		$[22]^{9}\ [33]^{8}\ [42]^{6}$	210
		$[2]^{3}\ [22]^{3}\ [222]^{1}\ [33]^{8}\ [6]^{8}$	105
24	1435	$[22]^{9}\ [33]^{8}\ [42]^{6}$	105
		$[2]^{2}\ [22]^{3}\ [3]^{2}\ [32]^{4}\ [322]^{6}\ [4]^{2}\ [43]^{4}$	105
		$[2]^{3}\ [22]^{1}\ [222]^{3}\ [3]^{2}\ [322]^{2}\ [4]^{2}\ [42]^{6}\ [43]^{4}$	105
		$[22]^{7}\ [222]^{6}\ [3]^{2}\ [322]^{2}\ [4]^{2}\ [43]^{4}$	105
		$[2]^{5}\ [22]^{7}\ [222]^{3}\ [3]^{2}\ [32]^{4}\ [322]^{2}$	105
		$[2]^{2}\ [22]^{1}\ [222]^{6}\ [3]^{2}\ [32]^{4}\ [322]^{2}\ [42]^{6}$	105
		$[22]^{9}\ [3]^{2}\ [322]^{6}\ [42]^{6}$	105
		$[2]^{3}\ [22]^{3}\ [222]^{9}\ [3]^{2}\ [322]^{6}$	35
		$[2]^{6}\ [22]^{9}\ [3]^{4}\ [32]^{12}\ [33]^{4}$	70
36	245	$[22]^{9}\ [222]^{6}\ [3]^{4}\ [33]^{4}\ [6]^{12}$	70
		$[22]^{9}\ [3]^{4}\ [33]^{4}\ [42]^{18}$	70
		$[22]^{3}\ [3]^{10}\ [322]^{6}\ [33]^{16}$	35
40	126	$[2]^{1}\ [22]^{5}\ [222]^{5}\ [4]^{10}\ [42]^{10}\ [5]^{4}\ [52]^{4}$	126
42	120	$[222]^{7}\ [33]^{14}\ [6]^{14}\ [7]^{6}$	120
		$[2]^{7}\ [22]^{9}\ [222]^{3}\ [3]^{8}\ [32]^{8}\ [4]^{6}\ [42]^{6}$	105
48	315	$[2]^{3}\ [22]^{9}\ [222]^{7}\ [33]^{8}\ [4]^{6}\ [42]^{6}\ [6]^{8}$	105
		$[2]^{5}\ [22]^{9}\ [222]^{9}\ [3]^{2}\ [32]^{4}\ [322]^{6}\ [4]^{2}\ [42]^{6}\ [43]^{4}$	105
60	63	$[22]^{15}\ [3]^{20}\ [5]^{24}$	21
		$[22]^{15}\ [33]^{20}\ [5]^{24}$	42
		$[2]^{6}\ [22]^{9}\ [222]^{6}\ [3]^{4}\ [32]^{12}\ [33]^{4}\ [42]^{18}\ [6]^{12}$	70
72	175	$[2]^{3}\ [22]^{3}\ [222]^{9}\ [3]^{10}\ [32]^{24}\ [322]^{6}\ [33]^{16}$	35
		$[2]^{6}\ [22]^{3}\ [3]^{10}\ [32]^{12}\ [322]^{6}\ [33]^{16}\ [4]^{6}\ [43]^{12}$	35
		$[22]^{21}\ [3]^{10}\ [322]^{6}\ [33]^{16}\ [42]^{18}$	35
		$[2]^{10}\ [22]^{15}[3]^{20}\ [32]^{20}\ [4]^{30}\ [5]^{24}$	21
120	105	$[22]^{15}\ [222]^{10}\ [33]^{20}\ [4]^{30}\ [5]^{24}\ [6]^{20}$	42
		$[2]^{1}[22]^{15}\ [222]^{15}\ [3]^{20}\ [32]^{20}\ [5]^{24}\ [52]^{24}$	21
		$[22]^{25}\ [3]^{20}\ [322]^{20}\ [42]^{30}\ [5]^{24}$	21
144	35	$[2]^{9}\ [22]^{21}\ [222]^{9}\ [3]^{10}\ [32]^{36}\ [322]^{6}\ [33]^{16}\ [4]^{6}\ [42]^{18}\ [43]^{12}$	35
168	30	$[22]^{21}\ [33]^{56}\ [42]^{42}\ [7]^{48}$	30
240	21	$[2]^{11}\ [22]^{25}\ [222]^{15}\ [3]^{20}\ [32]^{40}\ [322]^{20}\ [4]^{30}\ [42]^{30}\ [5]^{24}\ [52]^{24}$	21
360	7	$[22]^{45}\ [3]^{40}\ [33]^{40}\ [42]^{90}\ [5]^{144}$	7
720	7	$[2]^{15}\ [22]^{45}\ [222]^{15}\ [3]^{40}\ [32]^{120}\ [33]^{40}\ [4]^{90}\ [42]^{90}\ [5]^{144}\ [6]^{120}$	7
2520	1	$[22]^{105}\ [3]^{70}\ [322]^{210}\ [33]^{280}\ [42]^{630}\ [5]^{504}\ [7]^{720}$	1
5040	1	$[2]^{21}\ [22]^{105}\ [222]^{105}\ [3]^{70}\ [32]^{420}\ [322]^{210}\ [33]^{280}\ [4]^{210}\ [42]^{630}\ [43]^{420}\ [5]^{504}\ [52]^{504}\ [6]^{840}\ [7]^{720}$	1

5. Conclusions

In this article, we calculate the 11300 subgroups of S_7 by group-theoretical approach and represent all the subgroups in an alternative way for further analysis and applications. Although the total number of the subgroups of S_7 is considerable large, we provide an explanation by several analytical formulae of $N(m)$ and n_α, where $N(m)$ denotes the number of subgroups of order m and n_α is the number of elements in the class [α]. The research shows the power of the group-theoretical approach and will be quite useful in analyzing the properties of the permutation groups. It will also be helpful in the study of the finite groups with the familiar theorem of Cayley. Further, how to apply this method to simplify the computer program is also an interesting subject to study in the future.

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