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Uniform Approximation of the Generalized Cut Function by Erlang Cumulative Distribution Function and Application in Applied Insurance Mathematics

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Abstract: In this paper we study the uniform approximation of the generalized cut function by sigmoidal Erlang cumulative distribution function (Ecdf). The results are relevant for applied insurance mathematics and are intended for the actuary when preparing the strategy "Insurance responsibility". Numerical examples are presented using CAS MATHEMATICA.

Keywords: Erlang Cumulative Distribution Function (Ecdf), Generalized Cut Function Associated to the (Ecdf), Uniform Approximation

1. Introduction

One of the most significant goals of any insurance risk activity is to achieve a satisfactory model for the probability distribution of the total claim amount [1].

The analysis of the collective risk model assuming Erlang loss, when the claim frequency follows the discrete generalized Lindley distribution, is considered in [1]–[3].

We study the uniform approximation of the cut function by Erlang cumulative distribution function (Ecdf). We find an expression for the error of the uniform approximation.

The estimates obtained give more insights on the parameters in the strategy "Insurance responsibility" [4]–[7].

2. Preliminaries

2.1. The Erlang Cumulative Distribution Function (Ecdf)

Based on the exponential distribution A. K. Erlang used the stage method to construct the so–called Erlang distribution function of order k with (Ecdf) defined by:

$$F(t;k;\lambda)=1-\sum_{n=0}^{k-1}\frac{e^{-\lambda t}(\lambda t)^n}{n!};\quad t,\lambda\geq 0, \tag{1}$$

where k is called the shape parameter (integer), and the parameter λ is called the rate parameter.

The distribution is now used in the fields of statistic processes, teletraffic engineering, biomathematics, etc.

We have

$$F'(t;k;\lambda)=\frac{e^{-\lambda t}\lambda^k t^{k-1}}{(k-1)!}, \tag{2}$$

$$F''(t;k;\lambda)=\frac{e^{-\lambda t}\lambda^k t^{k-2}}{(k-2)!}\left(1-\frac{t\lambda}{k-1}\right). \tag{3}$$

From (3) we find that (1) has an inflection point at:

$$t^*=\frac{k-1}{\lambda}, \tag{4}$$

where $k>1$.

2.2. The Generalized Cut Function Associated to the (Ecdf)

The associate to the (Ecdf) cut function C_{Ecdf} is defined by

$$C_{Ecdf}(t)=\begin{cases}0, & \text{if} \quad t<t_1,\\ F'\left(\left(\frac{k-1}{\lambda}\right);k;\lambda\right)\times & \\ \left(t-\frac{k-1}{\lambda}\right)+ & \\ F\left(\left(\frac{k-1}{\lambda}\right);k;\lambda\right), & \text{if} \quad t_1\le t<t_2,\\ 1, & \text{if} \quad t\ge t_2.\end{cases} \tag{5}$$

3. Approximation of the Cut Function (5) by Function (1)

We next focus on the approximation of the cut function $C_{Ecdf}(t)$ by (Ecdf).

Note that the slope of the function $C_{Ecdf}(t)$ on the interval $\Delta=[t_1,t_2]$ is $F'(t^*;k;\lambda)$.

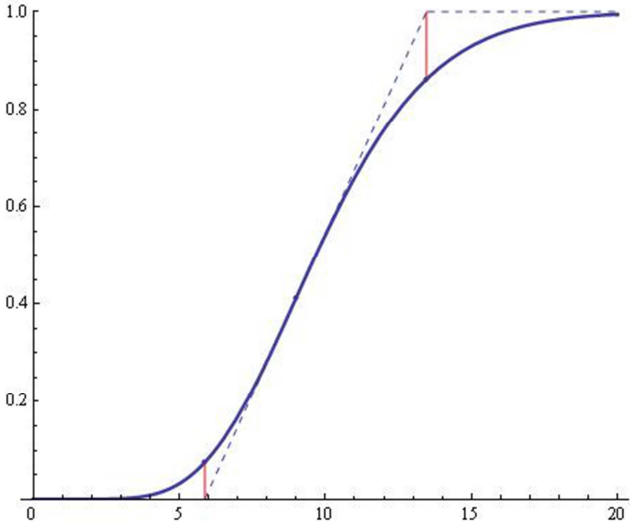

Figure 1. *The cut and the (Ecdf) functions with* $k=10$, $\lambda=1$, $t^*=9$, $t_1=5.86851$, $t_2=13.4583$, $F(t_1;10;1)=0.0751968$, $F(t_2;10;1)=0.862373$, *uniform distance* $\rho=1-0.862373=0.137627$.

In addition

$$F(t^*;k;\lambda)=1-\frac{\Gamma(k,k-1)}{\Gamma(k)},$$

where $\Gamma(k)$ is a complete gamma function, and $\Gamma(k,k-1)$ is the incomplete gamma function.

On the other hand

$$F'(t^*;k;\lambda)=\frac{\left(\frac{k-1}{e}\right)^{k-1}\lambda}{(k-1)!}.$$

The straight line

$$y=F'\left(\left(\frac{k-1}{\lambda}\right);k;\lambda\right)\left(t-\frac{k-1}{\lambda}\right)+F\left(\left(\frac{k-1}{\lambda}\right);k;\lambda\right)$$

crosses the lines $y=0$ and $y=1$ at the points

$$t_1=\frac{k-1}{\lambda}-\frac{\left(1-\frac{\Gamma(k,k-1)}{\Gamma(k)}\right)(k-1)!}{\lambda\left(\frac{k-1}{e}\right)^{k-1}}$$

and

$$t_2=\frac{k-1}{\lambda}+\frac{\frac{\Gamma(k,k-1)}{\Gamma(k)}(k-1)!}{\lambda\left(\frac{k-1}{e}\right)^{k-1}}$$

respectively.

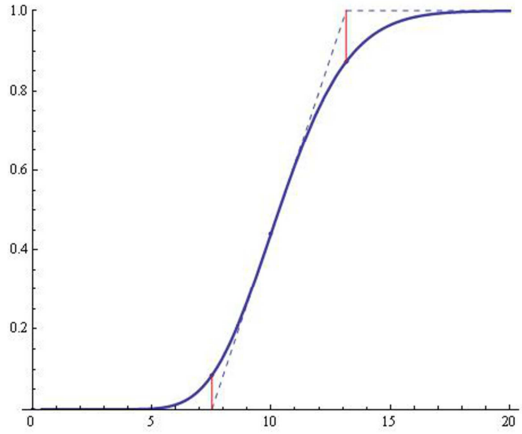

Figure 2. *The cut and the (Ecdf) functions with* $k=21$, $\lambda=2$, $t^*=10$, $t_1=7.5184$, $t_2=13.1468$, $F(t_1;21;2)=0.084519$, $F(t_2;21;2)=0.873205$, *uniform distance* $\rho=1-0.873205=0.126795$.

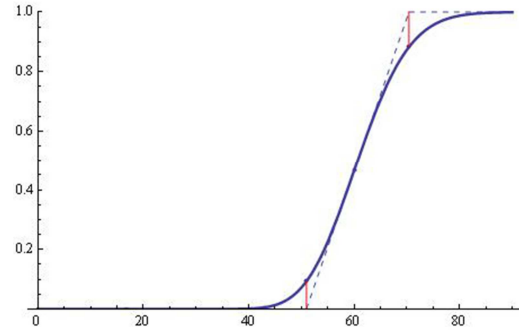

Figure 3. *The cut and the (Ecdf) functions with* $k=61$, $\lambda=1$, $t^*=60$, $t_1=50.9446$, $t_2=70.3878$, $F(t_1;61;1)=0.0929865$, $F(t_2;61;1)=0.882485$, *uniform distance* $\rho=1-0.882485=0.117515$.

Then, noticing that the largest uniform distance ρ between the cut and (Ecdf) functions is achieved at the endpoints of the underlying interval Δ we have the

following

Theorem. The function defined by (1): i) is the (Ecdf) function of best uniform one-sided approximation to function $C_{Ecdf}(t)$ in the interval Δ; ii) approximates the cut function $C_{Ecdf}(t)$ in uniform metric with an error

$$\rho = \max\{F(t_1;k;\lambda), 1-F(t_2;k;\lambda)\}. \quad (6)$$

We propose a software module within the programming environment *CAS Mathematica* for the sensitivity analysis of the considered approximation model.

The module offers the following possibilities:

i) generation of the cut function $C_{Ecdf}(t)$ associated to the Erlang cumulative distribution function under user-defined values for k and λ;

ii) automatic check of the condition that guarantees the existence of cumulative curve;

iii) computing the value of the best uniform approximation of cut function $C_{Ecdf}(t)$ by (Ecdf);

iv) computing the parameters for the actuary when preparing the strategy "Insurance responsibility";

v) software tools for animation and visualization.

4. Application and Conclusions

The results are relevant for applied insurance mathematics.

For example, when preparing the strategy "Risk in Perspective", the actuary approximately fixes: the probability distribution, for example - Erlang distribution (based on accumulated statistics for the study insured event); number of damaged objects (random variable); probability of losses for this number of objects and total losses, depending on the number of damaged objects.

In the above-mentioned strategy it is essential to sketch the curve analysis of cumulative probability of accumulation with increasing number of damaged objects and the amount of compensation likely to happen (strategy "Insurance Perspective").

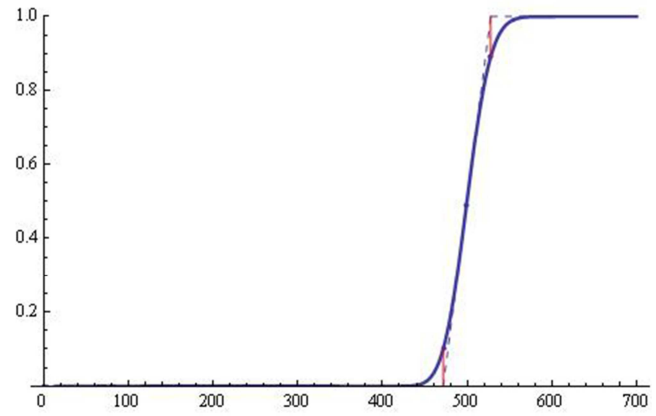

Figure 4. *The cut and the (Ecdf) functions with* $k = 500$, $\lambda = 1$, $t^* = 499$, $t_1 = 471.665$, $t_2 = 527.668$, $F(t_1;500;1) = 0.100808$, $F(t_2;500;1) = 0.890666$, *uniform distance* $\rho = 1 - 0.890666 = 0.109334$.

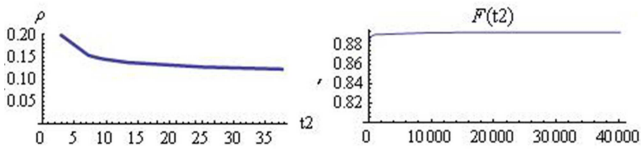

Figure 5. *The uniform distance* ρ *and* $F(t_2;k;\lambda)$ *for some* k. *The function* $F(t_2;k;\lambda)$ *visually depicts a "law of diminishing marginal returns".*

The detailed study of this sigmoid function, which is a good approximation of the cut function associated to the (Ecdf) gives good information for the actuary with respect to the minimum sample of the general aggregation of the damaged objects, whose losses must be covered, and thus the percentage of the insured event that are occurred (strategy "Insurance responsibility", according to the law of diminishing marginal returns), and at a later stage for the formation of a support plan for the formation insurance policy.

Certain interest is the inverse task at a fixed actuarial percent achieved liability insurance to determine the magnitude ρ (in this case, it is probability of losses after the number of the injured object - K of the sample).

Satisfactory answer to this question gives determining the value of the best uniform approximation of cut function $C_{Ecdf}(t)$ by (Ecdf) - the subject of current research.

Some results for Erlang distributed moments of impulses are given by Agarwal, Hristova, O'Regan, Kopanov in [8].

The Hausdorff ([24], [25]) and uniform approximation of the interval step function by the logistic and other sigmoid functions such as Burr cdf, generalized Burr function, Lindley cdf, transmuted Lindley function, exponentiated Lindley function, Gompertz function, transmuted Rayleigh function, etc., are discussed from various approximation, computational and modelling aspects in [9]–[28].

For the hyper–Erlang distribution model and its applications in wireless network and mobile computing system, see [22].

For the hypoexponential distribution, or the generalized Erlang distribution, see [23].

Based on the methodology proposed in the present note, the reader may formulate the corresponding approximation problems on his/her own.

Remark. Consider the following transmuted Erlang cumulative distribution function (tEcdf)

$$F^*(t;k;\lambda;\mu) = (1+\mu)F(t;k;\lambda) - \mu F^2(t;k;\lambda)$$

where $|\mu| \le 1$.

For some comparisons of the (tEcdf) and (Ecdf), see Fig. 6 – Fig. 8.

From this graphics it can be seen that the "saturation" is faster (for fixed k and λ) and μ increasing tends to 1.

This circumstance can be successfully used by the actuary of the insurance company.

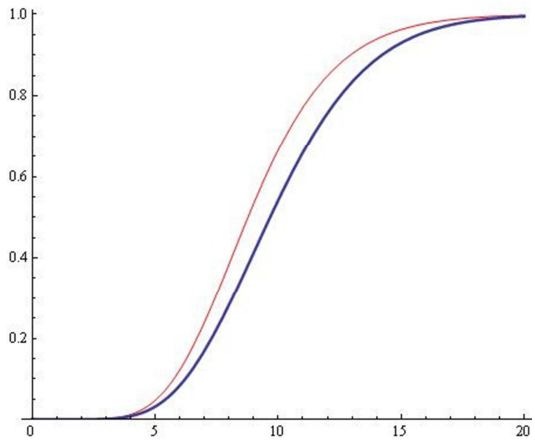

Figure 6. *The tEcdf (red) and Ecdf (thick) with* $k=10$, $\lambda=1$, $\mu=0.5$.

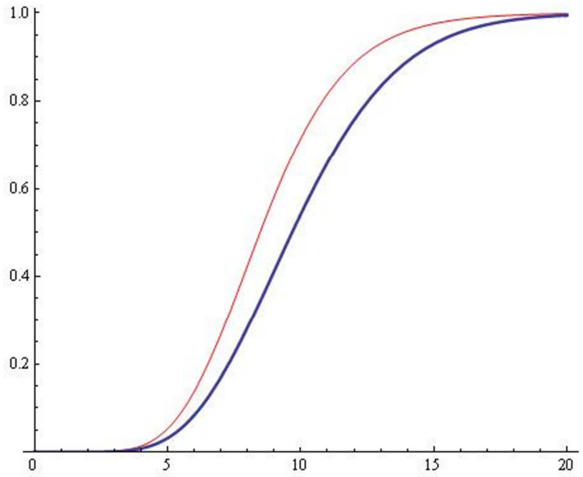

Figure 7. *The tEcdf (red) and ECDF (thick) with* $k=10$, $\lambda=1$, $\mu=0.7$.

Web application (intellectual property) to clarify the mathematical mechanism of insurance and reinsurance in all three phases "Risk in perspective", "Insurance perspective" and "Insurance liability" for insurance events governed by the Erlang Distribution is developed.

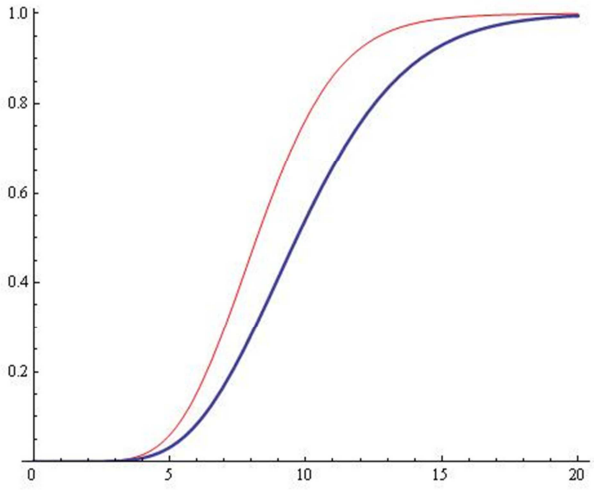

Figure 8. *The tEcdf (red) and ECDF (thick) with* $k=10$, $\lambda=1$, $\mu=0.9$.

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