



# Aboodh Transform Homotopy Perturbation Method for Solving Third Order Korteweg -DeVries Equation

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## To cite this article:

Khalid SulimanAboodh. Aboodh Transform Homotopy Perturbation Method for Solving Third Order Korteweg -De Vries Equation. *International Journal of Theoretical and Applied Mathematics*. Vol. 2, No. 2, 2016, pp. 35-39. doi: 10.11648/j.ijtam.20160202.12

**Received:** September 18, 2016; **Accepted:** November 21, 2016; **Published:** November 22, 2016

**Abstract:** This Paper is discussing the theoretical approach of Aboodh transform [1] coupled with Homotopy Perturbation Method [3] that can be applied to higher order partial differential equations for finding exact as well as approximate solutions of the equations. Here Homotopy Perturbation Method using Aboodh transform [1], [16] has been applied to Korteweg-de vries equation which is of third order homogenous partial differential equation.

**Keywords:** Homotopy Perturbation, Korteweg-DeVries Equation, Aboodh Transforms

## 1. Introduction

The korteweg-devries equation [4], [5]

$$\frac{\partial u}{\partial t} + a \frac{\partial^3 u}{\partial x^3} + bu \frac{\partial u}{\partial x} = 0 \quad (1)$$

was first derived by korteweg and vries(1895) to the water waves in shallow canal, when the study of water waves was of vital interest for applications in naval architecture and for the knowledge of tides and floods.

The canonical Korteweg-de Vries (KdV) equation

$$u_t + 6uu_x + u_{xxx} = 0$$

is widely recognized as a paradigm for the description of weakly nonlinear long waves in many branches of physics and engineering. Here  $u(x, t)$  is an appropriate field variable,  $t$  is the time coordinates, and  $x$  is the space coordinate in the relevant direction. It describes how waves evolve under the competing but comparable effects of weak nonlinearity and weak dispersion. Most scientific phenomena occur nonlinearly. We know that except a limited number, the majority of them don't have exact systematic arrangements. Therefore, these nonlinear Equations are to be solved using other methods. In recent years, many researchers have paid attention to study the solutions of

linear and nonlinear differential equations and also Fractional order differential equation by using various analytical methods. Recently, some nonlinear analytical techniques for solving nonlinear problems have been dominated by the perturbation method. Perturbation method is one of the most well known methods to solve nonlinear equations studied by a large number of researchers such as Bellman [2], Cole, 3- and O'Malley, 16]. Actually, these scientists had paid more attention to the mathematical aspects of the subject which included a loss of physical verification. This loss in the physical verification of the subject was recovered by Nayfeh [15] and Van Dyke [20]. But, like other nonlinear analytical methods, perturbation methods have their own particular limitations.

Firstly, almost all perturbation methods are based on an assumption that a small parameters must exist in the equation. This is so called small parameter assumption greatly restricts utilizations of perturbation techniques. As well known, an overwhelming majority of nonlinear problems have no small parameters at all. Secondly, the determination of small parameters seems to be a special art requiring special techniques. Asuitablechoice of small parameters leads to ideal results. However, an unsustainable choice of small parameters results in bad effects. Thirdly, even if there exist suitable parameters, the approximate solutions are obtained by the perturbation methods valid in most cases, only for the small values of the parameters. So it is necessary to develop a kind of new nonlinear analytical method which does not require small parameters at all. Homotopy is an

important part of differential topology. Homotopy techniques are generally connected to discover all bases of nonlinear algebraic equations. The homotopy technique, or the continuous mapping procedure, embeds a parameter the embedding parameter that typically ranges from zero to one. When the embedding parameter is zero, the equation is one of a direct framework, when it is one; the equation is the same as the first one. So the embedded parameter  $[0, 1]$  can be considered as a small parameter. The coupling method of the homotopy techniques is called the homotopy perturbation method. The Homotopy perturbation method (HPM) was presented by JiHuan He [He, 1999] of Shanghai University in 1998 which is the coupling method of the homotopy techniques and the perturbation technique. On the other hand, Homotopy perturbation transform method (HPTM) is combined form of the Laplace transform method with the homotopy perturbation method introduced by Y.khan and Q. Wu. The above methods find the solution without any discretisation or restrictive assumptions and avoid the round off errors. The HPM is a special case of the Homotopy analysis method (HAM) [Liao S., 1992] developed by Liao Shijun in 1992. The HAM uses a so-called convergence control parameter to guarantee the convergence of approximations series over a given interval of physical parameters.

## 2. Aboodh Transform on KDV Equation

Aboodh Transform is derived from the classical Fourier integral. Based on the mathematical simplicity of the Aboodh transform and its fundamental properties.

Aboodh transform was introduced by Khalid Aboodh to facilitate the process of solving ordinary and partial differential equations in the time domain. Typically, Fourier, Laplace and Sumudu transforms are the convenient mathematical tools for solving differential equations. Also Aboodh transform and some of its fundamental properties are used to solve differential equations.

The Aboodh transform denoted by the operator  $A(.)$  defined by the integral equations

$$A[f(t)] = K(v) = \frac{1}{v} \int_0^\infty f(t) e^{-vt} dt, \quad t \geq 0, k_1 \leq v \leq k_2$$

Consider the operator form of KDV equation (1) as

$$Du(x, t) + Ru(x, t) + Nu(x, t) = 0, \quad U(x, 0) = f(x) \quad (2)$$

Where  $D$  is a linear differential operator with respect to  $t$ ,  $R$  is a linear differential operator with respect to  $x$  and  $N$  is a nonlinear differential operator. On Taking Aboodh transform on both sides of equation (2) it gives,

$$A[Du(x, t)] + A[Ru(x, t)] + A[Nu(x, t)] = 0 \quad (3)$$

By using the differentiation property of Aboodh transforms and above initial conditions it yields

$$A[u(x, t)] = \frac{f(x)}{v^2} - \frac{1}{v} A \left[ a \frac{\partial^3 u}{\partial x^3} + bu \frac{\partial u}{\partial x} \right] \quad (4)$$

By using Aboodh inverse on both sides of equation (4), it yields

$$u(x, t) = f(x) - A^{-1} \left( \frac{1}{v} A \left[ a \frac{\partial^3 u}{\partial x^3} + bu \frac{\partial u}{\partial x} \right] \right) \quad (5)$$

Now by applying Homotopy Perturbation Method to (5), it gives

$$\sum_{n=0}^{\infty} p^n u_n(x, t) = f(x) - p \left[ A^{-1} \left\{ \frac{1}{v^2} A \left[ \sum_{n=0}^{\infty} a R u_n(x, t) + N b u_n(x, t) \right] \right\} \right] \quad (6)$$

The Nonlinear term in equation (6) can be decomposed as

$$N[u(x, t)] = \sum_{n=0}^{\infty} p^n H_n(u) \quad (7)$$

Where  $H_n(u)$  are given by:

$$H_n(u_0, u_1, u_2, \dots, u_n) = \frac{1}{n!} \frac{\partial}{\partial p^n} \left[ N \left( \sum_{i=0}^{\infty} p^i u_i \right) \right]_{p=0}, \quad n = 0, 1, 2, \dots,$$

Finally, the equation (6) becomes

$$\sum_{n=0}^{\infty} p^n u_n(x, t) = f(x) - p \left[ A^{-1} \left\{ \frac{1}{v^2} A \left[ \sum_{n=0}^{\infty} p^n b H_n(u) + \sum_{n=0}^{\infty} p^n a \frac{\partial^3 u_n}{\partial x^3} \right] \right\} \right] \quad (8)$$

Where  $H_n(u)$  are He's polynomials. The first few components of He's polynomial are given by

$$\begin{aligned} H_0(u) &= u_0 \frac{\partial u_0}{\partial x} \\ H_1(u) &= u_0 \frac{\partial u_1}{\partial x} + u_1 \frac{\partial u_0}{\partial x} \\ &\dots \end{aligned}$$

Hence, comparing the coefficient of like powers of  $p$ , it gives

$$\begin{aligned} p^0 : u_0(x, t) &= f(x) \\ p^1 : u_1(x, t) &= -A^{-1} \left\{ \frac{1}{v^2} A [Ra + bH_0(u)] \right\} \\ p^2 : u_2(x, t) &= -A^{-1} \left\{ \frac{1}{v^2} A [Ra + bH_1(u)] \right\} \\ p^3 : u_3(x, t) &= -A^{-1} \left\{ \frac{1}{v^2} A [Ra + bH_2(u)] \right\} \\ &\dots \end{aligned}$$

Hence, the approximate solution of equation (1) is

$$u(x, t) = u_0(x, t) + u_1(x, t) + u_2(x, t) + \dots \quad (9)$$

and the (9) is the approximate solution of KDV equation which converges to the exact solution.

### 3. Special Cases of Korteweg-Devries Equation

CASE1: For  $a = 1, b = 1$  Equation (2) becomes

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0 \quad (10)$$

We consider this problem with initial condition

$$u(x, 0) = (1 - x) \quad (11)$$

Using the differentiation property of Aboodh transforms and initial condition (11), it gives

$$A[u(x, t)] = \frac{(1-x)}{v^2} - \frac{1}{v} A \left[ u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} \right] \quad (12)$$

By taking inverse of Aboodh transform on both sides of (12), it gives

$$u(x, t) = (1-x) - A^{-1} \left[ \frac{1}{v} A \left[ u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} \right] \right] \quad (13)$$

By applying Homotopy Perturbation Method in equation (13) yields

$$\sum_{n=0}^{\infty} p^n u_n(x, t) = (1-x) - p \left[ A^{-1} \left\{ \frac{1}{v^2} A \left[ \sum_{n=0}^{\infty} p^n H_n(u) + \sum_{n=0}^{\infty} p^n \frac{\partial^3 u_n}{\partial x^3} \right] \right\} \right] \quad (14)$$

Where  $H_n(u)$  are He's Polynomial to be determined. Now the He's Polynomial are

$$\begin{aligned} H_0(u) &= u_0 \frac{\partial u_0}{\partial x} \\ H_1(u) &= u_0 \frac{\partial u_1}{\partial x} + u_1 \frac{\partial u_0}{\partial x} \\ H_2(u) &= u_0 \frac{\partial u_2}{\partial x} + u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_0}{\partial x} \\ &\dots\dots\dots \end{aligned}$$

On comparing the coefficient of like power of  $p$  on both sides,

$$p^0 : u_0(x, t) = 1 - x$$

$$H_0(u) = (1-x)(-1) = -(1-x)$$

$$p^1 : u_1(x, t) = -A^{-1} \left\{ \frac{1}{v^2} A \left[ H_0(u) + \frac{\partial^3 u_0}{\partial x^3} \right] \right\} = -A^{-1} \left\{ \frac{1}{v^2} A [-(1-x) + 0] \right\} = (1-x)t$$

$$H_1(u) = u_0 \frac{\partial u_1}{\partial x} + u_1 \frac{\partial u_0}{\partial x} = (1-x)(-t) + (1-x)t(-1) = -2(1-x)t$$

$$p^2 : u_2(x, t) = -A^{-1} \left\{ \frac{1}{v^2} A \left[ H_1(u) + \frac{\partial^3 u_1}{\partial x^3} \right] \right\} = -A^{-1} \left\{ \frac{1}{v^2} A [-2(1-x)t + 0] \right\} = 2(1-x) \frac{t^2}{2!} = (1-x)t^2$$

$$H_2(u) = u_0 \frac{\partial u_2}{\partial x} + u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_0}{\partial x} = (1-x)(-t^2) + (1-x)t(-t) + (1-x)t^2(-1) = -3(1-x)t^2$$

$$p^3 : u_3(x, t) = -A^{-1} \left\{ \frac{1}{v^2} A \left[ H_2(u) + \frac{\partial^3 u_2}{\partial x^3} \right] \right\} = -A^{-1} \left\{ \frac{1}{v^2} A [-3(1-x)t^2 + 0] \right\} = 6(1-x) \frac{t^3}{3!} = (1-x)t^3$$

Similarly the higher terms  $p^4, p^5, \dots$  can be determined. Therefore

$$\begin{aligned} u(x, t) &= u_0(x, t) + u_1(x, t) + u_2(x, t) + \dots\dots\dots \\ &= 1-x + (1-x)t + (1-x)t^2 + (1-x)t^3 + \dots\dots \\ &= (1-x) [1 + t + t^2 + t^3 + \dots\dots] \end{aligned} \quad (15)$$

Equation (15) is the approximate solution of (10) which converges to the exact solution  $u(x, t) = \frac{1-x}{1-t}$ .

CASE2: For  $a = -1, b = -6$  Equation (2) becomes

$$\frac{\partial u}{\partial t} - 6u \frac{\partial u}{\partial x} - \frac{\partial^3 u}{\partial x^3} = 0 \quad (16)$$

We consider this problem with initial condition

$$u(x, 0) = (1 - x) \quad (17)$$

taking Aboodh transform on equation (16) subject to initial condition (17), gives

$$A[u(x, t)] = \frac{(1-x)}{v^2} + \frac{1}{v} A \left[ 6u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} \right] \quad (18)$$

By taking inverse of Aboodh transform on both sides of (18), it yields,

$$u(x, t) = 1-x + A^{-1} \left\{ \frac{1}{v} A \left[ 6u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} \right] \right\} \quad (19)$$

Now we apply Homotopy Perturbation Method to equation (19) yields

$$\sum_{n=0}^{\infty} p^n u_n(x, t) = (1-x) - p \left[ A^{-1} \left\{ \frac{1}{v^2} A \left[ \sum_{n=0}^{\infty} 6p^n H_n(u) + \sum_{n=0}^{\infty} p^n \frac{\partial^3 u_n}{\partial x^3} \right] \right\} \right] \quad (20)$$

Where  $H_n(u)$  are He's Polynomial.

$$\begin{aligned}
H_0(u) &= u_0 \frac{\partial u_0}{\partial x} \\
H_1(u) &= u_0 \frac{\partial u_1}{\partial x} + u_1 \frac{\partial u_0}{\partial x} \\
H_2(u) &= u_0 \frac{\partial u_2}{\partial x} + u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_0}{\partial x} \\
&\dots\dots\dots
\end{aligned}$$

Comparing the coefficient of like power of  $p$  on both sides

$$\begin{aligned}
p^0 : u_0(x, t) &= 1 - x \\
H_0(u) &= (1 - x)(-1) = -(1 - x) \\
p^1 : u_1(x, t) &= A^{-1} \left\{ \frac{1}{v^2} A \left[ 6H_0(u) + \frac{\partial^3 u_0}{\partial x^3} \right] \right\} = A^{-1} \left\{ \frac{1}{v^2} A [-6(1 - x) + 0] \right\} = -6(1 - x)t \\
H_1(u) &= u_0 \frac{\partial u_1}{\partial x} + u_1 \frac{\partial u_0}{\partial x} = (1 - x)(6t) - 6(1 - x)t(-1) = 12(1 - x)t \\
p^2 : u_2(x, t) &= A^{-1} \left\{ \frac{1}{v^2} A \left[ 6H_1(u) + \frac{\partial^3 u_1}{\partial x^3} \right] \right\} = A^{-1} \left\{ \frac{1}{v^2} A [72(1 - x)t + 0] \right\} = 72(1 - x) \frac{t^2}{2!} = 36(1 - x)t^2 \\
H_2(u) &= u_0 \frac{\partial u_2}{\partial x} + u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_0}{\partial x} = (1 - x)(-36t^2) - 6(1 - x)t(6t) + 36(1 - x)t^2(-1) = -108(1 - x)t^2 \\
p^3 : u_3(x, t) &= A^{-1} \left\{ \frac{1}{v^2} A \left[ 6H_2(u) + \frac{\partial^3 u_2}{\partial x^3} \right] \right\} = A^{-1} \left\{ \frac{1}{v^2} A [-648(1 - x)t^2 + 0] \right\} = -648(1 - x) \frac{t^3}{3!} = -108(1 - x)t^3 \\
H_3(u) &= u_0 \frac{\partial u_3}{\partial x} + u_2 \frac{\partial u_1}{\partial x} + u_1 \frac{\partial u_2}{\partial x} + u_3 \frac{\partial u_0}{\partial x} = 432(1 - x)t^3 \\
p^4 : u_4(x, t) &= A^{-1} \left\{ \frac{1}{v^2} A \left[ 6H_3(u) + \frac{\partial^3 u_3}{\partial x^3} \right] \right\} = A^{-1} \left\{ \frac{1}{v^2} A [2592(1 - x)t^3 + 0] \right\} = 2592(1 - x) \frac{t^4}{4!} = 108(1 - x)t^4
\end{aligned}$$

Similarly to the higher terms  $p^4, p^5, \dots$  we have

$$\begin{aligned}
u(x, t) &= u_0(x, t) + u_1(x, t) + u_2(x, t) + \dots \\
&= (1 - x) - 6(1 - x)t + 36(1 - x)t^2 - 108(1 - x)t^3 + 108(1 - x)t^4 + \dots \\
&= (1 - x) [1 - 6t + 36t^2 - 108t^3 + 108t^4 - \dots] \\
&= (1 - x) [1 - 6t + 36t^2 - 108t^3 [1 - t + t^2 - \dots] \dots] \tag{21}
\end{aligned}$$

$$= (1 - x) \left[ 1 - 6t + 36t^2 - \frac{108t^3}{(1 + t)} \right] \tag{22}$$

Therefore, the approximate solution of equation (16) converges to the exact solution.

Solutions of Korteweg-devries (KDV) with initial are calculated to present an exact result of the KDV Equation.

## 4. Conclusion

In this paper, Korteweg-devries equation is solved by Homotopy Perturbation Method using Aboodh transform to get its solution in exact form for the given initial conditions.

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