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Error Analysis of Newly Developed Numerical Methods for Solving System of Nonlinear Equations

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Abstract: Solution methods are the major tools in research especially in the area of applied mathematics. This is because, most real-life problems result into system of nonlinear equations, and the right solution method with less computational error is required to obtain an approximated solution to these system of nonlinear equations. The introduction of the Broyden method set the groundwork for the development of several other methods, many of which are referred to as Broyden-like approaches by various researchers. In most cases, these methods have proven to be superior to the original classical Broyden method in terms of the number of iterations and CPU time needed to acquire a solution. Using the solutions of the traditional Broyden method as a point of comparison, this study aimed to examine the error associated with two newly developed numerical methods, the Trapezoidal-Simpson-3/8 (TS-3/8) and Midpoint-Simpson-3/8 (MS-3/8) methods. Results gathered after applying the classical Broyden, MS-3/8 and TS-3/8 methods to solve some bench-mark problems involving system of nonlinear equations and estimating the errors associated with each of the methods considered in the study, using the formula of the approximate error, showed that the error associated with the MS-3/8 method was minimal compared to that of the Broyden and the TS-3/8 methods. At the end of the study, the results gathered suggested the MS-3/8 technique as the most highly advised numerical approach among the other methods. This means that, MS-3/8 method is a more accurate solution method for solving system of nonlinear equations considering the results in this paper.

Keywords: Broyden Method, Newton-Raphson method, Quadrature Rules, Simpson – 1/3 Rule, Simpson - 3/8 Rule, Nonlinear Systems, Error Analysis

1. Introduction

Finding equation solutions is a critical task in mathematical computations. Many practical problems are answered by the roots of equations. Finding the most efficient numerical method for the purpose is critical because the accuracy of the result is critical for most practical problems [2]. When a problem requires the solution of a system of nonlinear equations after modeling, it becomes even more difficult.

Solving nonlinear equations, which is one of the fundamental challenges in mathematics, can be used to solve a variety of issues [1]. One of the most challenging issues in numerical computations is the solution of nonlinear equations, particularly for a wide range of engineering applications and numerous scientific disciplines [10]. Researchers have worked very hard, and a lot of helpful theories and algorithms are suggested to answer systems of nonlinear equations [9]. There are still some issues to be resolved with such methods, though. The convergence and performance characteristics of the majority of conventional numerical techniques, including Newton's approach, can be extremely sensitive to the original solution guess. However, for the majority of nonlinear equations, it is extremely challenging to choose an acceptable initial solution guess [8]. If the original guess of the solution is unreasonable, the algorithm may fail or the results may be incorrect. To solve systems of nonlinear equations, a variety of conventional numerical techniques and algorithms are used [6, 11, 13], which can solve the issue of choosing a reasonable initial guess of the solution. However, when there are multiple nonlinear equations to solve at the same time, the algorithms become too complex or expensive to compute with.

The Newton-Raphson scheme continues to be a favorite among the groups of numerical schemes for solving nonlinear systems of equations. However, the Newton-Raphson technique has some drawbacks, the most significant of which is the requirement to compute the inverse Jacobian matrix iteratively. This renders it particularly ineffective for large-scale issues [17], and this serves as a motivation for this research work.

The quasi-Newton Broyden method has undergone significant modifications and enhancements, which have inspired other researchers to create new techniques for quickly solving nonlinear systems of equations [1]. Numerous authors continue to offer various Newton-like methods [18, 5, 3, 4], using the secant methods, or quadrature formulas.

One of these methods introduced the central finite difference to roughly estimate the inverse Jacobian matrix, which led to the proposal of two improved classes of schemes [1]. Before using the Broyden method to solve the issue, the Steepest Decent method was presented in another research paper and used to get good and sufficient initial guesses (starting values) [12].

Formulating iterative schemes using quadrature rules has been a prominent trend of new methods created for the computation of solutions of systems of nonlinear equations for the past few years. Some references to developed techniques based on quadrature rules include: [14-17, 7].

The Newton Cotes quadrature rules are a set of numerical integration formulas based on the integrand's evaluation at evenly separated points and this was named after Roger Cotes and Isaac Newton [4], they approximated data using local order k polynomials. By evaluating a function at k nodes $(x_1, x_2, \dots, x_n)$ and weighting those nodes with n weights $w_1, w_2, \dots, w_n$, the Newton-Cotes quadrature formulas estimate the integral of a function $\int_a^b f(x)dx$. The Mid-point, Trapezoidal, and Simpson's rules are the three most popular Newton-Cotes quadrature algorithms. The Newton-Cotes formula's basic version is;

$$\int_a^b f(x)dx = \sum_{k=1}^{n} w_k f(x_k) \tag{1}$$

The Taylor's series expansion of a function (of a single variable) f(x) about the point x_1 can be used to determine the Newton's method:

$$f(x) = f(x_1) + (x - x_1)f'(x_1) + \frac{1}{2!}(x - x_1)^2 f''(x_1) + \cdots \tag{2}$$

where f is assessed at x_1, along with its first and second derivatives, f' and f'' . For a function with numerous variables, $F: R^n \to R^n$, the equation (2) can be demonstrated [16], to equivalently result into:

$$F(x) = F(x_k) + \int_{x_k}^{x} F'(t)dt \tag{3}$$

The Jacobian J is the matrix of partial derivatives $F'(t)$ that appears in equation (3), where $\int_{x_k}^{x} F'(t)dt$ is a multiple integral as in (4):

$$\int_{x_k}^{x} F'(t)dt = \int_{x_{k,1}}^{1} \int_{x_{k,2}}^{2} \cdots \int_{x_{k,n}}^{n} F'(x_1, x_2, \dots, x_n) dx_n dx_{n-1} \dots dx_1 \tag{4}$$

As a substitute, one could consider the multiple integral as a nested series of one-dimensional integrals and apply the one-dimensional quadrature rule to each argument individually [9]. Therefore, using the weighted combination of quadrature formulas, we can estimate $\int_{x_k}^{x} F'(t)dt$. Using Newton Cotes formulae of order zero to one, the authors [5, 4, 3, 17, 16] and the references therein have suggested a number of approaches for approximating the indefinite integral in equation (4). Using a weighted combination of the Trapezoidal, Simpson, and Midpoint quadrature rules, a variation of the Broyden-like method was suggested in a related work. This yielded the Broyden-like method known as TSMM, [17], which is given as;

$$m_k = x_k - B_K^{-1} F(x_k)$$

$$x_{k+1} = x_k - 24[5B(x_k) + 14B(z_k) + 5B(m_k)]^{-1} F(x_k) \tag{5}$$

where: $z_k = \frac{x_k + m_k}{2}, k = 0, 1, \dots$

The Classical Broyden (CB), Trapezoidal-Broyden (TB), and Midpoint-Simpson-Broyden methods (MSB) were compared to the TSMM, and the TSMM approach outperformed all of them [17]. The following year, the same author carried out additional related research that led to the development of a reliable Broyden-like technique known as the Midpoint-Trapezoidal (MT) method. The method's incremental framework is provided as;

$$m_k = x_k - B_K^{-1} F(x_k)$$

$$x_{k+1} = x_k - 4[B(x_k) + 2B(z_k) + B(m_k)]^{-1} F(x_k) \tag{6}$$

for $z_k = \frac{x_k + m_k}{2}, k = 0, 1, \dots$

The Classical Broyden (CB), Trapezoidal-Broyden, and Midpoint-Simpson-Broyden (MSB) methods were compared to the MT method, and it was found that the MT method did significantly better than all of them [16]. The consistent use of the three common quadrature formulas is a noteworthy feature of all the aforementioned Broyden-like techniques. (i.e. Trapezoidal, Midpoint and Simpson rules). Further improved techniques were expected to be created with the help of the improved versions of some of these common quadrature rules.

This research uses a weighted combination of the quadrature rules Trapezoidal, Midpoint, Simpson, Simpson's 1/3, and Simpson's 3/8 to approximate the integral in Equation (4).

A research aimed a formulating new Broyden-like method adopted the process of combining quadrature rules, and this yielded the TS-1/3, MS-1/3, SS-1/3, TS-3/8, MS-3/8 and SS-3/8 Broyden methods [7].

In this study the following objectives are achieved: (i) Computation of the approximate error of the selected numerical methods (ii) Comparison of the errors associated with each numerical method in order to find out the most accurate method. Further to this, the section 2.0 describes the New Alternative Broyden-like methods while section 3.0 explains how the numerical test were carried out in the research. The results and discussions with regards to the error analysis of the methods are discussed in section 4.0 and the final conclusion is discussed in section 5.0.

2. New Alternative Broyden–Like Methods

In this section, we will show our new schemes. Consider the following finding, the proof for which can be found in Azure et al, 2021 [7].

2.1. The MS-3/8 Method [7]

This method was formulated by carrying out a weighted quadrature combination of the Midpoint quadrature rule which is given by the formula:

$$\int_a^b F(x) \approx (b-a)F\left(\frac{a+b}{2}\right) \tag{7}$$

And the Simpson $3/8$ quadrature rule, represented by the formula:

$$\int_a^b F(x) \approx \left(\frac{b-a}{8}\right)\left(f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right) \tag{8}$$

The resulting numerical scheme for the MS-3/8 method is as shown below;

$$x_{k+1} = x_k - 16[B(x_k) + 14B(z_k) + B(m_k)]^{-1}F(x_k) \tag{9}$$

For $m_k = x_k - B_K^{-1}F(x_k)$ *and* $z_k = \frac{x_k+m_k}{2}$, *where* $k = 0, 1, \ldots$

The proof of the MS-3/8 method can be found in [7].

2.2. TS-3/8 Method [7]

In a similar way as in the derivation of the MS-3/8 scheme, the TS-3/8 method was formulated by combining two weighted quadrature rules namely; the Trapezoidal quadrature rule, which is given by;

$$\int_a^b F(x) \approx \left(\frac{b-a}{2}\right)\left(F(a) + F(b)\right) \tag{10}$$

and the Simpson $3/8$ quadrature rule, given by

$$\int_a^b F(x) \approx \left(\frac{b-a}{8}\right)\left(f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right) \tag{11}$$

The TS-3/8 method is thus given as;

$$x_{k+1} = x_k - 16[5F'(x_k) + 6F'(z_k) + 5F'(m_k)]^{-1}F(x_k) \tag{12}$$

with $m_k = x_k - B_K^{-1}F(x_k)$ and $z_k = \frac{x_k+m_k}{2}, k = 0, 1, \ldots,$ and its proof can be found in [7].

3. Numerical Tests

In order to determine the approximate error associated with the selected methods, the bench-mark problem was solved analytically to obtain its exact solution. The selected numerical methods being the Classical Broyden Method (CB), Trapezoidal–Simpson 3/8 Method (TS-3/8) and Midpoint-Trapezoidal 3/8 (MT-3/8), were also used to solve the same benchmark problem, hence making it possible for the computation of the approximate errors of each of the methods. The computation was performed in MATLAB R2020b using double precision arithmetic on a computer with the following specifications: CPU: AMD EI-2100APU with Radeon TM Graphics 1.00GHz, installed memory (RAM): 4.00GB, and system type is 64 - bit Operating System, x 64 - based processor.

4. Results and Discussion

This sub-section analyses the computational error associated with the MS-3/8 method in comparison with other methods such as the Broyden and the TS-3/8 methods. The study considered a simple system of nonlinear equations with two unknowns as in equation (13).

$$\left.\begin{aligned} f_1 &= 3x_1^2 - x_2^2 = 0 \\ f_2 &= 3x_1x_2^2 - x_1^3 - 1 = 0 \end{aligned}\right\} \tag{13}$$

With $x^0 = (1,1)^T$

The exact solution that was obtained after equation (13) was solved using the substitution method was; $x_1 = 0.5$ and $x_2 = 0.866025$.

The table 1 summarises the results that were obtained when the equation (13) was solved using the MS-3/8, Broyden and TS-3/8 methods. Results were obtained with the help of Matlab codes using the software version R2020b.

The table 1 also include the approximate error analysis for each of the methods. The error was estimated by subtracting the approximated values which were obtained using the numerical methods such as the MS-3/8, Broyden and the TS-3/8 methods.

The results in the table 1 showed that for the varied number of iterations such as K=1, 5, 10, 20, 50, 100 and 1000, the MS-3/8 and TS-3/8 methods converged, however, the MS-3/8 method had results which were very close to the exact solution for all the number of iterations compared to the TS-3/8 method.

On the other hand, the Broyden method did not converge for iterations k=1 and 5, but estimated results for iterations k=10, 20, 50, 100 and 1000 which were all very close to the exact solution, however, a comparison between the approximate errors of the Broyden and MS-3/8 methods from the table 1 showed that the MS-3/8 method had the least error for all the iterations considered, compared with the Broyden and TS-3/8 methods.

Table 1. *Comparison of Approximate Errors of Numerical Methods.*

k	Exact Value $X = (x_1, x_2)$	Broyden Method $X_1 = (x_1, x_2)$	Error e=|X - X_1|	TS-3/8 $X_2 = (x_1, x_2)$
1	0.5, 0.866025	No convergence	-	19.648152968607381, 13.810208536181882
5	0.5, 0.866025	No convergence	-	-1.148487745563099, 1.881711958146404
10	0.5, 0.866025	0.500000094006117, 0.866025655476792	1.0e-06 * 0.093991439742425, 0.253659208415691	0.278599390997886, -0.463231655644685
20	0.5, 0.866025	0.500000094006117, 0.866025655476792	1.0e-06 * 0.093991439742425, 0.253659208415691	0.500004715370853, -0.866024184552828
50	0.5, 0.866025	0.500000094006117, 0.866025655476792	1.0e-06 * 0.093991439742425, 0.253659208415691	0.499999965435269, -0.866025443646975
100	0.5, 0.866025	0.500000094006117, 0.866025655476792	1.0e-06 * 0.093991439742425, 0.253659208415691	0.499999965435269 -0.866025443646975
1000	0.5, 0.866025	0.500000094006117, 0.866025655476792	1.0e-06 * 0.093991439742425, 0.253659208415691	0.499999965435269 -0.866025443646975

Table 1. *Continued.*

k	Error e=|X - X_2|	MS-3/8 $X_3=(x_1, x_2)$	Error e=|X - X_3|
1	19.148152968592704, 12.944183134364298	0.524848464313792, 0.850986375423251	0.024848464299115, 0.015039026394333
5	1.648487745577777, 1.015686556328821	0.500000022328748, 0.866025268162215	1.0e-06 * 0.022314070591278, 0.133655368550478
10	0.221400609016791, 1.329257057462268	0.500000022328748, 0.866025268162215	1.0e-06 * 0.022314070591278, 0.133655368550478
20	0.000000034579409, 1.732050845464559	0.500000022328748, 0.866025268162215	1.0e-06 * 0.022314070591278, 0.133655368550478
50	0.000000034579409, 1.732050845464559	0.500000022328748, 0.866025268162215	1.0e-06 * 0.022314070591278, 0.133655368550478
100	0.000000034579409, 1.732050845464559	0.500000022328748, 0.866025268162215	1.0e-06 * 0.022314070591278 0.133655368550478
1000	0.000000034579409 1.732050845464559	0.500000022328748 0.866025268162215	1.0e-06 * 0.022314070591278 0.13365A5368550478

5. Conclusion

The results displayed after the comparison of the approximate errors of the methods clearly show that the Broyden method will need several number of iterations to converge to a solution, hence, computing for the approximate error for fewer number of iterations using the Broyden method is not possible, However, with the same Broyden method, there is a recurring solution at the tenth iteration up until the thousandth iteration, and the solution obtained is closer to the exact solution hence with a marginal approximate error.

In the case of the TS-3/8 method, it converged at the first iteration but the solution differ greatly from the exact solution, hence causing a great approximate error. It can however be seen that, approximate error reduces with an increase in the number of iterations.

The MS-3/8 method on the other hand, maintained an infinitesimally marginal approach error right from the first iteration to the last iteration considered in the study. This gives a confirmation that the MS-3/8 method is proven to be the most accurate methods among the other method tested in this study.

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