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On the Dual Nature of Forced Transverse Vibrations of Bridges Under the Action Moving Load

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Abstract: In this paper, forced transverse vibrations of an elastic hinge-supported Timoshenko beam are considered, taking into account the rotational motion caused by a periodically oscillating concentrated load moving along the beam at a constant speed v. This problem is of practical interest in connection with the study of forced transverse vibrations of bridges. The bridge span is considered here as a Timoshenko beam of constant transverse cross-section. The problem is solved by the method proposed earlier using combined conditions, including dynamic action on the Timoshenko beam and rotational motion relative to the bending wave front. The solution of the problem is built in the form of a number of own forms of vibrations. Two types of forced transverse vibrations and new resonance frequencies are obtained. The purpose of this study is to assess the effect of the identified new forced transverse vibrations for bridges and compare these results. With the solutions obtained by previous authors. To show at which new resonant frequency obtained in bridges new resonance phenomena arise. New dynamic phenomena in bridges caused by a periodically oscillating concentrated load moving along the beam at a constant speed, play an important role in bridge design. This work is a new calculation scheme for the design of bridges.

Keywords: Bridge, Centrifugal Force, Cross Section, Gravity, Transverse Oscillations, Natural Frequencies, Natural Forms

1. Introduction

Investigations of the transverse vibration of bridges under the action of moving loads have been carried out by many authors. Among these scientific works, we note, in particular, the article by G. G. Stokes [1], where a new problem of the movement of a load along a massive beam (Stokes problem) was formulated. The problem was not fully resolved. AN Krylov investigated the problem of the dynamics of massive beams under the influence of a moving force and gave a complete solution to this problem [2]. The solution of the Stokes problem, further, was considered in the works of SP Timoshenko [3-5, 6], where the critical velocities of the cargo movement were determined by the Fourier method and using generalized coordinates.

In the work of A. L. Florence, equations of the Timoshenko type are used to study the vibrations of a semi-infinite beam, along which a concentrated transverse force moves at a constant speed [7]. The solutions are constructed using the Laplace transform method. J. D. Achenbach, C. T.Sun [8] considered the problem of motion with a constant velocity V of a concentrated force along an infinite Timoshenko beam lying on an elastic foundation. By replacing the variables v=x - Vt, the equations of motion are reduced to a system of two ordinary differential equations for the angle of rotation and deflection, which are solved by the Fourier transform.

In the works of Tang Tang Sing-Chih [9], Bo1ey B. A. and Chi-Chang Chao [10], the problem of vibrations of the Timoshenko beam girder under the action of a moving force, which was previously solved using integral transformations of A. L. Florence [7], is investigated by the method of characteristics numerically. The calculation results are in good agreement with the analytical solutions. It was found that the computational work costs less when using the method of characteristics than when solving by the method of integral transformations.

H. H. Pan [11] examined a similar problem with viscous considerations. A lot of literature on this issue can be found in the works of P. M. Mathews [12], E. I. Grigolyuk and I. T.

Selezov [13]. The idea and basic equations of bending vibrations of a cantilever elastic rod with regard to rotational motion are given in the previous work [14], when a displacement Asinωt is applied to one end, while the other end is free. Later, in the study of Mkrtchyan [15], the forced transverse vibrations of an elastic hinged-supported rod were studied, taking into account the rotational motion, caused by a periodically oscillating concentrated load. A number of valuable results have been obtained in this direction. This article touches upon the problem of vibrations of the Timoshenko beam under the action of a moving force, which takes into account rotational motions.

The main task, is solved by dividing it into two tasks, each of which deals with forced transverse vibrations of an elastic hinged-supported beam Timoshenko, caused by different parts of the combined effect, including the dynamic effect on the beam Timoshenko and the rotational motion relative to the front of the bending wave. Much attention is paid to numerical results. Assessment of the influence of rotational motion on the dynamic deflections of elastic hinge-supported is given beam Timoshenko.

2. Mathematical Formulation of Problem 1 and Its Solution

Consider an elastic homogeneous hinge-supported Timoshenko beam having l length. Let at the moment $t = 0_+$ the beam is acted upon by a periodically oscillating concentrated load moving along the beam at a constant velocity ν. It is assumed that at the initial moment of time the beam was at rest, and the moving load was at its left end (the problem of a steam locomotive moving along a railway bridge). Let us attribute it to the Cartesian coordinate system x, y, z, the x axis of which is directed along the neutral line of the non-deformed beam, y and z axes are along the axes of symmetry of the cross section. The oscillations of the rod occur in avertical plane (xz plane), the geometry of which is presented in Figure 1.

The following problem is posed and solved: it is required to determine the forced transverse oscillations of this beam, resulting from the application of a moving load in the form

$$P(x,\mathrm{t}) = \begin{cases} \dfrac{\mathrm{p}\sin\omega t}{2\varepsilon}, \nu t - \varepsilon < x < \nu t + \varepsilon, \\ 0 - \text{for other values of } x, \end{cases} (\varepsilon \to 0).$$

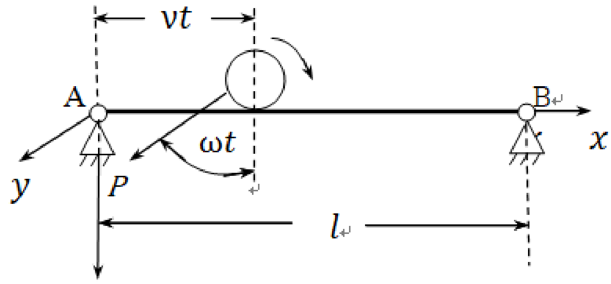

Figure 1. *The geometry of the problem.*

For the considered task, the forced transverse oscillations of the beam, taking into account the effects of transverse shear and inertia of rotation, are described by the following equations with the boundary and initial conditions [15]:

$$\mathrm{E}J\frac{\partial^2\psi}{\partial x^2} + kGF\left(\frac{\partial w}{\partial x} - \psi\right) = \rho J\frac{\partial^2\psi}{\partial t^2} \quad (1)$$

$$kGF\left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial\psi}{\partial x}\right) + P(x,\mathrm{t}) = \rho F\frac{\partial^2 w}{\partial t^2}$$

$$x=0,l\!: w = 0;\ \frac{\partial\psi}{\partial x} = 0 \quad (2)$$

$$t=0\!: w = 0;\ \frac{\partial w}{\partial t} = 0$$

$$\psi = 0;\ \frac{\partial\psi}{\partial t} = 0 \quad (3)$$

where t - is the time; l - is the length of the beam; F - is the cross-sectional area; J - is the moment of inertia of the cross section; δ() is the delta - Dirac function;

P−centrifugal force; ρ - is density of the material; E, G - are Young and shear modules, respectively; k - is a dimensionless numerical coefficient, depending on the cross-sectional shape of the rod; w - is the displacement of the center of bending of the section in the direction of the z axis (deflection); ψ - is the angle of rotation of the section around the center of the bend; ω - is the circular frequency.

The boundary conditions (2) will be satisfiedif the solution of system (1) is presented in the form:

$$w = \sum_{n=1}^{\infty} w_n(t)\sin\mu_n x \quad (4)$$

$$\psi = \sum_{n=1}^{\infty} \psi_n(t)\cos\mu_n x \quad (5)$$

where $\mu_n = n\pi/l$. Substituting the values of the functions w and ψ from (4) and (5) into system (1), obtain the following system of ordinary differential equations with respect to the desired functions w_n and ψ_n:

$$\rho\mathrm{J}\frac{d^2\psi_n}{dt^2} + (\mathrm{EJ}\mu_n^2 + kGF)\psi_n - kGF\,\mu_n w_n = 0$$

$$\rho F\frac{d^2 w_n}{dt^2} + kGF\mu_n^2 w_n - kGF\,\mu_n\psi_n = \mathrm{p}l^{-1}[\sin(\mu_n\nu + \omega)t + \sin(\mu_n\nu - \omega))t] \quad (6)$$

The general solution of system (6) with initial conditions (3) has the form

$$w_n(t) = \mathrm{A}_n\sin\omega_{n1}t + \mathrm{B}_n\sin\omega_{n2}t + \mathrm{p}l^{-1}[\Gamma_n^+\mathrm{H}_n^+\sin(\mu_n\nu + \omega)t + \Gamma_n^-\mathrm{H}_n^-(\mu_n\nu - \omega)t] \quad (7)$$

$$\psi_n(t) = \left(\mu_n - \frac{\rho\omega_{n1}^2}{kG\mu_n}\right)\mathrm{A}_n\sin\omega_{n1}t + \left(\mu_n - \frac{\rho\omega_{n2}^2}{kG\mu_n}\right)\mathrm{B}_n\sin\omega_{n2}t + \mathrm{p}l^{-1}\mu_n[\mathrm{H}_n^+\sin(\mu_n\nu + \omega)t + \mathrm{H}_n^-\sin(\mu_n\nu - \omega)t] \quad (8)$$

where

$$A_n = \frac{p}{\rho l \omega_{n1}(\omega_{n1}^2-\omega_{n2}^2)}\{[kG\mu_n^2 - (kG\mu_n^2 - \rho\omega_{n2}^2)\Gamma_n^+](\mu_n\nu + \omega) H_n^+ + [kG\mu_n^2 - (kG\mu_n^2 - \rho\omega_{n2}^2)\Gamma_n^-](\mu_n\nu - \omega)H_n^-\}$$

$$B_n = -\frac{p}{\rho l \omega_{n2}(\omega_{n1}^2-\omega_{n2}^2)}\{[kG\mu_n^2 - (kG\mu_n^2 - \rho\omega_{n1}^2)\Gamma_n^+](\mu_n\nu + \omega) H_n^+ + [kG\mu_n^2 - (kG\mu_n^2 - \rho\omega_{n1}^2)\Gamma_n^-](\mu_n\nu - \omega)H_n^-\}$$

$$H_n^{\pm} = \{ EJ\mu_n^4 - \left[\rho F + \left(\rho J + \frac{\rho EJ}{kG}\right)\mu_n^2\right](\mu_n\nu \pm \omega)^2 + \frac{\rho^2 J}{kG}(\mu_n\nu \pm \omega)^4\}^{-1}$$

$$(H_n^{\pm})^{-1} \neq 0, \Gamma_n^{\pm} = 1 + \frac{EJ\mu_n^2}{kGF} - \frac{\rho J(\mu_n\nu \pm \omega)^2}{kGF}.$$

ω_{n1}, ω_{n2} – represent the natural frequency of the beam Timoshenko and are described by the expression

$$\omega_{ni} = (2\gamma)^{-1/2}\left(\lambda_n + (-1)^i(\lambda_n^2 - 4\gamma k_n^2)^{1/2}\right)^{1/2} \tag{9}$$

$$\lambda_n = 1 + \left(\frac{J}{F} + \frac{EJ}{kGF}\right)\mu_n^2, k_n = \sqrt{\frac{EJ}{\rho F}}\mu_n^2, \gamma = \frac{\rho J}{kGF}$$

Hereinafter, i=1, 2.

Substituting the values of the functions $w_n(t)$, $\psi_n(t)$ from (7), (8) into the corresponding equations (4), (5), we find that the forced transverse oscillations of the beam under consideration are determined by the law

$$w = \frac{p}{l}\sum_{n=1}^{\infty}[\Gamma_n^+ H_n^+ \sin(\mu_n\nu + \omega)t + \Gamma_n^- H_n^- \sin(\mu_n\nu - \omega)t] \sin \mu_n x \tag{10}$$

$$\psi = \frac{p}{l}\sum_{n=1}^{\infty} \mu_n [H_n^+ \sin(\mu_n\nu + \omega)t + H_n^- \sin(\mu_n\nu - \omega)t] \cos \mu_n x \tag{11}$$

When the denominator n − x of the terms of series (10), (11) becomes equal to zero, the frequency of the disturbing force approaches one of the values:

$$\omega_{ni}^{\kappa} = \pm\mu_n\nu \pm \omega_{ni}, (\kappa = 1,2,3,4),$$

which are determined from the conditions $(H_n^{\pm})^{-1}$=0. In this case, obtain a state of resonance.

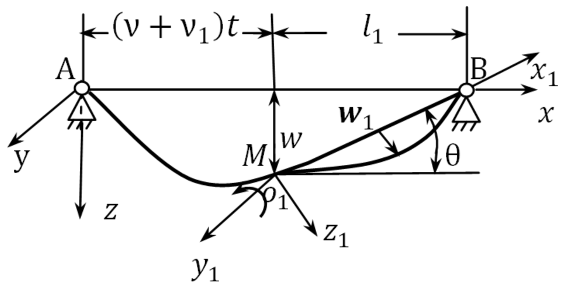

Figure 2. *Perturbed part of the beam.*

3. Mathematical Formulation of Problem 2

Let us consider, within the framework of the refined Timoshenko's theory, the rotational motion of the non-perturbed part of the beam, relative to the bending wave front (Figure 2) (in Figure 2 the same notation is used as in in [15]). Rotational motion occurs in the vertical plane xz, around the y_1 axis and it is carried out by the rotational moment M (t), which is formed on the front of the bending wave. The following problem is posed and solved: it is required to determine in this case the forced transverse oscillations w_1(t, x) and ψ_1(t, x) resulting from the rotational motion of the unperturbed part of the beam, as well as the rotational moment M(t) ensuring the specified movement of this part of the beam Timoshenko.

The linear - differential equation of motion and land the system of equations of transverse vibrations of an elastic hinge-supported beam Timoshenko have the form [15]:

$$\rho F \int_0^l x[x\ddot{\theta} - \ddot{w}_1(t,x)]\, dx = M(t) + \rho g F \int_0^l [x\cos\theta + w_1(t,x)\sin\theta]\, dx \tag{12}$$

$$EJ\frac{\partial^2\psi_1}{\partial x^2} + kGF\left(\frac{\partial w_1}{\partial x} - \psi_1\right) = \rho J\frac{\partial^2\psi_1}{\partial t^2}$$

$$kGF\left(\frac{\partial^2 w_1}{\partial x^2} - \frac{\partial\psi_1}{\partial x}\right) + q(x,t) = \rho F\frac{\partial^2 w_1}{\partial t^2} \tag{13}$$

with boundary and initial conditions:

$$x = 0, l: w_1 = 0;\ \partial\psi_1/\partial x = 0 \tag{14}$$

$$t = 0: \begin{array}{l} w_1 = 0;\ \partial w_1/\partial t = 0 \\ \psi_1 = 0;\ \partial\psi_1/\partial t = 0 \end{array} \tag{15}$$

where q(t, x) are the additional forces that have arisen from the rotational motion of the beam Timoshenko and whose dependence on the coordinate x and time t has the form

$$q(x,t)=\begin{cases}\rho F\left[x\ddot{\theta}-g\cos\theta\right] \text{ for } x'\le x\le l\\ 2ml/(\nu+\nu_1)<t<(2m+1)l/(\nu+\nu_1)\\ 0-\text{for other values of x, t,}\end{cases}$$

If the bending wave along the beam Timoshenko propagates in the opposite direction, then q(t, x) is represented as

$$q(x,t)=\begin{cases}-\rho F\left[x\ddot{\theta}+g\cos\theta\right] \text{ for } 0\le x\le x'\\ 2ml/(\nu-\nu_1)<t<(2m+1)l/(\nu-\nu_1)\\ 0-\text{for other values of x, t,}\end{cases}$$

m=0,1, …

where

$$\theta(t)=\psi(t, x') \tag{16}$$

x' (t) - the law of propagation in the forward and reverse direction of the leading and back edge of the bending wave along the beam Timoshenko is given by the equations

$$\begin{gathered} x'=(\nu+\nu_1)t-2ml\\ 2ml/(\nu+\nu_1)<t<(2m+1)l/(\nu+\nu_1)\\ m=0,1,\ldots \end{gathered}\tag{17}$$

$$\begin{gathered} x'=2(m+1)l-(\nu+\nu_1)t\\ (2m+1)l/(\nu+\nu_1)<t<2(m+1)l/(\nu+\nu_1)\\ m=0,1,\ldots \end{gathered}\tag{18}$$

$$\begin{gathered} x'=(\nu-\nu_1)t-2ml\\ 2ml/(\nu-\nu_1)<t<(2m+1)l/(\nu-\nu_1)\\ m=0,1,\ldots \end{gathered}\tag{19}$$

$$\begin{gathered} x'=2(m+1)l-(\nu-\nu_1)t\\ (2m+1)l/(\nu-\nu_1)<t<2(m+1)l/(\nu-\nu_1)\\ m=0,1,\ldots \end{gathered}\tag{20}$$

With the help of (11), (16) and (17) in the segment $x'\le x\le l$, $2ml/(\nu+\nu_1)<t<(2m+1)l/(\nu+\nu_1)$, m=0,1,… can bring $\ddot{\theta}(t)$ to the following form:

$$\ddot{\theta}(t)=\sum_{n=1}^{\infty}\ddot{\theta}_n(t)$$

$$\begin{gathered}\ddot{\theta}_n(t)=-\mathrm{p}(2l)^{-1}\mu_n\mathrm{H}_n^{+}\{[\mu_n(2\nu+\nu_1)+\omega]^2\sin[(\mu_n(2\nu+\nu_1)+\omega)t-2\mu_n ml]\\ -(\mu_n\nu_1-\omega)^2\sin[(\mu_n\nu_1-\omega)t-2\mu_n ml]\}-\\ -\mathrm{p}(2l)^{-1}\mu_n\mathrm{H}_n^{-}\{[\mu_n(2\nu+\nu_1)-\omega]^2\sin[(\mu_n(2\nu+\nu_1)-\omega)t-2\mu_n ml]-(\mu_n\nu_1+\omega)^2\sin[(\mu_n\nu_1+\omega)t-2\mu_n ml]\}\end{gathered}$$

After determining the bend $w_1(t,x)$ and the angle of rotation of the beam element $\psi_1(t,x)$ the requiredrotation moment M(t), providing the specified movement in this part of the beam, is calculated by the equation (12).

4. Solution of the Problem 2

The solution of the system (13) (without taking into account the beam Timoshenko own weight) will be sought in such a form that the boundary conditions (14) and the initial conditions (15) are completely satisfied, namely: we assume that:

$$w_1 = \sum_{n=1}^{\infty} w_{1n}(t) \sin \mu_n x,$$

$$2ml/(\nu + \nu_1) < t < (2m+1)l/(\nu + \nu_1),\ m=0,1,\dots \quad (21)$$

$$\psi_1 = \sum_{n=1}^{\infty} \psi_{1n}(t) \cos \mu_n x.$$

Substituting the values of w_1 and ψ_1 from (21) into system (13), we obtain the following system of ordinary differential equations for the desired functions w_{1n}, ψ_{1n}:

$$\rho J \frac{d^2\psi_{1n}}{dt^2} + (EJ\mu_n^2 + kGF)\psi_{1n} - kGF\mu_n w_{1n} = 0,$$

$$\rho F \frac{d^2 w_{1n}}{dt^2} + kGF\mu_n^2 w_{1n} - kGF\mu_n \psi_{1n} = q_n(t), \quad (22)$$

where

$$q_n(t) = 2l^{-1}\rho F \ddot{\theta}_n(t) D_n(x'),$$

$$D_n(x') = \begin{cases} \dfrac{1}{\mu_n^2}\{\beta_n - [\sin \mu_n x' - \mu_n x' \cos \mu_n x']\} \\ x' \le x \le l,\ 2ml/(\nu+\nu_1) < t < (2m+1)l/(\nu+\nu_1), \\ -\dfrac{1}{\mu_n^2}\{[\sin \mu_n x' - \mu_n x' \cos \mu_n x']\} \\ 0 \le x \le x',\ (2m+1)l/(\nu+\nu_1) < t < 2(m+1)l/(\nu+\nu_1), \end{cases}$$

$$\beta_n = \sin \mu_n l - \mu_n l \cos \mu_n l$$

Having constructed the solution of the system (22) according to the method of variation of constants under initial conditions (15), we conclude that the elastic transverse oscillations of the beam Timoshenko in time of motion are determined by the law:

$$w_1 = \sum_{n=1}^{\infty} \left[\int_0^t f_{1n}(\tau) \sin \omega_{n1}(t-\tau)d\tau + \int_0^t f_{2n}(\tau) \sin \omega_{n2}(t-\tau)d\tau \right] \sin \mu_n x$$

$$(x' \le x \le l,\ 2ml/(\nu+\nu_1) < t < (2m+1)l/(\nu+\nu_1)) \quad (23)$$

$$\psi_1 = \sum_{n=1}^{\infty} \left[\int_0^t g_{1n}(\tau) \sin \omega_{n1}(t-\tau)d\tau + \int_0^t g_{2n}(\tau) \sin \omega_{n2}(t-\tau)d\tau \right] \cos \mu_n x$$

$$f_{1n}(t) = \ddot{\theta}_n(t) D_n(x') P_{1n},\ f_{2n}(t) = \ddot{\theta}_n(t) D_n(x') P_{2n}$$

$$g_{1n}(t) = \left(\mu_n - \frac{\rho\omega_{n1}^2}{kG\mu_n}\right) f_{1n}(t),\ g_{2n}(t) = \left(\mu_n - \frac{\rho\omega_{n2}^2}{kG\mu_n}\right) f_{2n}(t)$$

$$P_{1n} = \frac{2(kG\mu_n^2 - \rho\omega_{n2}^2)}{\rho l \omega_{n1}(\omega_{n1}^2 - \omega_{n2}^2)},\ P_{2n} = \frac{2(kG\mu_n^2 - \rho\omega_{n1}^2)}{\rho l \omega_{n2}(\omega_{n1}^2 - \omega_{n2}^2)}.$$

Solutions (23) after transformations can be represented in the form of necessary and free oscillations caused by the disturbing forces q(t, x). Omitting the details, present the forced transverse vibrations from the general solution (23) in the form

$$w_1 = \frac{p}{2l} \sum_{n=1}^{\infty} \mu_n^{-1} [P_{1n} u_n(\omega_{n1}, t) + P_{2n} u_n(\omega_{n2}, t)] \sin \mu_n x$$

$$(x' \le x \le l,\ 2ml/(\nu+\nu_1) < t < (2m+1)l/(\nu+\nu_1)) \quad (24)$$

$$\psi_1 = \frac{p}{2l} \sum_{n=1}^{\infty} \mu_n^{-1} [P_{1n}^* u_n(\omega_{n1}, t) + P_{2n}^* u_n(\omega_{n2}, t)] \cos \mu_n x$$

$$P_{1n}^* = -\left(\mu_n - \frac{\rho\omega_{n1}^2}{kG\mu_n}\right) P_{1n},\ P_{2n}^* = -\left(\mu_n - \frac{\rho\omega_{n2}^2}{kG\mu_n}\right) P_{2n}$$

$$u_n(\lambda, t) = -\frac{\lambda\beta_n H_n^+ [\mu_n(2\nu+\nu_1) + \omega]^2}{\lambda^2 - [\mu_n(2\nu+\nu_1) + \omega]^2} \sin[(\mu_n(2\nu+\nu_1) + \omega)t - 2\mu_n m l] +$$

$$+\frac{\lambda\beta_n H_n^+(\mu_n\nu_1-\omega)^2}{\lambda^2-(\mu_n\nu_1-\omega)^2}\sin[(\mu_n\nu_1-\omega)t-2\mu_n ml]-\frac{\lambda\beta_n H_n^-[\mu_n(2\nu+\nu_1)-\omega]^2}{\lambda^2-[\mu_n(2\nu+\nu_1)-\omega]^2}\sin[(\mu_n(2\nu+\nu_1)-\omega)t-2\mu_n ml]+$$

$$+\frac{\lambda\beta_n H_n^-(\mu_n\nu_1+\omega)^2}{\lambda^2-(\mu_n\nu_1+\omega)^2}\sin[(\mu_n\nu_1+\omega)t-2\mu_n ml]-\frac{\lambda\mu_n H_n^+}{\lambda^2-(\mu_n\nu+\omega)^2}\{[(\mu_n\nu+\omega)^2+\mu_n^2(\nu+\nu_1)^2]x'(t)\sin(\mu_n\nu+\omega)t-$$

$$-\frac{2(\nu+\nu_1)(\mu_n\nu+\omega)[\lambda^2+\mu_n^2(\nu+\nu_1)^2]}{\lambda^2-(\mu_n\nu+\omega)^2}\cos(\mu_n\nu+\omega)t\}-$$

$$-\frac{\lambda H_n^+2^{-1}[\mu_n(2\nu+\nu_1)+\omega]^2}{\lambda^2-[\mu_n(3\nu+2\nu_1)+\omega]^2}\{\mu_n x'(t)\sin[(\mu_n(3\nu+2\nu_1)+\omega)t-4\mu_n ml]+$$

$$+\frac{\lambda^2-[\mu_n(3\nu+2\nu_1)+\omega][\mu_n(5\nu+4\nu_1)+\omega]}{\lambda^2-[\mu_n(3\nu+2\nu_1)+\omega]^2}\cos[(\mu_n(3\nu+2\nu_1)+\omega)t-4\mu_n ml]\}+$$

$$+\frac{\lambda H_n^+2^{-1}(\mu_n\nu-\omega)^2}{\lambda^2-[\mu_n(\nu+2\nu_1)-\omega]^2}\{\mu_n x'(t)\sin[(\mu_n(\nu+2\nu_1)-\omega)t-4\mu_n ml]+$$

$$+\frac{\lambda^2-[\mu_n(\nu+2\nu_1)-\omega][\mu_n(3\nu+4\nu_1)-\omega]}{\lambda^2-[\mu_n(\nu+2\nu_1)-\omega]^2}\cos[(\mu_n(\nu+2\nu_1)-\omega)t-4\mu_n ml]\}-$$

$$-\frac{\lambda\mu_n H_n^-}{\lambda^2-(\mu_n\nu-\omega)^2}\{[(\mu_n\nu-\omega)^2+\mu_n^2(\nu+\nu_1)^2]x'(t)\sin(\mu_n\nu-\omega)t-$$

$$-\frac{2(\nu+\nu_1)(\mu_n\nu-\omega)[\lambda^2+\mu_n^2(\nu+\nu_1)^2]}{\lambda^2-(\mu_n\nu-\omega)^2}\cos(\mu_n\nu-\omega)t\}-$$

$$-\frac{\lambda H_n^-2^{-1}[\mu_n(2\nu+\nu_1)-\omega]^2}{\lambda^2-[\mu_n(3\nu+2\nu_1)-\omega]^2}\{\mu_n x'(t)\sin[(\mu_n(3\nu+2\nu_1)-\omega)t-4\mu_n ml]+$$

$$+\frac{\lambda^2-[\mu_n(3\nu+2\nu_1)-\omega][\mu_n(5\nu+4\nu_1)-\omega]}{\lambda^2-[\mu_n(3\nu+2\nu_1)-\omega]^2}\cos[(\mu_n(3\nu+2\nu_1)-\omega)t-4\mu_n ml]\}+$$

$$+\frac{\lambda H_n^-2^{-1}(\mu_n\nu+\omega)^2}{\lambda^2-[\mu_n(\nu+2\nu_1)+\omega]^2}\{\mu_n x'(t)\sin[(\mu_n(\nu+2\nu_1)+\omega)t-4\mu_n ml]+$$

$$+\frac{\lambda^2-[\mu_n(\nu+2\nu_1)+\omega][\mu_n(3\nu+4\nu_1)+\omega]}{\lambda^2-[\mu_n(\nu+2\nu_1)+\omega]^2}\cos[(\mu_n(\nu+2\nu_1)+\omega)t-4\mu_n ml\}.$$

Forced oscillations in the region $0\le x\le x'$, $(2m+1)l/(\nu+\nu_1)<t<2(m+1)l/(\nu+\nu_1)$ (m=0, 1,...) of the elastic hinge-supported beam can be obtained using equations (24) replacing β_n, ν, ν_1, m, with 0, −ν, $-\nu_1$, $-(m+1)$.

When the denominator n − x of the terms of series (24) becomes equal to zero, the frequency of the disturbing force approaches one of the values:

$$\tilde\omega_{ni}^{\kappa}=\pm\mu_n\nu_1\pm\omega_{ni},\bar\omega_{ni}^{\kappa}=\pm\mu_n(\nu+2\nu_1)\pm\omega_{ni},\bar{\bar\omega}_{ni}^{\kappa}=\pm\mu_n(3\nu+2\nu_1)\pm\omega_{ni},$$

$$\omega_{ni}^{\kappa}=\pm\mu_n\nu\pm\omega_{ni}\ (\kappa=1,2,3,4).$$

which are determined from the conditions

$$\omega_{ni}^2-(\mu_n\nu\pm\omega)^2=0,\ \omega_{ni}^2-[\mu_n(\nu+2\nu_1)\pm\omega]^2=0,$$

$$\omega_{ni}^2-(\mu_n\nu_1\pm\omega)^2=0,\ \omega_{ni}^2-[\mu_n(3\nu+2\nu_1)\pm\omega]^2=0.$$

In this case, we obtain a state of resonance.

Comparing problem (1)–(3) with problem (12)–(15), we obtain new values

$$\tilde\omega_{ni}^{\kappa}=\pm\mu_n\nu_1\pm\omega_{ni},\ \bar\omega_{ni}^{\kappa}=\pm\mu_n(\nu+2\nu_1)\pm\omega_{ni},$$

$$\bar{\bar\omega}_{ni}^{\kappa}=\pm\mu_n(3\nu+2\nu_1)\pm\omega_{ni}\ (\kappa=1,2,3,4),\tag{25}$$

for the state of resonance.

The quantities $\tilde\omega_{ni}^{\kappa}$, $\bar\omega_{ni}^{\kappa}$, $\bar{\bar\omega}_{ni}^{\kappa}$ are enumerated so that the relations were true

$$\tilde\omega_{ni}^3=-\tilde\omega_{ni}^2,\ \tilde\omega_{ni}^4=-\tilde\omega_{ni}^1,\ \bar\omega_{ni}^3=-\bar\omega_{ni}^2,\ \bar\omega_{ni}^4=-\bar\omega_{ni}^1,$$

$$\overline{\overline{\omega}}_{ni}^3 = -\overline{\omega}_{ni}^2, \overline{\omega}_{ni}^4 = -\overline{\omega}_{ni}^1.$$

Numerical Example

To illustrate the effectiveness of the obtained results, when a steam locomotive passes the bridge, consider the following numerical values of the parameters for the iron superstructures of four single-track railway bridges with spans of 18.3 m, 36.6 m, 73.2 m, 109.7 m. Having velocity ν equal to 36.6 m/sec, the number of revolutions of the driving wheels of the locomotive is equal to 8 (ω=50.2) per second and the diameter of the driving wheels is 1.45 m[1]).

The calculation results are given in the following two tables and in expression (26). The tables show the values, resonance frequencies $\tilde{\omega}_{ni}^{\kappa}$, $\overline{\omega}_{ni}^{\kappa}$, $\overline{\overline{\omega}}_{ni}^{\kappa}$ as a function of the span of the bridge, for values n=1, i=1, calculated by formulas (25).

$$\max_t |w_1/w| \approx 1{,}4 \tag{26}$$

The following parameters are taken from book 1, p. 248

Table 1. *The resonance frequencies $\tilde{\omega}_{11}^1$, $\overline{\omega}_{11}^1$, $\overline{\overline{\omega}}_{11}^1$ for of the bridge.*

	$\boldsymbol{n=1, i=1}$		
l	$\tilde{\omega}_{11}^1 = +\mu_1\nu_1 + \omega_{11}$	$\overline{\omega}_{11}^1 = +\mu_1(\nu + 2\nu_1) + \omega_{11}$	$\overline{\overline{\omega}}_{11}^1 = +\mu_1(3\nu + 2\nu_1) + \omega_{11}$
18,3м	166,7	283,79	296,35
36,6м	92,27	156,67	162,95
73,2м	54,92	92,9	96,12
109,7м	36,58	61,88	64

Table 2. *The resonance frequencies $\tilde{\omega}_{11}^2$, $\overline{\omega}_{11}^2$, $\overline{\overline{\omega}}_{11}^2$ for of the bridge.*

	$\boldsymbol{n=1, i=1}$		
l	$\tilde{\omega}_{11}^2 = +\mu_1\nu_1 - \omega_{11}$	$\overline{\omega}_{11}^2 = +\mu_1(\nu + 2\nu_1) - \omega_{11}$	$\overline{\overline{\omega}}_{11}^2 = +\mu_1(3\nu + 2\nu_1) - \omega_{11}$
18,3м	54,83	171,89	184,45
36,6м	30,25	94,65	100,93
73,2м	16,1	55,89	59,1
109,7м	11,9	37,2	39,3

The results of calculations, shown in these tables, show that in all considered cases, the resonance frequencies $\tilde{\omega}_{ni}^{\kappa}$, $\overline{\omega}_{ni}^{\kappa}$, $\overline{\overline{\omega}}_{ni}^{\kappa}$ decrease with an increase in the span structure of the bridge. Expression (26) shows the maximum absolute value w_1/w for a span of 73.2m at the point x =40m. The values w and w_1 represent the values of the forced vibrations of the beam, calculated, respectively, by formulas (10) and (24). Expression (26) shows that the largest absolute value of $|w_1|$ is 1.4 times more than $|w|$.

5. Conclusion

Based on the above studies, the following conclusion can be made. From table 2 it can be seen that for bridges having small spans the resonant frequency is so high that the synchronization of the pulsating force and the forced oscillation (24) cannot be obtained at any velocity. Taking, for example, 8 (ω=50.2) revolutions of the driving wheels per second as the highest limit and taking the resonant frequencies from the above table, can conclude that the resonance is hardly possible for spans less than 18.3 m. For longer spans, the resonance phenomenon should be taken into account.

It should be noted that all our calculations were based on the assumption that a pulsating force moves along the bridge. In actual conditions, we have rolling masses that will cause the resonant frequency of the bridge to change in accordance with the change in the position of the loads. This variability of the resonant frequency, which is especially noticeable at small spans, is very favorable, since the pulsating load will no longer give resonance during the entire passage over the bridge, and its increasing effect will not be so noticeable. If several moving loads act on the bridge, the vibrations caused by them should be overlapped. The main results of the studies described herein can be successfully applied by designers to solve a number of important problems for bridges.

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