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Collocation Techniques for Block Methods for the Direct Solution of Higher Order Initial Value Problems of Ordinary Differential Equations

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Abstract: This paper presents the derivation techniques of block method for solving higher order initial value problems of ordinary differential equations directly. The method was developed via interpolation and collocation of the shifted Legendre polynomials as basis function. The method is capable of providing the numerical solution at several points simultaneously.

Keywords: Collocation, Interpolation, Shifted Legendre Polynomials, Block Method, Higher Order, Direct Solution, Initial Value Problems

1. Introduction

This work considers solving an ordinary differential equation (ODE) of the ω^{th} order ($\omega \geq 2$). There are currently two well-known techniques for solving higher order ODEs. The first is to reduce it to a system of first order ordinary differential equations and then solve using predictor corrector or Runge-Kutta method. The second technique is a block method; it is formulated in terms of linear multistep methods. It preserves the traditional advantage of one step methods, of being self-starting and permitting easy change of step length (Lambert, 1973). Their advantage over Runge-Kutta methods lies in the fact that they are less expensive in terms of the number of functions evaluation for a given order. The method generates simultaneous solutions at all grid points as suggested by many researchers such as Anake (2011), Onumanyi and Okunuga (1985), Onumanyi and Yusuph (2002), Onumanyi et al., (1993, 1994), Awoyemi (2001, 2005 and 1991), Areo et al. (2008), Fatunla (1991, 1995), Lambert (1991), Awoyemi and Kayode (2005), Awari et al. (2014), Okunuga and Ehigie (2009), Adesanya et al., (2009, 2008), Serisina et al., (2004), Owolabi (2015), Yahaya and Badmus (2009) and much recently by Warren and Zill (2013) and Omar and Kuboye (2015). These methods solve higher order initial value problems of ordinary differential equations without going through the process of reduction.

This present method is aimed at developing a general block method for the direct solution of higher order ($\omega \geq 2$) initial value problems of ordinary differential equations, with the block approach, the non-self-starting nature associated with the predictor corrector method has been eliminated. Unlike the approach in predictor corrector method where additional equations were supplied from a different formulation, all our additional equations are obtained from the same continuous formulation.

2. Derivation of the k-Step Block Method

Consider solving an ordinary differential equation (ODE) of the ω^{th} order ($\omega \geq 2$)

$$\begin{gathered} y^{\omega} = f(x, y, y', y'', \ldots, y^{\omega-1}) \\ y(a) = y_0, y'(a) = y_1, y''(a) = y_2, \ldots, y^{\omega-1}(a) = y_{\omega-1}, x \in [a, b] \end{gathered} \tag{1}$$

where $y^{\omega} = \frac{d^{\omega}y}{dx^{\omega}}$.

Consider an approximate solution of (1) given by the shifted Legendre polynomials of the form;

$$y(x)=\sum_{i=0}^{m} C_i P_i(t) \tag{2}$$

where $C_i \in \mathbb{R}$, $y \epsilon C^{\omega}(a,b)$. The ω^{th} derivative of (2) gives

$$y^{\omega}=\sum_{i=0}^{m} C_i P_i^{\omega}(t) \tag{3}$$

Substituting (3) into (1) gives

$$y^{\omega}=\sum_{i=0}^{m} C_i P_i^{\omega}(t)=f(x,y(x),y^{'}(x),y^{''}(x),y^{'''}(x),\ldots y^{\omega-1}(x)) \tag{4}$$

Evaluating (4) at $x_{n+r}, r=0(1)k$ and (2) at $x_{n+r}, r=0(1)k-1$ respectively; gives a system of nonlinear equations of the form

$$AX=B \tag{5}$$

where

$$\begin{bmatrix} P_0(0) & P_1(0) & P_2(0) & \cdots & P_m(0) \\ P_0((k-1)h) & P_1((k-1)h) & P_2((k-1)h) & \cdots & P_m((k-1)h) \\ P_0^{\omega}(0) & P_1^{\omega}(0) & P_2^{\omega}(0) & \cdots & P_m^{\omega}(0) \\ P_0^{\omega}(h) & P_1^{\omega}(h) & P_2^{\omega}(h) & \cdots & P_m^{\omega}(h) \\ P_0^{\omega}(2h) & P_1^{\omega}(2h) & P_2^{\omega}(2h) & \cdots & P_m^{\omega}(2h) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ P_0^{\omega}(kh) & P_1^{\omega}(kh) & P_2^{\omega}(kh) & \cdots & P_m^{\omega}(kh) \end{bmatrix} \begin{bmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \\ C_4 \\ \vdots \\ C_m \end{bmatrix} = \begin{bmatrix} y_n \\ y_{k-1} \\ f_n \\ f_{n+1} \\ f_{n+2} \\ \vdots \\ f_{n+k} \end{bmatrix}$$

Solving for $C_i's, i=0(1)2k-1$ in (5) using inverse of a matrix method which are then substituted into (2) to produce a continuous linear multistep method;

$$y(x)=\sum_{j=1}^{k-1} \alpha_j(x) y_{n+j} + h^{\omega} \sum_{j=1}^{k} \beta_j(x) f_{n+j} \tag{6}$$

where $\alpha_j(x)$ and $\beta_j(x)$ are coefficients to be determined. Evaluating $\alpha_j(x)$ and $\beta_j(x)$ at x_{n+k} and its first derivatives evaluated at the points $x_{n+r}, r=0(1)k$and substituting in (6) gives the discrete block method of the form

$$A^{(0)} w_s = \sum_{i=0}^{k} B^{(0)} q_s + h^{\omega} \sum_{i=0}^{k} D^{(0)} F_s \tag{7}$$

Where $A^{(0)}$, $B^{(0)}$ and $D^{(0)}$ are square matrices and

$$w_s=[y_{n+1}, y_{n+2}, \ldots, y_{n+k}, hy'_{n+1}, hy'_{n+2}, \ldots, hy'_{n+k}]^T$$

$$q_s=[y_n, hy'_n]^T$$

$$F_s=[f_n, f_{n+1}, \ldots, f_{n+2}, \ldots, f_{n+k}]^T$$

which can be modified to obtain explicitly values for $y_{n+1}, y_{n+2}, \ldots, y_{n+k}, y'_{n+1}, y'_{n+2}, y'_{n+3}, \ldots$ and y'_{n+k} needed for the implementation of an ωth order initial value problem of ordinary differential equation.

3. Numerical Example

Consider a specific case with $k=5, \omega=2$ and approximating the solution of (1) by the shifted Legendre polynomials of the form (2).

Evaluating (4) at $x_{n+r}, r=(1)k$ and (2) at x_n and x_{n+4} respectively; gives a system of nonlinear equations of the form

$$AX=B \ (8)$$

where

$$A=\begin{bmatrix} p_0(0) & p_1(0) & p_2(0) & p_3(0) & p_4(0) & p_5(0) & p_6(0) \\ p_0(4h) & p_1(4h) & p_2(4h) & p_3(4h) & p_4(4h) & p_5(4h) & p_6(4h) \\ p_0^{''}(h) & p_1^{''}(h) & p_2^{''}(h) & p_3^{''}(h) & p_4^{''}(h) & p_5^{''}(h) & p_6^{''}(h) \\ p_0^{''}(2h) & p_1^{''}(2h) & p_0^{''}(h) & p_3^{''}(2h) & p_4^{''}(2h) & p_5^{''}(2h) & p_6^{''}(2h) \\ p_0^{''}(3h) & p_1^{''}(3h) & p_0^{''}(h) & p_3^{''}(3h) & p_4^{''}(3h) & p_5^{''}(3h) & p_6^{''}(3h) \\ p_0^{''}(4h) & p_1^{''}(4h) & p_0^{''}(h) & p_3^{''}(4h) & p_4^{''}(4h) & p_5^{''}(4h) & p_6^{''}(4h) \\ p_0^{''}(5h) & p_1^{''}(5h) & p_0^{''}(5h) & p_3^{''}(5h) & p_4^{''}(5h) & p_5^{''}(5h) & p_6^{''}(5h) \end{bmatrix}$$

$$X=\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \end{bmatrix}, B=\begin{bmatrix} y_n \\ y_{n+4} \\ f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \\ f_{n+5} \end{bmatrix}$$

Solving for $C_i's, i=0(1)6$ in (8) using inverse of a matrix method and then substituting into (6) to give a continuous linear multistep method;

$$y(x)=\sum_{j=0}^{4} \alpha_j(x) y_{n+j} + h^2 \sum_{j=0}^{5} \beta_j(x) f_{n+j} \tag{9}$$

where

$$\left[\begin{array}{c}
\alpha_0(x)=1-\frac{1}{4}t \\[2ex]
\alpha_4(x)=1-\frac{1}{4}t \\[2ex]
\beta_0(x)=\frac{1}{2}t^2-\frac{137}{360h}t^3+\frac{5}{32h^2}t^4-\frac{17}{480h^3}t^5+\frac{1}{240h^4}t^6-\frac{1}{5040h^5}t^7-\frac{94}{315}ht \\[2ex]
\beta_1(x)=\frac{5}{6h}t^3-\frac{77}{144h^2}t^4+\frac{71}{450h^3}t^5-\frac{7}{360h^4}t^6+\frac{1}{1008h^5}t^7-\frac{356}{315}ht \\[2ex]
\beta_2(x)=\frac{13}{360h^4}t^6+\frac{107}{144h^2}t^4-\frac{5}{6h}t^3-\frac{1}{504h^5}t^7-\frac{44}{315}ht-\frac{59}{315h^3}t^5 \\[2ex]
\beta_3(x)=\frac{5}{9h}t^3-\frac{13}{24h^2}t^4+\frac{49}{240h^3}t^5-\frac{1}{360h^4}t^6+\frac{1}{504h^5}t^7-\frac{152}{315}ht \\[2ex]
\beta_4(x)=\frac{61}{288h^2}t^4+\frac{4}{63}ht-\frac{5}{24h}t^3+\frac{11}{3720h^4}t^6-\frac{1}{1008h^5}t^7-\frac{41}{480h^3}t^5 \\[2ex]
\beta_5(x)=\frac{1}{5040h^5}t^7-\frac{5}{144h^2}t^4-\frac{4}{315}ht+\frac{1}{30h}t^3+\frac{1}{3360h^4}t^6+\frac{7}{480h^3}t^5
\end{array}\right] \tag{10}$$

Evaluating (10) at $t=-1,-2,-3$ and 5 with its first derivative evaluated at $t=0,-1,-2,-3,-4$ and 5 with $t=(x_n-x)$ and the results substituted in (9), to give the following discrete schemes needed for the implementation of the main scheme

$$\begin{array}{l}
y_{n+1}=\frac{3}{4}y_n-\frac{337}{480}h^2f_{n+1}-\frac{53}{120}h^2f_{n+2}-\frac{71}{240}h^2f_{n+3}-\frac{1}{240}h^2f_{n+4}-\frac{1}{480}h^2f_{n+5}+\frac{1}{4}y_{n+4}-\frac{13}{240}h^2f_n \\[2ex]
y_{n+2}=\frac{1}{2}y_n-\frac{8}{15}h^2f_{n+1}-\frac{13}{15}h^2f_{n+2}-\frac{8}{15}h^2f_{n+3}-\frac{1}{30}h^2f_{n+4}+\frac{1}{2}y_{n+4}-\frac{1}{30}h^2f_n \\[2ex]
y_{n+3}=\frac{1}{4}y_n-\frac{127}{480}h^2f_{n+1}-\frac{29}{60}h^2f_{n+2}-\frac{161}{240}h^2f_{n+3}-\frac{1}{15}h^2f_{n+4}+\frac{1}{480}h^2f_{n+5}+\frac{3}{4}y_{n+4}-\frac{1}{60}h^2f_n \\[2ex]
y_{n+4}=y_n+\frac{1424}{315}h^2f_{n+1}+\frac{176}{315}h^2f_{n+2}+\frac{608}{315}h^2f_{n+3}-\frac{16}{63}h^2f_{n+4}+\frac{16}{315}h^2f_{n+5}+4hy'_n+\frac{376}{315}h^2f_n \\[2ex]
y_{n+5}=-\frac{1}{4}y_n+\frac{23}{96}h^2f_{n+1}+\frac{13}{24}h^2f_{n+2}+\frac{11}{16}h^2f_{n+3}+\frac{15}{16}h^2f_{n+4}+\frac{7}{96}h^2f_{n+5}+\frac{5}{4}y_{n+4}+\frac{1}{48}h^2f_n \\[2ex]
y'_{n+1}=\frac{1}{4h}y_{n+4}-\frac{1}{4h}y_n++\frac{317}{10080}hf_n-\frac{1403}{10080}hf_{n+1}-\frac{3497}{5040}hf_{n+2}-\frac{149}{1008}hf_{n+3}-\frac{571}{10080}hf_{n+4}+\frac{61}{10080}hf_{n+5} \\[2ex]
y'_{n+2}=\frac{1}{4h}y_{n+4}-\frac{1}{4h}y_n++\frac{4}{315}hf_n+\frac{191}{630}hf_{n+1}+\frac{1}{63}hf_{n+2}-\frac{103}{315}hf_{n+3}-\frac{1}{315}hf_{n+4}-\frac{1}{630}hf_{n+5} \\[2ex]
y'_{n+3}=\frac{1}{4h}y_{n+4}-\frac{1}{4h}y_n++\frac{41}{2016}hf_n+\frac{481}{2016}hf_{n+1}+\frac{2887}{5040}hf_{n+2}+\frac{1159}{5040}hf_{n+3}-\frac{683}{10080}hf_{n+4}+\frac{61}{10080}hf_{n+5} \\[2ex]
y'_{n+4}=\frac{1}{4h}y_{n+4}-\frac{1}{4h}y_n++\frac{4}{315}hf_n+\frac{92}{315}hf_{n+1}+\frac{124}{315}hf_{n+2}+\frac{296}{315}hf_{n+3}+\frac{118}{315}hf_{n+4}-\frac{4}{315}hf_{n+5} \\[2ex]
y'_{n+5}=\frac{1}{4h}y_{n+4}-\frac{1}{4h}y_n++\frac{317}{10080}hf_n+\frac{1733}{10080}hf_{n+1}+\frac{3671}{5040}hf_{n+2}+\frac{1943}{5040}hf_{n+3}+\frac{2753}{2016}hf_{n+4}+\frac{3197}{10080}hf_{n+5}
\end{array} \tag{11}$$

4. Analysis of the Block Method

4.1. Order and Error Constants

Expanding the block solution of (11) in Taylor's series and collecting like terms in powers of h, the following result is obtained;

$$\breve{C}_0 = \breve{C}_1 = \cdots = \breve{C}_6 = (0,0,0,0,0,0,0,0,0,0)^T,$$

$$\breve{C}_7 = \left(0,0,0,0,0,0,0,-\frac{29}{2240},-\frac{8}{945},-\frac{275}{12096}\right)^T$$

$$\breve{C}_8 = \left(-\frac{199}{24192},-\frac{19}{945},-\frac{141}{4480},-\frac{8}{189},-\frac{1375}{24192},-\frac{863}{60480},-\frac{37}{3780}\right)^T.$$

Hence the block method has varying order of $\breve{p} = 5$ and 6 with varying error constants of $\breve{C}_7 = \left(-\frac{29}{2240},-\frac{8}{945},-\frac{275}{12096}\right)^T$ and $\breve{C}_8 = \left(-\frac{199}{24192},-\frac{19}{945},-\frac{141}{4480},-\frac{8}{189},-\frac{1375}{24192},-\frac{863}{60480},-\frac{37}{3780}\right)^T$.

Consistency

Following Lambert (1973, 1991), the block method is consistent since it has orders $\breve{p} = 5,6 > 1$

Zero stability

The block solution of the block method (11) is said to be zero stable if the roots $z_r;\ r = 1,\dots,n$ of the first characteristic polynomial $p(z)$, defined by

$$p(z) = det|zQ - T|$$

satisfies $|z_r| \leq 1$ and every root with $|z_r| = 1$ has multiplicity not exceeding two in the limit as $h \to 0$

From the block method (11), we have

$$Q = \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&1 \end{bmatrix}$$

$$\text{and} T = \begin{bmatrix} 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0 \end{bmatrix}$$

Substituting, we get

$$z^{10} - z^8 = 0$$

$$z = (0,0,0,0,0,0,0,0,-1,1)$$

This shows that the block method is zero stable, since all roots with modulus one do not have multiplicity exceeding the order of the differential equation in the limit as $h \to 0$.

Convergence

According to Lambert (1991), the block method is convergent since it is both consistent and zero stable.

Region of absolute stability

Reformulating (11) as a General Linear Multistep Method (GLMM) containing a partition of matrices A, B, C and DI where

$$A = \begin{bmatrix} 1&0&0&-\frac{1}{4}&0\\ 0&1&0&-\frac{1}{2}&0\\ 0&0&1&\frac{3}{4}&0\\ 0&0&0&1&0\\ 0&0&0&-\frac{5}{4}&1 \end{bmatrix} \quad B = \begin{bmatrix} 0&0&0&0&\frac{3}{4}\\ 0&0&0&0&\frac{1}{2}\\ 0&0&0&0&\frac{1}{4}\\ 0&0&0&0&1\\ 0&0&0&0&-\frac{1}{4} \end{bmatrix}$$

$$C = \begin{bmatrix} \frac{1403}{10080}&\frac{3497}{5040}&\frac{149}{1008}&-\frac{571}{10080}&\frac{61}{10080}\\ \frac{191}{630}&\frac{1}{63}&-\frac{103}{315}&-\frac{1}{315}&-\frac{1}{630}\\ \frac{481}{2016}&\frac{2887}{5040}&\frac{1159}{5040}&-\frac{683}{10080}&\frac{61}{10080}\\ \frac{92}{315}&\frac{124}{315}&\frac{296}{315}&\frac{118}{315}&-\frac{4}{315}\\ \frac{1733}{10080}&\frac{3671}{5040}&\frac{1943}{5040}&\frac{2753}{2016}&\frac{3197}{10080} \end{bmatrix}$$

$$DI = \begin{bmatrix} -\frac{337}{480}&-\frac{53}{120}&-\frac{71}{240}&-\frac{1}{240}&-\frac{1}{480}\\ -\frac{8}{15}&-\frac{13}{15}&-\frac{8}{15}&-\frac{1}{30}&0\\ -\frac{127}{480}&-\frac{29}{60}&-\frac{161}{240}&-\frac{1}{15}&\frac{1}{480}\\ \frac{1424}{315}&\frac{176}{315}&\frac{608}{315}&-\frac{16}{63}&\frac{16}{315}\\ \frac{23}{96}&\frac{13}{24}&\frac{11}{16}&\frac{15}{16}&\frac{7}{96} \end{bmatrix}$$

Substituting these matrices into the stability polynomial $r(A - Cz - DIz^2) - B$, the stability matrix whose determinant and it first derivative are respectively obtained as

$$\frac{123}{70}r^3z+\frac{3398363}{453600}r^3z^2+\frac{97283}{16200}r^3z^3+\frac{226567}{43200}r^3z^4+\frac{9021191}{3628800}r^3z^5+\frac{1199267}{604800}r^3z^6+r^5-2r^4$$

$$+r^3-\frac{17}{450}r^5z^9-\frac{1}{126}r^5z^{10}-\frac{7303}{12600}r^5z^7-\frac{175993}{1134000}r^5z^8-\frac{73}{210}r^5z+\frac{12664}{4725}r^5z^2-\frac{1427}{113400}r^5z^3$$

$$+\frac{91279}{34020}r^5z^4+\frac{590657}{1134000}r^5z^5+\frac{1091239}{10206000}r^5z^6-\frac{2837197}{8164800}r^4z^7-\frac{1976459}{2721600}r^4z^8-\frac{2651}{1008}r^4z$$

$$-\frac{480973}{37800}r^4z^2-\frac{46535353}{5103000}r^4z^3-\frac{34704703}{2721600}r^4z^4-\frac{29705461}{3628800}r^4z^5-\frac{890205719}{163296000}r^4z^6$$

and

$$\frac{123}{70}r^3+\frac{3398363}{226800}r^3z+\frac{97283}{5400}r^3z^2+\frac{226567}{10800}r^3z^3+\frac{9021191}{725760}r^3z^4+\frac{1199267}{100800}r^3z^5+r^5$$

$$-2r^4-\frac{17}{50}r^5z^8-\frac{5}{63}r^5z^9-\frac{7303}{1800}r^5z^6-\frac{175993}{141750}r^5z^7-\frac{73}{210}r^5+\frac{25328}{4725}r^5z-\frac{1427}{37800}r^5z^2$$

$$+\frac{91279}{8505}r^5z^3+\frac{590657}{226800}r^5z^4+\frac{1091239}{1701000}r^5z^5-\frac{2837197}{1166400}r^4z^6-\frac{1976459}{340200}r^4z^7-\frac{2651}{1008}r^4$$

$$-\frac{480973}{18900}r^4z-\frac{46535353}{1701000}r^4z^2-\frac{34704703}{680400}r^4z^3-\frac{29705461}{725760}r^4z^4-\frac{890205719}{27216000}r^4z^5$$

The region of absolute stability is shown in Figure 1

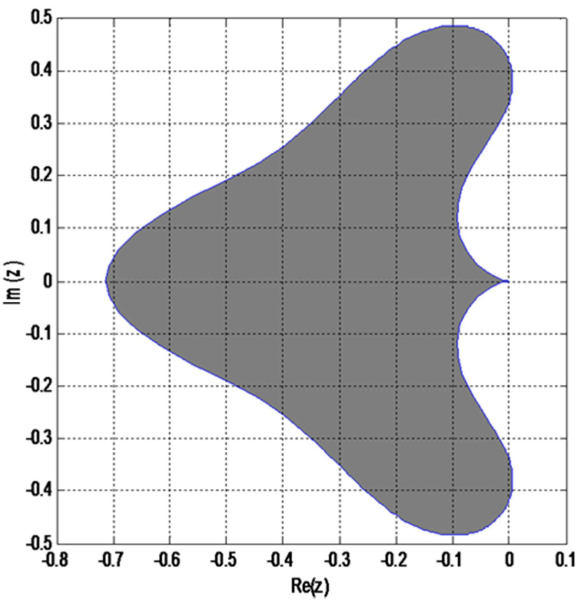

Figure 1. *The Region of Absolute Stability of the Block Method (11).*

5. Conclusion

In this paper, it has been shown that continuous collocation methods for solving ordinary differential equations can equally be derived through the approach in this study. In this study, a new block method approach which is capable of solving higher order initial value problems of ordinary differential equations is presented, with the block approach; the non self starting nature associated with the predictor corrector method and the Runge Kutta method has been eliminated. Unlike the approach in predictor corrector method where additional equations were supplied from a different formulation, all the required additional equations are obtained from the same continuous formulation. The basic property of the method was investigated and was found to be zero stable, consistent and convergent. The absolute stability region of the block method was also investigated and revelations showed that the newly constructed method is not A-stable as revealed by Figure 1. This method is very simple and effective for a wide-range of ordinary differential equations.

References

[1] Adesanya, A. O., Anake, T. A., Bishop, S. A. and Osilagun, J. A. (2009). Two Steps Block Method for the Solution of General Second Order Initial Value Problems of Ordinary Differential Equation. ASSET 8(1): 59-68.

[2] Adesanya, A. O., Anake, T. A. and Udoh, M. O. (2008). Improved Continuous Method for Direct Solution of General Second Order Ordinary Differential Equations. Journal of the Nigeria Association of Mathematical Physics.13: 59-62.

[3] Anake, T. A. (2011). Continuous Implicit Hybrid One –Step Methods for the Solution of Initial Value Problems of Second Order Ordinary Differential Equations. PhD Thesis, Covenant University, Ota, Ogun State, Nigeria.

[4] Awarri, Y. S., Chima, E. E., Kamoh, N. M. and Oladele, F. L. (2014). A Family of Implicit Uniformly Accurate Order Block Integrators for the Solution of Second Order Differential Equations; International Journal of Mathematics and Statistics Invention (IJMSI).2: 34-46.

[5] Areo, E. A., Ademiluyi, R. A. and Babatola, P. O. (2008). Accurate Collocation Multistep Method for integration of first order ordinary differential equations. Journal of Modern Mathematics and Statistics. 2(1): 1-6.

[6] Awoyemi, D. O. (1991). A Class of Continuous Linear Multistep Method for General Second Order Initial Value Problems in Ordinary Differential Equations. International Journal of Computer Mathematics.72: 29-37.

[7] Awoyemi, D. O and Kayode, S. J. (2005). An Implicit Collocation Method for Direct Solution of Second Order Ordinary Differential Equations; Journal of Nigeria Association of Mathematical Physics.24:70-78.

[8] Awoyemi, D. O. (2001). A New Sixth-Order Algorithm for General Second Order Ordinary Differential Equation. International Journal of computational mathematics. 77: 117-124.

[9] Serisina, U. W., Kumleng, G. M. and Yahaya, Y. A. (2004). A New Butcher Type Two-Step Block Hybrid Multistep Method for Accurate and Efficient Parallel Solution of Ordinary Differential Equations. Abacus Mathematics Series31:1-7.

[10] Fatunla, S. O. (1991). Block Method for Second Order Ordinary Differential Equations. International Journal of Computer Mathematics, 41:(1&2), 55-63.

[11] Fatunla, S. O. (1995). A Class of Block Method for Second Order Initial Value Problems. International Journal of Computer Mathematics, 55:(1&2), 119-133.

[12] Lambert, J. D. (1991). Numerical Methods for Ordinary Differential Equations. John Wiley and Sons New York, 293pp.

[13] Lambert, J. D. (1973). Computational Methods in Ordinary Differential Equations. New York: John Wiley and Sons.

[14] Okunuga, S. A. and Ohigie, J. (2009). A New Derivation of Continuous Multistep Methods Using Power Series as Basis Function. Journal of Modern Mathematics and Statistics 3(2): 43-50.

[15] Okunuga, S. A and Onumanyi, P. (1985). An Accurate Collocation Method for Solving Ordinary Differential Equations. AMSE, France, 4(4): 45-48.

[16] Onumanyi, P. A. Woyemi, D. O., Jator, S. N. and Sirisena, U. W. (1994). New Linear Multistep Methods with Continuous Coefficients for First Order Initial Value Problems. Journal of Nigeria Mathematics Society.13:37-51.

[17] Onumanyi, P and Yusuph, Y. (2002). New Multiple FDMs Through Multistep Collocation for $y'' = f(x, y)$. Abacus, 29: 92-100.

[18] Onumanyi, P and Okunuga, S. A. (1986). Collocation Tau Method for the Solution of Ordinary Differential Equations. Abacus, 17: 51-60.

[19] Onumanyi, P., Oladele, J. O., Adeniyi, R. B and Awoyemi, D. O. (1993). Derivation of Finite Method by Collocation. Abacus, 23(2): 72-83.

[20] Onumanyi, P., Sirisena, W. U. & Jator, S. N. (1999). Continuous Finite Difference Approximations for Solving Differential Equations. International Journal of Computer and Mathematics, 72, 15-27.

[21] Owolabi, K. M. (2015) ARebust Optimal Order Formula for Direct Integration of Second Order Orbital Problems. Advances in Physics Theories and Applications 42: p21-26.

[22] Warren, W. S and Zill, D. G. (2013). Differential Equations with Boundary- Value Problem; Eight edition books/Cole, Cengage Learning.

[23] Yahaya, Y. A. and Badmus, A. M. (2009). A Class of Collocation Methods for General Second Order Ordinary Differential Equations, African Journal of Mathematics and Computer Science Research, 2(4): 069-072.