International Journal of Applied Mathematics and Theoretical Physics
2022; 8(1): 1-23
http://www.sciencepublishinggroup.com/j/ijamtp
doi: 10.11648/j.ijamtp.20220801.11
ISSN: 2575-5919 (Print); ISSN: 2575-5927 (Online)

Direct Approach for Solving Second Order Delay Differential Equations Through a Five-Step with Several Off-grid Points

Familua A. B.

Department of Mathematics and Statistics, First Technical University, Ibadan, Nigeria

Email address:
adefunke.familua@tech-u.edu.ng, funky4fam@gmail.com

To cite this article:
Familua A. B. Direct Approach for Solving Second Order Delay Differential Equations Through a Five-Step with Several Off-grid Points. *International Journal of Applied Mathematics and Theoretical Physics*. Vol. 8, No. 1, 2022, pp. 1-23. doi: 10.11648/j.ijamtp.20220801.11

Received: May 30, 2021; **Accepted**: October 15, 2021; **Published**: February 5, 2022

Abstract: This paper presents a direct approach for numerical solution of special second order delay differential equations (DDEs) directly without reduction to systems of low orders. The methods were generated using collocation approach via a combination of power series and exponential function. The approximate basis functions are interpolated at the first two grid points and collocated at both grid and off-grid points. The developed schemes and its derivatives were combined to form block methods to simultaneously solve second order Delay Differential Equations (DDEs) directly without the rigor of developing separate predictors. The required methods were obtained for step lengths of five with generalized number of hybrid points (3k). The basic properties of the methods were examined, the methods were found to have high order of accuracy of 21, low error constant, gives large interval of absolute stability, zero stable, consistence and convergent. The developed methods were applied to solve some special second order Delay Differential Equations. The methods also solve an engineering problem namely Matheiu's equation in order to test for the efficiency and accuracy of the new methods. The results obtained were compared with existing methods in the literature. The results obtained showed better performance than some existing methods. The stability domain of the method is showed in figure 1 whereas the efficiency curve of the application problem for linear and nonlinear is presented in figure 2.

Keywords: Block Method, Special Second Order Delay Differential Equations, Five-step, Off-grid Points, Power Series, Exponential Function

1. Introduction

In this paper, we considered the method of approximate solution of the special second order delay differential equations the form:

$$y''(t) = f(t, y(t-\tau)),\ \ y(t_0) = \alpha,\ y'(t_0) = \beta,\ t \geq t_0,\ \ \tau > 0 \quad (1)$$

where α is the initial function and τ is the delay term. Most of the methods for solving special second order ODEs can be adopted for solving special second order delay differential equation. There are two different ways to calculate the delay term in the developed method [5, 6].

Equation (1) is of importance to researchers because of its extensive relevance in engineering, Sciences, biological sciences, optimal control theory and other real life problem.

The conventional method of solving (1) is to reduce it to a system of first order differential equation [1, 4, 6]. The reduction of (1) to a system of first order equations leads to serious computational burden as well as wastage of computer and human efforts. However, these setbacks have been taken cared by some researchers [4, 18]. It has been reported in literature that the direct method of solving the above equation is more efficient in terms of speed and accuracy than the method of reduction to a system of first order ODES [5-13].

Several scholars developed numerical methods ranging from Adomian Decomposition method, Runge kutta method, and Predictor-corrector method [2-20]. However, their results are not good enough. The implementation of numerical method in predictor-corrector approach, however, has some setbacks which include lengthy computational time due to more function evaluations needed per step and computational burden which may affect the accuracy of the method in terms of error [4]. In overcoming the setbacks mentioned above, a five-step with several off-grid points method without predictor were considered, the first two grid points were interpolated for solving special second order DDEs. The numerical results generated were better than the previous method in the literature. Therefore, in this study a five-step block method with fifteen off-step points for solving (1) in order to improve the accuracy of the existing methods is proposed.

2. Derivation of Five-step Method with Fifteen off Step Points

This work considers an approximate solution that combines power series and exponential function of the form;

$$y(x)=\sum_{j=0}^{r+s-1} a_j x^j + a_{r+s}\sum_{j=0}^{r+s}\frac{\alpha^j x^j}{j!} \tag{2}$$

The second derivatives of (2) is given as,

$$y''(x)=\sum_{j=2}^{r+s-1} j(j-1)a_j x^{j-2} + a_{r+s}\sum_{j=2}^{r+s}\frac{\alpha^j x^{j-2}}{(j-2)!} \tag{3}$$

The approach discussed in the study of Familua A. B. et al. [6] is adopted by partitioning the five-step length (x_n, x_{n+5}) into fifteen sub-steps. Solving the matrix (2), with the Aid of Mathematica or Maple for the unknowns.

$$\text{AX=U} \tag{4}$$

where $A=\left[a_0 \ldots\, a_{n+k-1} \ldots\, a_{n+vi} \ldots\, a_{2k+vi}\right]$

$U=\left[y_n \ldots\, y_{n+k-1} \ldots\, f_n \ldots f_{n+vi} \cdots\, f_{n+k}\right]$ and

$$X=\begin{bmatrix}
1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 & x_{n+1}^7 & x_{n+1}^8 & x_{n+1}^9 & x_{n+1}^{10} & \cdots & x_{n+1}^{20} & x_{n+1}^{21} & \cdots & \left(1+\alpha x_{n+1}+\frac{(\alpha^2 x_{n+1}^2)}{2!}+\frac{(\alpha^3 x_{n+1}^3)}{3!}+..\frac{(\alpha^{22} x_n^{22})}{22!}\right)\\
1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 & x_{n+2}^5 & x_{n+2}^7 & x_{n+2}^7 & x_{n+2}^8 & x_{n+2}^9 & x_{n+2}^{10} & \cdots & x_{n+2}^{20} & x_{n+2}^{21} & \cdots & \left(1+\alpha x_{n+2}+\frac{(\alpha^2 x_{n+2}^2)}{2!}+\frac{(\alpha^3 x_{n+2}^3)}{3!}+..\frac{(\alpha^{22} x_{n+2}^{22})}{22!}\right)\\
0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & 30x_n^4 & 42x_n^5 & 56x_n^6 & 72x_n^7 & 90x_n^8 & 110x_n^9 & 132x_n^{10} & 156x_n^{11} & \cdots & \left(\alpha^2+\alpha^2 x_n+\frac{(\alpha^3 x_n^2)}{2!}+\frac{(\alpha^4 x_n^3)}{3!}+..\frac{210(\alpha^{14} x_n^{12})}{12!}\right)\\
0 & 0 & 2 & 6x_{n+1/4} & 12x_{n+1/4}^2 & 20x_{n+1/4}^3 & 30x_{n+1/4}^4 & 42x_{n+1/4}^5 & 56x_{n+1/4}^6 & 72x_{n+1/4}^7 & 90x_{n+1/4}^8 & 110x_{n+1/4}^9 & 132x_{n+1/4}^{10} & 156x_{n+1/4}^{11} & \cdots & \left(\alpha^2+\alpha^2 x_{n+1/4}+\frac{(\alpha^3 x_{n+1/4}^2)}{2!}+\frac{(\alpha^4 x_{n+1/4}^3)}{3!}+..\frac{210(\alpha^{14} x_{n+1/4}^{12})}{12!}\right)\\
0 & 0 & 2 & 6x_{n+1/2} & 12x_{n+1/2}^2 & 20x_{n+1/2}^2 & 30x_{n+1/2}^2 & 42x_{n+1/2}^2 & 56x_{n+1/2}^2 & 72x_{n+1/2}^7 & 90x_{n+1/2}^8 & 110x_{n+1/2}^9 & 132x_{n+1/2}^{10} & 156x_{n+1/2}^{11} & \cdots & \left(\alpha^2+\alpha^2 x_{n+1/2}+\frac{(\alpha^3 x_{n+1/2}^2)}{2!}+\frac{(\alpha^4 x_{n+1/2}^3)}{3!}+..\frac{210(\alpha^{14} x_{n+1/2}^{12})}{12!}\right)\\
0 & 0 & 2 & 6x_{n+3/4} & 12x_{n+3/4}^2 & 20x_{n+3/4}^3 & 30x_{n+3/4}^4 & 42x_{n+3/4}^5 & 56x_{n+3/4}^6 & 72x_{n+3/4}^7 & 90x_{n+3/4}^8 & 110x_{n+3/4}^9 & 132x_{n+3/4}^{10} & 156x_{n+3/4}^{11} & \cdots & \left(\alpha^2+\alpha^2 x_{n+3/4}+\frac{(\alpha^3 x_{n+3/4}^2)}{2!}+\frac{(\alpha^4 x_{n+3/4}^3)}{3!}+..\frac{210(\alpha^{14} x_{n+3/4}^{12})}{12!}\right)\\
0 & 0 & 2 & 6x_{n+1} & 12x_{n+1}^2 & 20x_{n+1}^3 & 30x_{n+1}^4 & 42x_{n+1}^5 & 56x_{n+1}^6 & 72x_{n+1}^7 & 90x_{n+1}^8 & 110x_{n+1}^9 & 132x_{n+1}^{10} & 156x_{n+1}^{11} & \cdots & \left(\alpha^2+\alpha^2 x_{n+1}+\frac{(\alpha^3 x_{n+1}^2)}{2!}+\frac{(\alpha^4 x_{n+1}^3)}{3!}+..\frac{210(\alpha^{14} x_{n+1}^{12})}{12!}\right)\\
0 & 0 & 2 & 6x_{n+5/4} & 12x_{n+5/4}^2 & 20x_{n+5/4}^3 & 30x_{n+5/4}^4 & 42x_{n+5/4}^5 & 56x_{n+5/4}^6 & 72x_{n+5/4}^7 & 90x_{n+5/4}^8 & 110x_{n+5/4}^9 & 132x_{n+5/4}^{10} & 156x_{n+5/4}^{11} & \cdots & \left(\alpha^2+\alpha^2 x_{n+5/4}+\frac{(\alpha^3 x_{n+5/4}^2)}{2!}+\frac{(\alpha^4 x_{n+5/4}^3)}{3!}+..\frac{210(\alpha^{14} x_{n+5/4}^{12})}{12!}\right)\\
0 & 0 & 2 & 6x_{n+3/2} & 12x_{n+3/2}^2 & 20x_{n+3/2}^3 & 30x_{n+3/2}^4 & 42x_{n+3/2}^5 & 56x_{n+3/2}^6 & 72x_{n+3/2}^7 & 90x_{n+3/2}^8 & 110x_{n+3/2}^9 & 132x_{n+3/2}^{10} & 156x_{n+3/2}^{11} & \cdots & \left(\alpha^2+\alpha^2 x_{n+3/2}+\frac{(\alpha^3 x_{n+3/2}^2)}{2!}+\frac{(\alpha^4 x_{n+3/2}^3)}{3!}+..\frac{210(\alpha^{14} x_{n+3/2}^{12})}{12!}\right)\\
0 & 0 & 2 & 6x_{n+7/4} & 12x_{n+7/4}^2 & 20x_{n+7/4}^3 & 30x_{n+7/4}^4 & 42x_{n+7/4}^5 & 56x_{n+7/4}^6 & 72x_{n+7/4}^7 & 90x_{n+7/4}^8 & 110x_{n+7/4}^9 & 132x_{n+7/4}^{10} & 156x_{n+7/4}^{11} & \cdots & \left(\alpha^2+\alpha^2 x_{n+7/4}+\frac{(\alpha^3 x_{n+7/4}^2)}{2!}+\frac{(\alpha^4 x_{n+7/4}^3)}{3!}+..\frac{210(\alpha^{14} x_{n+7/4}^{12})}{12!}\right)\\
0 & 0 & 2 & 6x_{n+2} & 12x_{n+2}^2 & 20x_{n+2}^3 & 30x_{n+2}^4 & 42x_{n+2}^5 & 56x_{n+2}^6 & 72x_{n+2}^7 & 90x_{n+2}^8 & 110x_{n+2}^9 & 132x_{n+2}^{10} & 156x_{n+2}^{11} & \cdots & \left(\alpha^2+\alpha^2 x_{n+2}+\frac{(\alpha^3 x_{n+2}^2)}{2!}+\frac{(\alpha^4 x_{n+2}^3)}{3!}+....\frac{210(\alpha^{14} x_{n+2}^{12})}{12!}\right)\\
0 & 0 & 2 & 6x_{n+9/4} & 12x_{n+9/4}^2 & 20x_{n+9/4}^3 & 30x_{n+9/4}^4 & 42x_{n+9/4}^5 & 56x_{n+9/4}^6 & 72x_{n+9/4}^7 & 90x_{n+9/4}^8 & 110x_{n+9/4}^9 & 132x_{n+9/4}^{10} & 156x_{n+9/4}^{11} & \cdots & \left(\alpha^2+\alpha^2 x_{n+9/4}+\frac{(\alpha^3 x_{n+9/4}^2)}{2!}+\frac{(\alpha^4 x_{n+9/4}^3)}{3!}+..\frac{210(\alpha^{14} x_{n+9/4}^{12})}{12!}\right)\\
0 & 0 & 2 & 6x_{n+5/2} & 12x_{n+5/2}^2 & 20x_{n+5/2}^3 & 30x_{n+5/2}^4 & 42x_{n+5/2}^5 & 56x_{n+5/2}^6 & 72x_{n+5/2}^7 & 90x_{n+5/2}^8 & 110x_{n+5/2}^9 & 132x_{n+5/2}^{10} & 156x_{n+5/2}^{11} & \cdots & \left(\alpha^2+\alpha^2 x_{n+5/2}+\frac{(\alpha^3 x_{n+5/2}^2)}{2!}+\frac{(\alpha^4 x_{n+5/2}^3)}{3!}+..\frac{210(\alpha^{14} x_{n+5/2}^{12})}{12!}\right)\\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & .\\
0 & 0 & 2 & 6x_{n+5} & 12x_{n+5}^2 & 20x_{n+5}^3 & 30x_{n+5}^4 & 42x_{n+5}^5 & 56x_{n+5}^6 & 72x_{n+5}^7 & 90x_{n+5}^8 & 110x_{n+5}^9 & 132x_{n+5}^{10} & 156x_{n+5}^{11} & \cdots & \left(\alpha^2+\alpha^2 x_{n+5}+\frac{(\alpha^3 x_{n+5}^2)}{2!}+\frac{(\alpha^4 x_{n+5}^3)}{3!}+..\frac{210(\alpha^{14} x_{n+5}^{12})}{12!}\right)
\end{bmatrix}$$

Solving (4) for a_j's, $j=0(1/4)k$ using Gaussian elimination method, to obtained values for the parameters: $a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, ..a_{22}$ and simplifying the resultant functions, to obtain a continuous scheme of the general form:

$$y(t)=\sum_{j=0}^{k-1}\alpha_j(t)y_{n+j}(t)+\sum_{j=0}^{k}\beta_j f_{n+j}(t) \tag{5}$$

Where $t=\dfrac{x-x_{n+k-1}}{h}$, and setting $x_n=0,\ \ x_{n+1}=h,\ x_{n+2}=2h,\ x_{n+3}=3h,\ \ x_{n+4}=4h,\ \ x_{n+5}=5h$

The coefficients of $\alpha_j(t)$ and $\beta_j(t)$ are:

$$\alpha_1(t)=(2-t)\,y_{n+1}\,,\alpha_2(t)=(-1+t)\,y_{n+2}$$

$$\beta_0(t)=\left(\begin{array}{l}\frac{6670985204447}{183891708000}h^2t^8+\frac{2031616}{1325826376875}t^{20}h^2+\frac{522460655692911}{1860808950000}h^2t^6-\frac{31591404263}{875674800}h^2t^7\\ -\frac{601088}{221524875}h^2t^{17}-\frac{16306688}{16489025}h^2t^{15}+\frac{947187488}{2210236875}h^2t^{14}-\frac{109542331}{11026125}h^2t^{11}+\frac{1}{2}h^2t^2\\ -\frac{65536}{2525383575}h^2t^{19}+\frac{217198974162341}{73503814333950000}h^2+\frac{24392729140594}{1292988571875}h^2t^{10}-\frac{55835135}{23279256}h^2t^3\\ -\frac{6829066740083219}{114339266741700000}h^2+\frac{24392729140594}{1292988571875}h^2t^{10}-\frac{55835135}{23279256}h^2t^3\\ -\frac{6829066740083219}{114339266741700000}h^2t+\frac{2097152}{2143861251406875}h^2t^{22}+\frac{64405504}{209341006875}h^2t^{18}\\ +\frac{1591946368}{86199238125}h^2t^{16}-\frac{13334148911}{787644000}h^2t^5-\frac{524288}{9280784638125}h^2t^{21}-\frac{58575736}{39092625}h^2t^{13}\\ +\frac{2965638101}{694645875}h^2t^{12}-\frac{573738838201}{19702683000}h^2t^9+\frac{665690574539}{87995587680}h^2t^4\end{array}\right)f_n$$

$$\beta_{\frac{1}{4}}(t)=\left(\begin{array}{l}\frac{129320217952}{736745625}h^2t^{11}-\frac{201578155}{2909907}h^2t^4+\frac{147767713390877}{1837595358348750}h^2\\ -\frac{8388608}{428772250281375}h^2t^{22}+\frac{40}{3}h^2t^3-\frac{273473923819}{723647925}h^2t^6+\frac{4525446601157}{9403553250}h^2t^9\\ -\frac{93847552}{3093594879375}t^{20}h^2+\frac{14432580976}{522419625}h^2t^{13}+\frac{1048576}{932519030625}h^2t^{21}\\ +\frac{15204352}{29777918625}h^2t^{19}-\frac{30554814464}{86199238125}h^2t^{16}+\frac{7954069504}{4219543125}h^2t^{15}-\frac{410186288716409}{73503814339500}h^2t\\ +\frac{647583200071}{1218979125}h^2t^7+\frac{155650048}{2960377875}h^2t^{17}-\frac{1761542144}{293077409625}h^2t^{18}-\frac{83701323187024}{258597714375}h^2t^{10}\\ -\frac{1133978824339}{1986484500}h^2t^8-\frac{107634184832}{13408770375}h^2t^{14}+\frac{1726815331}{8771490}h^2t^5-\frac{375499736456}{4862521125}h^2t^{12}\end{array}\right)f_{n+\frac{1}{4}}$$

$$\beta_{\frac{1}{2}}(t)=\left(\begin{array}{l}-\frac{209927696558189}{53635081500}h^2t^7+\frac{2524812122717}{964863900}h^2t^6+\frac{691923673409258}{258597714375}h^2t^{10}\\ -\frac{790066509824}{46414974375}h^2t^{15}+\frac{882507776}{3093594879375}t^{20}h^2-\frac{483965626799}{385945560}h^2t^5\\ +\frac{28675867373177}{6501222000}h^2t^8+\frac{278820067072}{86199238125}h^2t^{16}-\frac{927440896}{1915538625}h^2t^{17}\\ -\frac{12051283791452}{8104201875}h^2t^{11}+\frac{240376915}{612612}h^2t^4-\frac{190}{3}h^2t^3+\frac{1732080468999623}{51452670033765000}h^2\\ +\frac{3426302732698}{4862521125}h^2t^{12}+\frac{108619136734570489}{102905340067530000}h^2t+\frac{1486389248}{26643400875}h^2t^{18}\\ -\frac{15631646672}{327557104875}h^2t^{19}-\frac{30717197904091}{7956852750}h^2t^9+\frac{1486389248}{26643400875}h^2t^{18}\\ -\frac{1563164672}{327557104875}h^2t^{19}-\frac{30717979404091}{7956852750}h^2t^9+\frac{4194304}{22566960541125}h^2t^{22}\\ -\frac{8388608}{789054564375}h^2t^{21}-\frac{126908353024}{522419625}h^2t^{13}+\frac{960113692864}{13408770375}h^2t^{14}\end{array}\right)f_{n+\frac{1}{2}}$$

$$\beta_{\frac{3}{4}}(t)=\left(\begin{array}{l}
\frac{282809557473}{141891750}h^2t^9+\frac{167140968448}{1719073125}h^2t^{15}-\frac{536132747264}{28733079375}h^2t^{16}\\
+\frac{41386339823}{7422030}h^2t^5+\frac{3075997696}{109185701625}h^2t^{19}+\frac{235838383792}{174139875}h^2t^{13}\\
-\frac{5422547129984}{13408770375}h^2t^{14}-\frac{403177472}{237968836875}t^{20}h^2-\frac{5942930682376}{1620840375}h^2t^{12}\\
-\frac{8388608}{7522320180375}h^2t^{22}-\frac{1219684686711616}{86199238125}h^2t^{10}+\frac{784123904}{278326125}h^2t^{17}\\
+\frac{7226925674912}{900466875}h^2t^{11}-\frac{1659274556891297}{317609074282500}h^2t+\frac{181325998653287}{329824807908750}h^2\\
-\frac{253309835}{153153}h^2t^4-\frac{31956729856}{97692469875}h^2t^{18}-\frac{91327826418263}{4125775500}h^2t^8\\
-\frac{26481829554443}{2170943775}h^2t^6+\frac{760}{3}h^2t^3+\frac{254652014806639}{13408770375}h^2t^7+\frac{24117248}{379915160625}h^2t^{21}
\end{array}\right)f_{n+\frac{3}{4}}$$

$$\beta_1(t)=\left(\begin{array}{l}
\frac{2416136925728}{1489863375}h^2t^{14}+\frac{2097152}{44248922375}h^2t^{22}+\frac{32613777155288012}{2143861251406875}h^2t\\
-\frac{1931087090971484}{2143861251406875}h^2+\frac{48037888}{673985125}t^{20}h^2+\frac{4620778157850863}{86199238125}h^2t^{10}\\
-\frac{6095449503232}{15471658125}h^2t^{15}+\frac{132757372928}{97692469875}h^2t^{18}-\frac{10216796381885189}{137918781000}h^2t^9\\
-\frac{567495135029}{30270240}h^2t^5-\frac{126106625024}{10854718875}h^2t^{17}-\frac{10385254712}{19348875}h^2t^{13}\\
-\frac{54001664}{201131555625}h^2t^{21}+\frac{2192673509248}{28733079375}h^2t^{16}+\frac{259776295}{48048}h^2t^4-\frac{1615}{2}h^2t^3\\
+\frac{1279566748471577}{15891876000}h^2t^8+\frac{2394154240249}{56756700}h^2t^6-\frac{1429274624}{12131744625}h^2t^{19}\\
-\frac{83540342424641}{2701400625}h^2t^{11}+\frac{23244571515317}{1620840375}h^2t^{12}-\frac{4822528390933063}{71513442000}h^2t^7
\end{array}\right)f_{n+1}$$

$$\beta_{\frac{5}{4}}(t)=\left(\begin{array}{l}
\frac{3724950708224}{3094331625}h^2t^{15}+\frac{3299873151363674}{10719306257034375}h^2+\frac{165267917296652}{893918025}h^2t^7\\
-\frac{665624411176713792}{430996190625}h^2t^{10}+\frac{130581716992}{3618239625}h^2t^{17}-\frac{230751993856}{54273594375}h^2t^{18}\\
-\frac{33554432}{2212447111875}h^2t^{22}-\frac{6758134513664}{28733079375}h^2t^{16}-\frac{22932685474592}{540280125}h^2t^{12}\\
-\frac{22378330760948}{197071875}h^2t^6+\frac{111961796416}{6965595}h^2t^{13}-\frac{1369440256}{60658723125}t^{20}h^2\\
-\frac{32284128426940897}{824562019771875}h^2t-\frac{334729237973849}{1489863375}h^2t^8-\frac{1054624684}{75075}h^2t^4+\frac{10336}{5}h^2t^3\\
-\frac{109682066574848}{22347950625}h^2t^{14}+\frac{171966464}{201131555625}h^2t^{21}+\frac{16273280826496}{180093375}h^2t^{11}\\
+\frac{5859609484286}{118243125}h^2t^5+\frac{3623188505490154}{17239847625}h^2t^9+\frac{8101298176}{21837140325}h^2t^{19}
\end{array}\right)f_{n+\frac{5}{4}}$$

$$\beta_{\frac{3}{2}}(t)=\left(\begin{array}{l}
\frac{31256303905633}{127702575}h^2t^6+\frac{88747585}{3003}h^2t^4-\frac{14841223774208}{5157219375}h^2t^{15}\\
+\frac{2424625692866959}{4875916500}h^2t^8+\frac{11995380591872}{1031443875}h^2t^{14}-\frac{2708033617485377}{5746615875}h^2t^9\\
+\frac{10166468608}{181976169375}t^{20}h^2-\frac{12920}{3}h^2t^3-\frac{62309858522848}{300155625}h^2t^{11}\\
-\frac{5414452011875513}{13408770375}h^2t^7-\frac{105689792512}{1206079875}h^2t^{17}+\frac{159815718978856}{1620840375}h^2t^{12}\\
-\frac{99658760192}{109185701625}h^2t^{19}+\frac{92316237824}{8881133625}h^2t^{18}-\frac{142606336}{67043851875}h^2t^{21}\\
-\frac{99658760192}{109185701625}h^2t^{19}+\frac{92316237824}{8881133625}h^2t^{18}-\frac{142606336}{67043851875}h^2t^{21}\\
+\frac{16777216}{442489422375}h^2t^{22}-\frac{6569490356096}{174139875}h^2t^{13}+\frac{3057542616556243}{38113088913900}h^2t\\
+\frac{16287510477824}{28733079375}h^2t^{16}+\frac{3057542616556243}{38113088913900}h^2t+\frac{16287510477824}{28733079375}h^2t^{16}\\
+\frac{30224671587015424}{86199238125}h^2t^{10}-\frac{598597400279}{5675670}h^2t^5-\frac{25115413711194419}{4287722502813750}h^2
\end{array}\right)f_{n+\frac{3}{2}}$$

$$\beta_{\frac{7}{4}}(t)=\left(\begin{array}{l}
\frac{7051602979456}{18376875}h^2t^{11}-\frac{871638333796}{2027025}h^2t^6+\frac{4191936512}{24613875}h^2t^{17}-\frac{958398464}{8665531875}t^{20}h^2\\
+\frac{2102363441764402}{2462835375}h^2t^9-\frac{55314654406452608}{86199238125}h^2t^{10}-\frac{33554432}{442489422375}h^2t^{22}\\
-\frac{31430513139712}{28733079375}h^2t^{16}+\frac{51680}{7}h^2t^3-\frac{22559210224175123}{164912403954375}h^2t\\
-\frac{8600158208}{422911125}h^2t^{18}+\frac{121634816}{28733079375}h^2t^{21}-\frac{14106252411392}{638512875}h^2t^{14}\\
-\frac{5749779560617}{6449625}h^2t^8-\frac{1420252371488}{77182875}h^2t^{12}+\frac{65357863923364}{91216125}h^2t^7\\
-\frac{1072361260}{21021}h^2t^4+\frac{241902426376874}{23558914850625}h^2+\frac{4003463168}{2228279625}h^2t^{19}\\
+\frac{12181192548352}{2210236875}h^2t^{15}+\frac{3543924586}{19305}h^2t^5+\frac{252500863936}{3553875}h^2t^{13}
\end{array}\right)f_{n+\frac{7}{4}}$$

$$\beta_2(t)=\left(\begin{array}{l}
-\frac{287679146326369}{2750517000}h^2t^7+\frac{3794025768704}{2210236875}h^2t^{16}-\frac{6328725629872}{58046625}h^2t^{13}\\
+\frac{218707668598549}{166698000}h^2t^8+\frac{1812865600451}{2910600}h^2t^6+\frac{16968060216256}{496621125}h^2t^{14}\\
+\frac{453994115455378}{1620840375}h^2t^{12}-\frac{307379806687}{1164240}h^2t^5+\frac{4194304}{34037647875}h^2t^{22}\\
+\frac{22088155136}{683164125}h^2t^{18}-\frac{20995}{2}h^2t^3-\frac{224660166656}{834978375}h^2t^{17}-\frac{8045330432}{2799633375}h^2t^{19}\\
+\frac{276692992}{1555351875}t^{20}h^2+\frac{86131482974007799}{445477662630000}h^2t-\frac{35405831794352741}{2450127144465000}h^2\\
-\frac{6720428568127123}{5304568500}h^2t^9+\frac{636892064729896}{6630710625}h^2t^{10}-\frac{105906176}{15471658125}h^2t^{21}\\
-\frac{1564790160015638}{2701400625}h^2t^{11}
\end{array}\right)f_{n+2}$$

$$\beta_{\frac{9}{4}}(t)=\left(\begin{array}{l}\frac{1947069028748608}{2701400625}h^2t^{11}-\frac{22071573947042}{29469825}h^2t^6-\frac{326127355899798109}{1429240834271250}h^2t\\+\frac{109860463689138181}{6431583754220625}h^2-\frac{9889400655182497}{6188663250}h^2t^8-\frac{23602664267135584}{19892131875}h^2t^{10}\\-\frac{16777216}{102112943625}h^2t^{22}+\frac{435088037763338}{343814625}h^2t^7-\frac{951136681984}{22544416125}h^2t^{18}\\+\frac{2060618015774383}{1326142125}h^2t^9+\frac{291367727104}{834978375}h^2t^{17}+\frac{31678529536}{8398900125}h^2t^{19}\\+\frac{170079447371776}{15471658125}h^2t^{15}-\frac{14667750600704}{6630710625}h^2t^{16}+\frac{140509184}{15471658125}h^2t^{21}\\-\frac{1707079866911728}{4862521125}h^2t^{12}+\frac{68915283131}{218295}h^2t^5-\frac{1623322370}{18711}h^2t^4+\frac{335920}{27}h^2t^3\\-\frac{9859760128}{41994500625}t^{20}h^2+\frac{23968756379744}{174139875}h^2t^{13}-\frac{174739641058048}{40226311125}h^2t^{14}\end{array}\right)f_{n+\frac{9}{4}}$$

$$\beta_{\frac{5}{2}}(t)=\left(\begin{array}{l}-\frac{3123262434907}{9922500}h^2t^5+\frac{325320704}{1272560625}t^{20}h^2+\frac{46553486176}{10247461875}h^2t^{18}\\-\frac{55823938310891}{74418750}h^2t^6-\frac{184756}{15}h^2t^3+\frac{478662793076836691}{2126143389825000}h^2t\\+\frac{8388608}{46414974375}h^2t^{22}+\frac{278198227344256}{2126143389825000}h^2t^{14}-\frac{9809932374016}{843908625}h^2t^{15}\\+\frac{10989420566249948}{9041878125}h^2t^{10}-\frac{1871183872}{458121825}h^2t^{19}-\frac{8388608}{843908625}h^2t^{21}\\-\frac{47779490540489}{37507050}h^2t^7-\frac{85065138176}{227721375}h^2t^{17}+\frac{1418806171136}{602791875}h^2t^{16}+\frac{271415923}{3150}h^2t^4\\-\frac{197263529429526737}{11693788644037500}h^2t^{17}-\frac{21894384515624}{29469825}h^2t^{11}+\frac{160989407168852}{442047375}h^2t^{12}\\-\frac{572157698977448}{361675125}h^2t^9+\frac{4993063878883}{3087000}h^2t^8-\frac{75074638256768}{522419625}h^2t^{13}\end{array}\right)f_{n+\frac{5}{2}}$$

$$\beta_{\frac{11}{4}}(t)=\left(\begin{array}{l}\frac{472086727462912}{46414974375}h^2t^{15}-\frac{477331021049608283}{2572633501688250}h^2t-\frac{936190797843163}{687629250}h^2t^8\\+\frac{5324004852281491}{3978426375}h^2t^9+\frac{828229808128}{2504935125}h^2t^{17}+\frac{1100915052743866}{1031443875}h^2t^7\\-\frac{532397470098176}{13408770375}h^2t^{14}-\frac{13735051157504}{6630710625}h^2t^{16}+\frac{417333248}{46414974375}h^2t^{21}\\+\frac{92114255872}{25196700375}h^2t^{19}-\frac{20559051411353344}{19892131875}h^2t^{10}+\frac{5148092204598976}{8104201875}h^2t^{11}\\+\frac{335920}{33}h^2t^3-\frac{16777216}{102112943625}h^2t^{22}-\frac{1522722596863568}{486251125}h^2t^{12}\\+\frac{714195429543328}{5746615875}h^2t^{13}-\frac{49476610}{693}h^2t^4+\frac{89480656953122311}{6431583754220625}h^2\\-\frac{911652552704}{22544416125}h^2t^{18}+\frac{57116932489}{218295}h^2t^5-\frac{3220176896}{13998166875}t^{20}h^2-\frac{683136605234}{1091475}h^2t^6\end{array}\right)f_{n+\frac{11}{4}}$$

$$\beta_3(t)=\left(\begin{array}{l}\frac{4194304}{34037647875}h^2t^{22}-\frac{20995}{3}h^2t^3-\frac{81881089536178171}{8575445005627500}h^2+\frac{17042815920589}{39293100}h^2t^6\\+\frac{713050462037}{5602905000}h^2t-\frac{382906412032}{52093125}h^2t^{15}+\frac{361040715706498}{1620840375}h^2t^{12}\\-\frac{15474220949104}{174139875}h^2t^{13}+\frac{2390622208}{13998166875}t^{20}h^2-\frac{36803706896914}{81860625}h^2t^{11}\\-\frac{556829673738883}{750141000}h^2t^7+\frac{4827002076440266}{6630710625}h^2t^{10}+\frac{223216500736}{7514805375}h^2t^{18}\\-\frac{1048576}{156279375}h^2t^{21}-\frac{6112012288}{25302375}h^2t^{17}-\frac{2061369344}{763536375}h^2t^{19}\\+\frac{3326719119104}{2210236875}h^2t^{16}-\frac{50276226952183}{53581500}h^2t^9-\frac{57393564859}{317520}h^2t^5\\+\frac{272709215}{5544}h^2t^4+\frac{15687330117079931}{16503102000}h^2t^8+\frac{383688114583872}{13408770375}h^2t^{14}\end{array}\right)f_{n+3}$$

$$\beta_{\frac{13}{4}}(t)=\left(\begin{array}{l}-\frac{24701037343349}{45147375}h^2t^8+\frac{1572537049088}{10854718875}h^2t^{17}+\frac{707422385168768}{2701400625}h^2t^{11}\\+\frac{11610754673253194}{2143861251406875}h^2-\frac{159034114048}{8881133625}h^2t^{18}-\frac{154680834453804773}{2143861251406875}h^2t\\-\frac{3519642921748}{14189175}h^2t^6+\frac{9345087931434466}{17239847625}h^2t^9-\frac{25803657920512}{28733079375}h^2t^{16}\\+\frac{3032039657152}{58046625}h^2t^{13}-\frac{36395836878448448}{86199238125}h^2t^{10}+\frac{877620599922688}{201131555625}h^2t^{15}\\+\frac{59544436736}{36395233875}h^2t^{19}+\frac{1903183861465804}{4469590125}h^2t^7-\frac{33554432}{442489422375}h^2t^{22}\\+\frac{97525314046}{945945}h^2t^5+\frac{826277888}{3003}h^2t^4-\frac{25116195486208}{1489863375}h^2t^{14}\\+\frac{51680}{13}h^2t^3-\frac{2103443456}{20219574375}t^{20}h^2\end{array}\right)f_{n+\frac{13}{4}}$$

$$\beta_{\frac{7}{2}}(t)=\left(\begin{array}{l}-\frac{2283729108544}{18376875}h^2t^{11}+\frac{273632995}{21021}h^2t^4-\frac{12920}{7}h^2t^3+\frac{17206645671805264}{86199238125}h^2t^{10}\\+\frac{26089169531}{225225}h^2t^6+\frac{40814116864}{4652022375}h^2t^{18}-\frac{628933962108187}{2462835375}h^2t^9\\-\frac{1538216973812297}{612531786116250}h^2+\frac{446169088}{8665531875}t^{20}h^2+\frac{5173348992256}{638512875}h^2t^{14}\\-\frac{1854630353}{38610}h^2t^5-\frac{2597933882113}{13030875}h^2t^7+\frac{1597568491352}{25727625}h^2t^{12}\\-\frac{1738391552}{24613875}h^2t^{17}-\frac{88823479936}{3553875}h^2t^{13}+\frac{286782317454699839}{8575445005627500}h^2t\\+\frac{16777216}{442489422375}h^2t^{22}+\frac{12518324993024}{28733079375}h^2t^{16}-\frac{4656535748608}{2210236875}h^2t^{15}\\-\frac{8388608}{4104725625}h^2t^{21}-\frac{1795162112}{2228279625}h^2t^{19}+\frac{218729515074143}{851350500}h^2t^8\end{array}\right)f_{n+\frac{7}{2}}$$

$$\beta_{\frac{15}{4}}(t)=\left(\begin{array}{l}
-\frac{209136916908544}{67043851875}h^2t^{14}-\frac{2121225656787857}{170147718365625}h^2t-\frac{1298920510971971}{13408770375}h^2t^8 \\
+\frac{2839244107648}{60031125}h^2t^{11}+\frac{201070954839532}{2681754075}h^2t^7+\frac{10336}{15}h^2t^3 \\
-\frac{693799868421788}{15962821875}h^2t^6-\frac{33554432}{2212447111875}h^2t^{22}-\frac{1680144072704}{488462349375}h^2t^{18} \\
+\frac{8536927538674}{88409475}h^2t^9-\frac{32632747781297792}{430996190625}h^2t^{10}-\frac{121778892}{25025}h^2t^4 \\
+\frac{1386217472}{4367428065}h^2t^{19}-\frac{4861959102464}{28733079375}h^2t^{16}-\frac{3709861888}{181976169375}t^{20}h^2 \\
+\frac{4194304}{5157219375}h^2t^{21}+\frac{204864793035326}{218761352184375}h^2+\frac{6380711218334}{354729375}h^2t^5 \\
-\frac{38443782481696}{1620840375}h^2t^{12}+\frac{1668876156608}{174139875}h^2t^{13}+\frac{33236738048}{1206079875}h^2t^{17} \\
+\frac{840227528704}{1031443875}h^2t^{15}
\end{array}\right)f_{n+\frac{15}{4}}$$

$$\beta_4(t)=\left(\begin{array}{l}
-\frac{9401651886117719}{34301780022510000}h^2+\frac{50050254308007539}{13720712009004000}h^2t+\frac{2903596039159}{227026800}h^2t^6 \\
+\frac{34996600503769}{1222452000}h^2t^8+\frac{4199050416224}{4469590125}h^2t^{14}+\frac{425713023751}{60031125}h^2t^{12} \\
+\frac{137162915}{96096}h^2t^4-\frac{2339144704}{278326125}h^2t^{17}-\frac{1580989524586597}{71513442000}h^2t^7 \\
+\frac{1939077829702658}{86199238125}h^2t^{10}-\frac{10667884544}{109185701625}h^2t^{19}+\frac{29425664}{4666055625}t^{20}h^2 \\
-\frac{38517984424}{13395375}h^2t^{13}+\frac{343164354 56}{32564156625}h^2t^{18}-\frac{3810882646528}{15471658125}h^2t^{15} \\
-\frac{12303207377}{2328480}h^2t^5-\frac{12691824108353}{900466875}h^2t^{11}-\frac{50855936}{201131555625}h^2t^{21} \\
+\frac{1476273662848}{28733079375}h^2t^{16}
\end{array}\right)f_{n+4}$$

$$\beta_{\frac{17}{4}}(t)=\left(\begin{array}{l}
\frac{260024604025601}{4287722502813750}h^2+\frac{4670881792}{206239658625}h^2t^{19}-\frac{8388608}{7522320180375}h^2t^{22} \\
-\frac{16153595}{51051}h^2t^4-\frac{337733617664}{28733079375}h^2t^{16}+\frac{868411291648}{15471658125}h^2t^{15} \\
+\frac{202375168}{3419236445625}h^2t^{21}-\frac{2159476736}{8881133625}h^2t^{18}-\frac{228620816651}{80405325}h^2t^6 \\
-\frac{6919605198266177}{8575445005627500}h^2t+\frac{760}{17}h^2t^3+\frac{8561316224864}{2701400625}h^2t^{11} \\
+\frac{12587947472}{19348875}h^2+t^{13}+\frac{220639325081381}{34479695250}h^2t^9+\frac{22006889280967}{4469590125}h^2t^7 \\
-\frac{3461296134211}{541768500}h^2t^8-\frac{2592218449816}{1620840375}h^2t^{12}-\frac{56098816}{38192529375}t^{20}h^2 \\
-\frac{317777853056}{1489863375}h^2t^{14}-\frac{434809250413936}{86199238125}h^2t^{10}+\frac{37721549347}{32162130}h^2t^5 \\
+\frac{20957333504}{10854718875}h^2t^{17}
\end{array}\right)f_{n+\frac{17}{4}}$$

$$\beta_{\frac{9}{2}}(t)=\begin{pmatrix} -\frac{1355169154116}{2701400625}h^2t^{11}+\frac{207635647700498}{258597714375}h^2t^{10}-\frac{6060508544}{58046625}h^2t^{13} \\ -\frac{7933357279}{42882840}h^2t^5-\frac{3411746816}{10854718875}h^2t^{17}+\frac{1450816404814057}{11433926674170000}h^2t \\ +\frac{2249064448}{9280784638125}t^{20}h^2-\frac{190}{27}h^2t^3+\frac{1244342117978}{4862521125}h^2t^{12}+\frac{1059094528}{26643400875}h^2t^{18} \\ +\frac{4194304}{22566960541125}h^2t^{22}+\frac{824594105}{16540524}h^2t^4-\frac{41779494660821}{53635081500}h^2t^7 \\ -\frac{350414279266459}{34479695250}h^2t^9-\frac{33554432}{3419236445625}h^2t^{21}+\frac{164294063872}{86199238125}h^2t^{16} \\ +\frac{11700090481031}{26051325300}h^2t^6-\frac{406061056}{109185701625}h^2t^{19}+\frac{1381373280832}{40226311125}h^2t^{14} \\ -\frac{490772113685617}{51452670033765000}h^2-\frac{140296374272}{15471658125}h^2t^{15}+\frac{592408267718153}{58510998000}h^2t^8 \end{pmatrix} f_{n+\frac{9}{2}}$$

$$\beta_{\frac{19}{4}}(t)=\begin{pmatrix} -\frac{2697929593169}{212614338982500}h^2t+\frac{5535773584}{522419625}h^2t^{13}+\frac{126353408}{327557104975}h^2t^{19} \\ -\frac{8388608}{428772250281375}h^2t^{22}-\frac{46871222912}{13408770375}h^2t^{14}-\frac{78118912}{3093594879375}t^{20}h^2 \\ -\frac{109379584}{26643400875}h^2t^{18}-\frac{20915725497664}{258597714375}h^2t^{10}+\frac{643509986291}{677008816233750}h^2 \\ +\frac{1052813312}{32564156625}h^2t^{17}+\frac{10567729954153}{103439085750}h^2t^9-\frac{32510829361}{723647925}h^2t^6 \\ +\frac{200278016}{194896477400625}h^2t^{21}+\frac{42981595136}{46414974375}h^2t^{15}-\frac{16836337664}{86199238125}h^2t^{16} \\ -\frac{125945164376}{4862521125}h^2t^{12}+\frac{40}{57}h^2t^3+\frac{413793204128}{8104201875}h^2t^{11}+\frac{1783376939}{96486390}h^2t^5 \\ +\frac{1046330784599}{13408770375}h^2t^7-\frac{14478505}{2909907}h^2t^4-\frac{18348828859}{180589500}h^2t^8 \end{pmatrix} f_{n+\frac{19}{4}}$$

$$\beta_5(t)=\begin{pmatrix} \frac{645242018886}{30151996875}h^2t^6-\frac{10292224}{6512831325}h^2t^{17}+\frac{3866624}{3093594879375}t^{20}h^2 \\ -\frac{5803979601547}{128631675084412500}h^2+\frac{154364955666811}{257263350168825000}h^2t+\frac{6063698587}{4862521125}h^2t^{12} \\ -\frac{267195784}{522419625}h^2t^{13}+\frac{5013017410969}{1292988571875}h^2t^{10}+\frac{275295799}{1163962800}h^2t^4 \\ +\frac{694142313941}{143026884000}h^2t^8-\frac{417321472}{9282994875}h^2t^{15}-\frac{2022480780283}{413756343000}h^2t^9 \\ -\frac{31398836201}{8581613040}h^2t^7-\frac{524288}{10257709336875}h^2t^{21}-\frac{65536}{3447969525}h^2t^{19} \\ -\frac{3975325483}{1620840375}h^2t^{11}+\frac{295190528}{1465387048125}h^2t^{18}+\frac{2097152}{2143861251406875}h^2t^{22}-\frac{1}{30}h^2t^3 \\ +\frac{820029568}{86199238125}h^2t^{16}+\frac{11342959136}{67043851875}h^2t^{14}-\frac{169704792667}{192972780000}h^2t^5 \end{pmatrix} f_{n+5}$$

Evaluating (5) at $t=1$ and other non-interpolating points to obtain the discrete schemes.

$$
\begin{aligned}
y_{n+5} = & -3y_{n+1} + 4y_{n+2} + \frac{1776200362021}{3233576548125} f_{n+\frac{17}{4}} h^2 - \frac{282768303467593}{317609074282500} f_{n+4} h^2 \\
& + \frac{1636807354759364}{510443155096875} f_{n+\frac{15}{4}} h^2 - \frac{1534492376964049}{714620417135625} f_{n+\frac{3}{2}} h^2 - \frac{4013330079761549}{714620417135625} f_{n+\frac{7}{2}} h^2 \\
& + \frac{4030809946125748}{3573102085678125} f_{n+\frac{5}{4}} h^2 - \frac{10574081731840829}{649654924668750} f_{n+\frac{5}{2}} h^2 - \frac{39958739073765847}{2858481668542500} f_{n+3} h^2 \\
& + \frac{42807077530291}{714620417135625} f_{n+\frac{3}{4}} h^2 + \frac{96607397039}{102088631019375} f_{n+\frac{1}{4}} h^2 - \frac{903282230385644}{102088631019375} f_{n+2} h^2 - \\
& \frac{70295523701659}{272236349385000} f_{n+1} h^2 + \frac{31370941992359722}{2143861251406875} f_{n+\frac{9}{4}} h^2 + \frac{5412695508501266}{306265893058125} f_{n+\frac{11}{4}} h^2 \\
& - \frac{81259662604807}{8575445005627500} f_{n+\frac{1}{2}} h^2 - \frac{1928183112769}{42877225028137500} h^2 f_n + \frac{25340560637687}{85754450056275000} f_{n+5} h^2 + \\
& \frac{2538906475947356}{238206805711875} f_{n+\frac{13}{4}} h^2 + \frac{41285430345559}{1225063572232500} f_{n+\frac{9}{2}} h^2 + \frac{4420272282791}{54970801318125} f_{n+\frac{19}{4}} h^2 \\
& + \frac{150770503748}{251354375} f_{n+\frac{7}{4}} h^2
\end{aligned} \tag{6}
$$

$$
\begin{aligned}
y_{n+\frac{1}{4}} = & +2y_{n+1} - y_{n+2} + \frac{260024604025601}{4287722502813750} f_{n+\frac{17}{4}} h^2 - \frac{9401651886117719}{34301780022510000} f_{n+4} h^2 + \\
& \frac{204864793035326}{218761352184375} f_{n+\frac{15}{4}} h^2 - \frac{25115413711194419}{4287722502813750} f_{n+\frac{3}{2}} h^2 - \frac{1538216973812297}{612531786116250} f_{n+\frac{7}{2}} h^2 + \\
& \frac{32998793151363674}{10719306257034375} f_{n+\frac{5}{4}} h^2 - \frac{197263529429526737}{11693788644037500} f_{n+\frac{5}{2}} h^2 - \frac{81881089536178171}{8575445005627500} f_{n+3} h^2 + \\
& \frac{181325998653287}{329824807908750} f_{n+\frac{3}{4}} h^2 + \frac{147767713390877}{1837595358348750} f_{n+\frac{1}{4}} h^2 - \frac{35405831794352741}{2450127144465000} f_{n+2} h^2 - \\
& \frac{1931087090971484}{2143861251406875} f_{n+1} h^2 + \frac{109860463689138181}{6431583754220625} f_{n+\frac{9}{4}} h^2 + \frac{89480656953122311}{6431583754220625} f_{n+\frac{11}{4}} h^2 + \\
& \frac{1732080468999623}{51452670033765000} f_{n+\frac{1}{2}} h^2 + \frac{217198974162341}{73503814333950000} h^2 f_n - \frac{5803979601547}{12863165084412500} f_{n+5} h^2 + \\
& \frac{11610754673253194}{214861251406875} f_{n+\frac{13}{4}} h^2 - \frac{49072113685617}{51452670033765000} f_{n+\frac{9}{2}} h^2 + \frac{643509986291}{677008816233750} f_{n+\frac{19}{4}} h^2 \\
& + \frac{241902426376874}{2355891450625} f_{n+\frac{7}{4}} h^2
\end{aligned} \tag{7}
$$

$$
\begin{aligned}
y_n = & -\frac{1165346871375235061}{285460502292725760000} f_{n+\frac{17}{4}} h^2 + \frac{63425889744225624841}{3425526027512709120000} f_{n+4} h^2 - \\
& \frac{67993560096219771161}{1070476883597721600000} f_{n+\frac{15}{4}} h^2 + \frac{23222015272538344885 1}{4281907534390886400000} f_{n+\frac{3}{2}} h^2 + \\
& \frac{3489003393266278751}{20390035878051840000} f_{n+\frac{7}{2}} h^2 - \frac{4215512703107358791}{5097508969512960000} f_{n+\frac{5}{4}} h^2 + \\
& \frac{1380701419844674854643}{1167792963924787200000} f_{n+\frac{5}{2}} h^2 + \frac{161053156894920468599}{244680430556622080000} f_{n+3} h^2 + \\
& \frac{4746050274878036731}{50375382757539840000} f_{n+\frac{3}{4}} h^2 + \frac{10035044503712785003}{2569144520634531840000} f_{n+\frac{1}{4}} h^2 + \frac{7}{4} y_{n+1} \\
& + \frac{595810677090655314031}{570921004585451520000} f_{n+2} h^2 + \frac{131960613347550817243}{489360861073244160000} f_{n+1} h^2 - \\
& \frac{3}{4} y_{n+2} - \frac{1555574248518383645243}{1284572260317265920000} f_{n+\frac{9}{4}} h^2 - \frac{19702553879395159271}{20390035878051840000} f_{n+\frac{11}{4}} h^2 + \\
& \frac{13490697484082615029}{190307001528483840000} f_{n+\frac{1}{2}} h^2 - \frac{332178049268088299}{734041291609866240000} h^2 f_n + \\
& \frac{17196021561981623}{5709210045854515200000} f_{n+5} h^2 - \frac{79430474751195735931}{214095376719544320000} f_{n+\frac{13}{4}} h^2 + \\
& \frac{469849509297764929}{734041291609866240000} f_{n+\frac{9}{2}} h^2 - \frac{1373284761560113}{21589449753231360000} f_{n+\frac{19}{4}} h^2 - \\
& \frac{21424479419808984823}{30585053817077760000} f_{n+\frac{7}{4}} h^2
\end{aligned} \tag{8}
$$

$$
\begin{aligned}
y_{n+\frac{1}{2}} = & \frac{164816051095897}{365885653573440000} f_{n+\frac{17}{4}} h^2 - \frac{1333029099308519}{650463384130560000} f_{n+4} h^2 + \frac{248198137318393}{35181312843600000} \\
& f_{n+\frac{15}{4}} h^2 + \frac{5721474872641379}{650463384130560000} f_{n+4} h^2 + \frac{248198137318393}{35181312843600000} f_{n+\frac{15}{4}} h^2 + \frac{5721474872641379}{731771307146880000} f_{n+\frac{3}{3}} h^2 - \\
& \frac{1994811603223883}{104538758163840000} f_{n+\frac{7}{2}} h^2 + \frac{5614765456396809}{457357066966800000} f_{n+\frac{5}{4}} h^2 - \frac{89852795332787831}{665246642860800000} f_{n+\frac{5}{2}} h^2 \\
& - \frac{217040696241913963}{2927085228587520000} f_{n+3} h^2 + \frac{24641348697487877}{365885653573440000} f_{n+\frac{3}{4}} h^2 - \frac{38593276046879}{365885653573440000} f_{n+\frac{1}{4}} h^2 + \\
& \frac{3}{2} y_{n+1} + \frac{202177335837757883}{1951390152391680000} f_{n+1} h^2 - \frac{1}{2} y_{n+2} - \frac{346510588471706363}{2927085228587520000} f_{n+2} h^2 + \\
& \frac{76533980126988277}{548828480360160000} f_{n+\frac{9}{4}} h^2 + \frac{60127585846657937}{548828480360160000} f_{n+\frac{11}{4}} h^2 + \frac{39482462140330967}{8781255685762560000} f_{n+\frac{1}{2}} h^2 + \\
& \frac{3635312175749}{12544650979660800000} h^2 f_n - \frac{28972503065557}{8781256857625600000} f_{n+\frac{1}{2}} h^2 + \frac{1267387453769443}{30490471131120000} f_{n+\frac{13}{4}} h^2 - \\
& \frac{618531049510793}{8781255685762560000} f_{n+\frac{9}{2}} h^2 + \frac{2554946603821}{365885653573440000} f_{n+\frac{19}{4}} h^2 + \frac{55053850717}{49616080000} f_{n+\frac{7}{4}} h^2
\end{aligned} \tag{9}
$$

$$\begin{aligned}
y_{n+\frac{3}{4}} = &-\frac{22654369252652773}{211576607581667328000} f_{n+\frac{17}{4}} h^2 + \frac{70464677681881210 69}{143872093155533783040 00} f_{n+4} h^2 \\
&-\frac{7627892271364016371}{446002911110430720000} f_{n+\frac{15}{4}} h^2 + \frac{83957590548446820859}{179840116444417228800 0} f_{n+\frac{15}{4}} h^2 \\
&+\frac{83957590548446820859}{1798401164444172288000} f_{n+\frac{3}{2}} h^2 + \frac{8313304966375274563}{1798401164444172288000} f_{n+\frac{7}{2}} h^2 \\
&+\frac{25194415175496473537}{642286130158632960000} f_{n+\frac{5}{4}} h^2 + \frac{168443536389951023203}{4904730448484106240000} f_{n+\frac{5}{2}} h^2 \\
&+\frac{18814684818221394523}{102765780825381273600 0} f_{n+3} h^2 + \frac{19147986366012960193}{356802328888344576000} f_{n+\frac{3}{4}} h^2 \\
&+\frac{133428616678258777}{10790406986665033728000} f_{n+\frac{1}{4}} h^2 + \frac{5}{4} y_{n+1} + \frac{242930882053447987057}{719360465777668915200 0} f_{n+2} h^2 \\
&+\frac{83679067514029357951 3}{143872093155533783040 00} f_{n+1} h^2 - \frac{1}{4} y_{n+2} - \frac{195973318926695158969}{539520349333251686400 0} f_{n+\frac{9}{4}} h^2 \\
&-\frac{147830822897630116507}{5395203493332516864000} f_{n+\frac{11}{4}} h^2 - \frac{4536740659895795209}{2158081397333006745600 0} f_{n+\frac{1}{2}} h^2 \\
&-\frac{91869300338939713}{215808139733300674560000} h^2 f_n + \frac{16713362704039907}{215808139733300674560000} f_{n+5} h^2 \\
&-\frac{9133147467189501409}{899200582222086144000} f_{n+\frac{13}{4}} h^2 + \frac{359626318206676487}{21580813973330067456000} f_{n+\frac{9}{2}} h^2 \\
&-\frac{2535989372893283}{1541486712380719104000} f_{n+\frac{19}{4}} h^2 - \frac{7924463380409086747}{899200582222086144000} f_{n+\frac{7}{4}} h^2
\end{aligned} \quad (10)$$

$$\begin{aligned}
y_{n+\frac{5}{4}} = &-\frac{7100961507017501}{352627679302778880000} f_{n+\frac{17}{4}} h^2 + \frac{246461960773163231}{26642980213987737600 00} f_{n+4} h^2 \\
&-\frac{2412974855070620539}{749333818518405120000 0} f_{n+\frac{15}{4}} h^2 - \frac{85963726840583489771}{2997335274073620480000} f_{n+\frac{3}{2}} h^2 \\
&+\frac{2644676218004337149}{2997335274073620480000} f_{n+\frac{7}{2}} h^2 - \frac{3464873754878545979}{82344375661363200000} f_{n+\frac{5}{4}} h^2 \\
&+\frac{18386609317773912119}{272485024915783680000 0} f_{n+\frac{5}{2}} h^2 + \frac{6069424569392463001}{171276301375633545600 00} f_{n+3} h^2 \\
&+\frac{1425942246965622913}{599467054814724096000 0} f_{n+\frac{3}{4}} h^2 + \frac{9382689417540059}{5994670548147240960000} f_{n+\frac{1}{4}} h^2 + \frac{3}{4} y_{n+1} \\
&+\frac{59707487491941105587}{119893410929448192000 0} f_{n+2} h^2 - \frac{383299750629505757}{87834000705454080000} f_{n+1} h^2 + \frac{1}{4} y_{n+2} \\
&-\frac{63778729685880254497}{899200582222086144000 0} f_{n+\frac{9}{4}} h^2 - \frac{6867956170762062461}{1284572260317260317659 20000} f_{n+\frac{11}{4}} h^2 \\
&-\frac{784197299510870783}{35968023288883445760000} f_{n+\frac{1}{2}} h^2 - \frac{21267081005088307}{359680232888834457600000} h^2 f_n \\
&+\frac{51866599293916993}{359680232888834457600000} f_{n+5} h^2 - \frac{974818911678805723}{49955587901227008000 0} f_{n+\frac{13}{4}} h^2 \\
&+\frac{16043155287641831}{51382890412690636800 00} f_{n+\frac{9}{2}} h^2 - \frac{37585967623159}{1223402152683110400 00} f_{n+\frac{19}{4}} h^2 \\
&-\frac{13953332100396697}{685261836779520000} f_{n+\frac{7}{4}} h^2
\end{aligned} \quad (11)$$

$$
\begin{aligned}
y_{n+\frac{3}{2}} = & -\frac{2143472528173}{274414240180080000} f_{n+\frac{17}{4}} h^2 + \frac{48326171107547}{1350962413194240000} f_{n+4} h^2 - \\
& \frac{12159509301781}{98005085778600000} f_{n+\frac{15}{4}} h^2 - \frac{12140163316241579}{2195313921440640000} f_{n+\frac{3}{2}} h^2 + \frac{743048992774061}{2195313921440640000} f_{n+\frac{7}{2}} h^2 \\
& - \frac{212584248470873}{6596496158175000} f_{n+\frac{5}{4}} h^2 + \frac{14857206028512593}{5987219785747200000} f_{n+\frac{5}{2}} h^2 \\
& + \frac{11788021515654803}{8781255685762560000} f_{n+3} h^2 + \frac{3923539218167}{32284028256480000} f_{n+\frac{3}{4}} h^2 \\
& + \frac{84246129299}{117606102934320000} f_{n+\frac{1}{4}} h^2 + \frac{1}{2} y_{n+1} - \frac{653609807858131}{1254465097966080000} f_{n+2} h^2 \\
& - \frac{6564677709887887}{2508930195932160000} f_{n+1} h^2 + \frac{1}{2} y_{n+2} - \frac{1003217251470443}{411621360270120000} f_{n+\frac{9}{4}} h^2 \\
& - \frac{235751881027543}{117606102934320000} f_{n+\frac{11}{4}} h^2 - \frac{273523042315907}{26343767057287680000} f_{n+\frac{1}{2}} h^2 \\
& - \frac{6962117829243}{263437670572876800000} h^2 f_n + \frac{1483798765357}{26343767052876800000} f_{n+5} h^2 \\
& - \frac{25537341905773}{34301780022510000} f_{n+\frac{13}{4}} h^2 + \frac{456993926699}{3763395293898240000} f_{n+\frac{9}{2}} h^2 \\
& - \frac{197183750219}{1646485441080480000} f_{n+\frac{19}{4}} h^2 - \frac{352534557484339}{10554393853080000} f_{n+\frac{7}{4}} h^2
\end{aligned} \tag{12}
$$

$$
\begin{aligned}
y_{n+\frac{7}{4}} = & -\frac{895585999606993}{153709501234544640000} f_{n+\frac{17}{4}} h^2 + \frac{641383973089092899}{23978682192588863840000} f_{n+4} h^2 \\
& - \frac{14222210535846961}{15292569085388800000} f_{n+\frac{15}{4}} h^2 - \frac{93229248337971215651}{2997335274073620480000} f_{n+\frac{3}{2}} h^2 \\
& + \frac{36291955154302369}{142730251146362880000} f_{n+\frac{7}{2}} h^2 - \frac{39445667680517000113}{2497779395061350400000} f_{n+\frac{5}{4}} h^2 \\
& + \frac{1453390637789073157}{8174550747473510400000} f_{n+\frac{5}{2}} h^2 + \frac{12087088713472486627}{11989341096294481920000} f_{n+3} h^2 \\
& + \frac{415334770760036203}{5994670548147240960000} f_{n+\frac{3}{4}} h^2 + \frac{1147730488105541}{2569144520634551840000} f_{n+\frac{1}{4}} h^2 + \frac{1}{4} y_{n+1} \\
& - \frac{1953814411069734973}{570921004585451520000} f_{n+2} h^2 - \frac{32664737652926352121}{23978682192588963840000} f_{n+1} h^2 + \frac{3}{4} y_{n+2} \\
& - \frac{12960622740212574307}{8992005822220861440000} f_{n+\frac{9}{4}} h^2 - \frac{1489679272405749793}{99911175802450160000} f_{n+\frac{11}{4}} h^2 \\
& - \frac{8326797518110909}{1332149010699386880000} f_{n+\frac{1}{2}} h^2 - \frac{866307726785401}{51382890412690636800000} h^2 f_n \\
& + \frac{166834500017077}{39964470320981606400000} f_{n+5} h^2 - \frac{839218087922306219}{1498667637036810240000} f_{n+\frac{13}{4}} h^2 \\
& + \frac{32504311318870337}{35968023288883445760000} f_{n+\frac{9}{2}} h^2 - \frac{123033232991491}{1383385511110901760000} f_{n+\frac{19}{4}} h^2 \\
& - \frac{97894398858213677}{235696447467520000} f_{n+\frac{7}{4}} h^2
\end{aligned} \tag{13}
$$

$$
\begin{aligned}
y_{n+\frac{9}{4}} =& \frac{125496099346445}{28774418631106756608} f_{n+\frac{17}{4}} h^2 - \frac{22242827337065459}{1106708408888721408000} f_{n+4} h^2 \\
&+ \frac{24330688528648043}{345846377777725440000} f_{n+\frac{15}{4}} h^2 + \frac{5451739990287741 8083}{179840116444417228 8000} f_{n+\frac{3}{3}} h^2 \\
&- \frac{3847881787300223}{19762650158727168000} f_{n+\frac{7}{2}} h^2 + \frac{71736124038148249829}{449600291111043072000 0} f_{n+\frac{5}{4}} h^2 \\
&- \frac{11235272280913565597}{490473044848106240000} f_{n+\frac{5}{2}} h^2 - \frac{119306257673178 3283}{1438720931555337830400} f_{n+3} h^2 \\
&- \frac{228585191225199821}{35968023288883445760 00} f_{n+\frac{3}{4}} h^2 - \frac{4125778666895021}{1079040698666503372800 0} f_{n+\frac{1}{4}} h^2 - \frac{1}{4} y_{n+1} \\
&+ \frac{37900659118874329758 1}{719360465777668915200 0} f_{n+2} h^2 + \frac{19119812966289839677}{143872093155533783040 00} f_{n+1} h^2 + \frac{5}{4} y_{n+2} \qquad (14) \\
&+ \frac{46200587142858211229}{539520349333251686400 0} f_{n+\frac{9}{4}} h^2 + \frac{7417436577967206671}{539520349333251686400 0} f_{n+\frac{11}{4}} h^2 \\
&+ \frac{23751900406562011}{431616279466601349120 0} f_{n+\frac{1}{2}} h^2 + \frac{33540745064777}{237151801904726016000 0} h^2 f_n \\
&- \frac{671783799135313}{21580813973330067456000 0} f_{n+5} h^2 + \frac{394657047914284733}{899200582222086144000} f_{n+\frac{13}{4}} h^2 \\
&- \frac{14567435492622409}{21580813973330067456000} f_{n+\frac{9}{2}} h^2 + \frac{715991121863761}{1079040698666503372800 0} f_{n+\frac{19}{4}} h^2 \\
&+ \frac{1256482064736044941}{25691445206345318400} f_{n+\frac{7}{4}} h^2
\end{aligned}
$$

$$
\begin{aligned}
y_{n+\frac{5}{2}} =& \frac{1099903106149}{121961884524480000} f_{n+\frac{17}{4}} h^2 - \frac{241436706700151}{5854170457175040000} f_{n+4} h^2 \\
&+ \frac{65244384273559}{457357066966800000} f_{n+\frac{15}{4}} h^2 + \frac{445279857934466 99}{731771307146880000} f_{n+\frac{3}{2}} h^2 \\
&- \frac{7262060875939}{18763366849920000} f_{n+\frac{7}{2}} h^2 + \frac{693792519696379}{21778907950800000} f_{n+\frac{5}{4}} h^2 \\
&+ \frac{5817572865585307}{199573992858240000 0} f_{n+\frac{5}{2}} h^2 - \frac{611603376182389}{418155032655360000} f_{n+3} h^2 \\
&- \frac{4729597666273}{365885653573440000} f_{n+\frac{3}{4}} h^2 - \frac{8668467399467}{1097656960720320000} f_{n+\frac{1}{4}} h^2 - \frac{1}{2} y_{n+1} \\
&+ \frac{110142738780205039}{975695076195840000} f_{n+2} h^2 + \frac{15630496619378569}{5854170457175040000} f_{n+1} h^2 + \frac{3}{2} y_{n+2} \qquad (15) \\
&+ \frac{36265962938945347}{548828480360160000} f_{n+\frac{9}{4}} h^2 + \frac{113571421928683}{60980942262240000} f_{n+\frac{11}{4}} h^2 \\
&+ \frac{216056435093}{19131276003840000} f_{n+\frac{1}{2}} h^2 + \frac{11661191183}{397341886233600000} h^2 f_n \\
&- \frac{63537913573}{9756950761958400000} f_{n+5} h^2 + \frac{76895589028199}{91471413393360000} f_{n+\frac{13}{4}} h^2 \\
&- \frac{12320873288153}{8781255685762560000} f_{n+\frac{9}{2}} h^2 + \frac{21707758339}{1568081372455760000} f_{n+\frac{19}{4}} h^2 \\
&+ \frac{8863592542506599}{91471413393360000} f_{n+\frac{7}{4}} h^2
\end{aligned}
$$

$$
\begin{aligned}
y_{n+\frac{11}{4}} = {} & \frac{12283587911210947}{856381506878177280000} f_{n+\frac{17}{4}} h^2 - \frac{25219017248173691}{380614003056967680000} f_{n+4} h^2 \\
& + \frac{2739379849377679}{11763482237337600000} f_{n+\frac{15}{4}} h^2 + \frac{3909997203713005489\,1}{428190753439088640000} f_{n+\frac{3}{2}} h^2 \\
& - \frac{39748328352396107}{61170107634155520000} f_{n+\frac{7}{2}} h^2 + \frac{730596069717716557 7}{152925269085388800000} f_{n+\frac{5}{4}} h^2 \\
& + \frac{2217717764835385608 1}{389264321308262400000 0} f_{n+\frac{5}{2}} h^2 - \frac{107218416713271763}{34954347219517440000} f_{n+3} h^2 \\
& - \frac{166401997578003703}{856381506878177280000} f_{n+\frac{3}{4}} h^2 - \frac{11199011629199}{9410785789870080000} f_{n+\frac{1}{4}} h^2 - \frac{3}{4} y_{n+1} \\
& + \frac{4211314235128162003 9}{244680430536622080000 0} f_{n+2} h^2 + \frac{4575514135769000542 7}{114184200917090304000 0} f_{n+1} h^2 + \frac{7}{4} y_{n+2} h^2 \qquad (16) \\
& + \frac{1685254053483349503 67}{128457226031726592000 0} f_{n+\frac{9}{4}} h^2 + \frac{1277813190194766055 7}{128457226031726569200 00} f_{n+\frac{11}{4}} h^2 \\
& + \frac{87343582328260103}{513828904126906368000 00} f_{n+\frac{1}{2}} h^2 + \frac{46419237197743}{104863041658552320000 0} h^2 f_n \\
& - \frac{523980848884093}{513828904126906368000 00} f_{n+5} h^2 + \frac{106801408577637853}{71365125573181440000} f_{n+\frac{13}{4}} h^2 \\
& - \frac{8759978771535309}{395253003174543360000} f_{n+\frac{9}{2}} h^2 + \frac{1401054498157}{6438958698332160000} f_{n+\frac{19}{4}} h^2 \\
& + \frac{156144801692197}{1075764264960000} f_{n+\frac{7}{4}} h^2
\end{aligned}
$$

$$
\begin{aligned}
y_{n+3} = {} & \frac{75074065811-}{4287722502813750} f_{n+\frac{17}{4}} h^2 - \frac{1358175628627}{17150890011255000} f_{n+4} h^2 + \frac{2874787291274}{10719306257034375} f_{n+\frac{15}{4}} h^2 \\
& + \frac{522165439891021}{4287722502813750} f_{n+\frac{3}{2}} h^2 - \frac{2965882385639}{4287722502813750} f_{n+\frac{7}{2}} h^2 \\
& + \frac{97538352622082}{1531329465290625} f_{n+\frac{5}{4}} h^2 + \frac{1386011651278363}{11693788644037500} f_{n+\frac{5}{2}} h^2 \\
& + \frac{6961017671029}{2450127144465000} f_{n+3} h^2 - \frac{1110762325159}{4287722502813750} f_{n+\frac{3}{4}} h^2 \\
& - \frac{20382802351}{12863167508441250} f_{n+\frac{1}{4}} h^2 - y_{n+1} + \frac{496387921673906}{2143861251406875} f_{n+2} h^2 \qquad (17) \\
& + \frac{2014097444041}{376942637610000} f_{n+1} h^2 + 2 y_{n+2} + \frac{1256844618618751}{6431583754220625} f_{n+\frac{9}{4}} h^2 \\
& + \frac{61543691500443}{918797679174375} f_{n+\frac{11}{4}} h^2 + \frac{1165440628523}{51452670033765000} f_{n+\frac{1}{2}} h^2 \\
& + \frac{15152490781}{257263350168825000} h^2 f_n - \frac{20423981}{1592961920550000} f_{n+5} h^2 \\
& + \frac{2656928768414}{2143861251406875} f_{n+\frac{13}{4}} h^2 - \frac{20155987771}{735038143395000} f_{n+\frac{9}{2}} h^2 \\
& + \frac{38410409}{141353489103750} f_{n+\frac{19}{4}} h^2 + \frac{31887180650978}{164912403954375} f_{n+\frac{7}{4}} h^2
\end{aligned}
$$

$$
\begin{aligned}
y_{n+\frac{13}{4}} = {} & \frac{351770307913783}{1332149010699386880000} f_{n+\frac{17}{4}} h^2 - \frac{2004879291362333591}{15985788128392642560000} f_{n+4} h^2 \\
& + \frac{33309168282118117}{71365125573181440000} f_{n+\frac{15}{4}} h^2 + \frac{304345952750763833099}{1998223516049080320000} f_{n+\frac{3}{2}} h^2 \\
& - \frac{101463359891403919}{666074505349693440000} f_{n+\frac{7}{2}} h^2 + \frac{13253726853437129083}{166518626337423360000} f_{n+\frac{5}{4}} h^2 \\
& + \frac{97858163975250102283}{544970049831567360000} f_{n+\frac{5}{2}} h^2 + \frac{4636992819279895121}{79928940641963212800} f_{n+3} h^2 \\
& - \frac{9999412440430429}{307419002469089280000} f_{n+\frac{3}{4}} h^2 - \frac{342849986257679}{172763013756554560000} f_{n+\frac{1}{4}} h^2 - \frac{5}{4} y_{n+1} \\
& + \frac{11081288401600137337}{380614003056967680000} f_{n+2} h^2 + \frac{1526873147582941099}{228368401834180608000} f_{n+1} h^2 + \frac{9}{4} y_{n+2} \\
& + \frac{155577981171601120441}{599467054814724096000} f_{n+\frac{9}{4}} h^2 + \frac{1247880116281403821}{9515350076424192000} f_{n+\frac{11}{4}} h^2 \\
& + \frac{2533887092655613}{888099340466257920000} f_{n+\frac{1}{2}} h^2 + \frac{94138623173953}{1262035904873103360000} h^2 f_n \\
& - \frac{48388695931897}{2664298021398773760000} f_{n+5} h^2 + \frac{827192074032094601}{99911175802454016000} f_{n+\frac{13}{4}} h^2 \\
& - \frac{1375581113640599}{342552602751270912000} f_{n+\frac{9}{2}} h^2 + \frac{467891309328109}{1198934109629448192000} f_{+\frac{19}{4}} h^2 \\
& + \frac{185564041470164839 1}{76854750617272320000} f_{n+\frac{7}{4}} h^2
\end{aligned}
\tag{18}
$$

$$
\begin{aligned}
y_{n+\frac{7}{2}} = {} & \frac{5346985885}{292708522858752} f_{n+\frac{17}{4}} h^2 - \frac{8285093914127}{13009267826112000} f_{n+4} h^2 + \frac{384079843}{16667531595000} f_{n+\frac{15}{4}} h^2 \\
& + \frac{26733789395890307}{146354261429376000} f_{n+\frac{3}{2}} h^2 + \frac{96727539273229}{20907751632768000} f_{n+\frac{7}{2}} h^2 \\
& + \frac{4370176639349827}{45357066966800000} f_{n+\frac{5}{4}} h^2 + \frac{31869625279029}{133049328572160000} f_{n+\frac{5}{2}} h^2 \\
& + \frac{13992661679101393}{117083409143500800} f_{n+3} h^2 - \frac{7091619042499}{18294282678672000} f_{n+\frac{3}{4}} h^2 \\
& - \frac{12339617713}{5226937908192000} f_{n+\frac{1}{4}} h^2 - \frac{3}{2} y_{n+1} + \frac{2927685913777307}{83631006531072000} f_{n+2} h^2 \\
& + \frac{3126535477148411}{390278030478336000} f_{n+1} h^2 + \frac{5}{2} y_{n+2} + \frac{17815748599121777}{54882848036016000} f_{n+\frac{9}{4}} h^2 \\
& + \frac{5371619907543899}{27441424018008000} f_{n+\frac{11}{4}} h^2 + \frac{11879045966119}{351250227130502400} f_{n+\frac{1}{2}} h^2 \\
& + \frac{219706728229}{2508930195932160000} h^2 f_n - \frac{14949112183}{924342703764480000} f_{n+5} h^2 \\
& + \frac{198604068117293}{304907113112000} f_{n+\frac{13}{4}} h^2 - \frac{5568588778061}{1756251137152512000} f_{n+\frac{9}{2}} h^2 \\
& + \frac{6067256953}{18294282678672000} f_{n+\frac{19}{4}} h^2 + \frac{8661152221}{29875045200} f_{n+\frac{7}{4}} h^2
\end{aligned}
\tag{19}
$$

$$\begin{aligned}
y_{n+\frac{15}{4}} =& \frac{17333982321064207}{233558592784957440000} f_{n+\frac{17}{4}} h^2 - \frac{462500316609274019}{934234371139829760000} f_{n+4} h^2 \\
&+ \frac{12582138269346823}{2014596090777600000} f_{n+\frac{15}{4}} h^2 + \frac{24930189845045358931}{116779296392478720000} f_{n+\frac{3}{2}} h^2 \\
&+ \frac{1025082015060458413}{16682756627496960000} f_{n+\frac{7}{2}} h^2 + \frac{4643884838645029277}{417068915687424000000} f_{n+\frac{5}{4}} h^2 \\
&+ \frac{96521717533951545643}{318488990161305600000} f_{n+\frac{5}{2}} h^2 + \frac{12252239930174748379}{66731026509987840000} f_{n+3} h^2 \\
&- \frac{107519707234076003}{233558592784957440000} f_{n+\frac{3}{4}} h^2 - \frac{2015460020925947}{70067577834872320000} f_{n+\frac{1}{4}} h^2 - \frac{7}{4} y_{n+1} \\
&+ \frac{19192353211225939 7953}{467117185569914880000} f_{n+2} h^2 + \frac{113852770745359613}{1213291391090688 0000} f_{n+1} h^2 + \frac{11}{4} y_{n+2} \\
&+ \frac{135536202300555975227}{350337889177436160000} f_{n+\frac{9}{4}} h^2 + \frac{12847068594568802471}{50048269882490880000} f_{n+\frac{11}{4}} h^2 \\
&+ \frac{57041215869140543}{1401351556709744640000} f_{n+\frac{1}{2}} h^2 + \frac{215961904334201}{2001930795299635200000} h^2 f_n \\
&- \frac{523980848884093}{14013515567097446400000} f_{n+5} h^2 + \frac{7405295321775261539}{583896481962393600000} f_{n+\frac{13}{4}} h^2 \\
&- \frac{1924530376807111}{200193079529963520000} f_{n+\frac{9}{2}} h^2 + \frac{85308744197449}{100096539764981760000} f_{n+\frac{19}{4}} h^2 \\
&+ \frac{216462057770376659}{6416444856729600000} f_{n+\frac{7}{4}} h^2
\end{aligned} \tag{20}$$

$$\begin{aligned}
y_{n+4} =& -\frac{106734910393}{47641361142750} f_{n+\frac{17}{4}} h^2 + \frac{59283388297741}{11433926674170000} f_{n+4} h^2 + \frac{229506014312374}{3573102085678125} f_{n+\frac{15}{4}} h^2 \\
&+ \frac{3471216657798801}{1429240834271250} f_{n+\frac{3}{2}} h^2 + \frac{8194185319361}{68059087346250} f_{n+\frac{7}{2}} h^2 \\
&+ \frac{152045747111158}{1191034028559375} f_{n+\frac{5}{4}} h^2 + \frac{1386011651278363}{3897929548012500} f_{n+\frac{5}{2}} h^2 \\
&+ \frac{678092936797609}{2858481668542500} f_{n+3} h^2 - \frac{715483528169}{1429240834271250} f_{n+\frac{3}{4}} h^2 \\
&- \frac{12423419161}{4287722502813750} f_{n+\frac{1}{4}} h^2 - 2 y_{n+1} + \frac{887881541164411}{1905654445695000} f_{n+2} h^2 \\
&+ \frac{7580195490796}{714620417135625} f_{n+1} h^2 + 3 y_{n+2} + \frac{981502273762601}{2143861251406875} f_{n+\frac{9}{4}} h^2 \\
&+ \frac{78462214322939}{238206805711875} f_{n+\frac{11}{4}} h^2 + \frac{27034436329}{63518148565000} f_{n+\frac{1}{2}} h^2 \\
&+ \frac{2572048441}{24501271444650000} h^2 f_n + \frac{158436017}{4764136114237500} f_{n+5} h^2 \\
&+ \frac{139949068666514}{714620417135625} f_{n+\frac{13}{4}} h^2 + \frac{17318760779}{1008875883015000} f_{n+\frac{9}{2}} h^2 \\
&- \frac{4464035971}{4287722502813750} f_{n+\frac{19}{4}} h^2 + \frac{39605886937802}{102088631019375} f_{n+\frac{7}{4}} h^2
\end{aligned} \tag{21}$$

$$
\begin{aligned}
y_{n+\frac{17}{4}} = {} & \frac{774571162593347857}{15370950123454464000} f_{n+\frac{17}{4}} h^2 + \frac{4417474892998216129}{6831533388201984000} f_{n+4} h^2 + \\
& \frac{229504790866667506039}{19213687654318080000} f_{n+\frac{15}{4}} h^2 + \frac{2134766931159931141 1}{7685475061727232000} f_{n+\frac{3}{2}} h^2 + \\
& \frac{151676797828404708 91}{7685475061727232000} f_{n+\frac{7}{2}} h^2 + \frac{389633999227126162 7}{274481252204544000 00} f_{n+\frac{5}{4}} h^2 + \\
& \frac{3138344032679709528 1}{6986795510661120000 0} f_{n+\frac{5}{2}} h^2 + \frac{1447446526675290059 9}{4391700035272704000 0} f_{n+3} h^2 - \\
& \frac{102687910351364293}{15370950123454464000} f_{n+\frac{3}{4}} h^2 - \frac{748690023492119}{15370950123454464000} f_{n+\frac{1}{4}} h^2 - \frac{9}{4} y_{n+1} + \\
& \frac{16687406674050854977 3}{30741900246908928000 0} f_{n+2} h^2 + \frac{254401327318605388 7}{20494600164605952000 0} f_{n+1} h^2 + \frac{13}{4} y_{n+2} + \\
& \frac{11436448466574761163 7}{23056425185181696000 0} f_{n+\frac{9}{4}} h^2 + \frac{8269507462007313880 7}{2306425185181696000 0} f_{n+\frac{11}{4}} h^2 + \\
& \frac{59209456659659063}{922257007407267840000} f_{n+\frac{1}{2}} h^2 + \frac{94138623173953}{485398424951193600000} h^2 f_n - \\
& \frac{3513094929934993}{9222570074072678400000} f_{n+5} h^2 + \frac{3018771928556295463}{128091251028787200000} f_{n+\frac{13}{4}} h^2 - \\
& \frac{173455776040871897}{922257007407267840000} f_{n+\frac{9}{2}} h^2 + \frac{241458987159493}{21958500176363520000} f_{n+\frac{19}{4}} h^2 + \\
& \frac{7493045477761357}{1757081632768000 0} f_{n+\frac{7}{4}} h^2
\end{aligned} \tag{22}
$$

$$
\begin{aligned}
y_{n+\frac{9}{2}} = {} & \frac{2101301736049189}{31361627449152000} f_{n+\frac{17}{4}} h^2 + \frac{55910866717133521}{501786039186432000} f_{n+4} h^2 + \frac{1226344659659087}{5600290615920000} f_{n+\frac{15}{4}} h^2 \\
& + \frac{17834369192018131}{62723254898304000} f_{n+\frac{3}{2}} h^2 + \frac{1691392461226981}{896046485472000} f_{n+\frac{7}{2}} h^2 \\
& + \frac{71810862288479}{430791585840000} f_{n+\frac{5}{4}} h^2 + \frac{57996477252085627}{171063422449920000} f_{n+\frac{5}{2}} h^2 \\
& + \frac{1208540142088501}{5120265705984000} f_{n+3} h^2 - \frac{315313098391}{1844801614656000} f_{n+\frac{3}{4}} h^2 \\
& + \frac{46088731681}{13440697478208000} f_{n+\frac{1}{4}} h^2 - \frac{5}{2} y_{n+1} + \frac{18242092490671459}{35841859941888000} f_{n+2} h^2 \\
& + \frac{431921157879109}{3859892609126400 0} f_{n+1} h^2 + \frac{7}{2} y_{n+2} + \frac{32862779020894001}{47042441173728000} f_{n+\frac{9}{4}} h^2 \\
& + \frac{28227243649922165 3}{47042441173728000} f_{n+\frac{11}{4}} h^2 - \frac{13491078122281}{752679058779648000} f_{n+\frac{1}{2}} h^2 \\
& - \frac{31016668933}{153607971179520000} h^2 f_n + \frac{22235591198363}{7526790587796480000} f_{n+5} h^2 \\
& + \frac{3237445049576261}{7840406862288000} f_{n+\frac{13}{4}} h^2 + \frac{3404624204950583}{752679058779648000} f_{n+\frac{9}{2}} h^2 \\
& - \frac{84736964779}{790629263424000} f_{n+\frac{19}{4}} h^2 + \frac{590070567002159}{1120058123184000} f_{n+\frac{7}{4}} h^2
\end{aligned} \tag{23}
$$

$$
\begin{aligned}
y_{n+\frac{19}{4}} &= \frac{27265500551439965}{290650693243502592} f_{n+\frac{17}{4}} h^2 + \frac{9349741521657168481}{33536618451173376000} f_{n+4} h^2 + \\
&\frac{4020057468845315869}{136242512457891840000} f_{n+\frac{15}{4}} h^2 + \frac{27510190765251123803}{544970049831567360000} f_{n+\frac{3}{2}} h^2 + \\
&\frac{13688634311704882529}{181656683277189120000} f_{n+\frac{7}{2}} h^2 + \frac{55951297934031943}{499056822190080000} f_{n+\frac{5}{4}} h^2 + \frac{254998647248460113923}{148628195408609280000} f_{n+\frac{5}{2}} h^2 \\
&+ \frac{9072052481944535071}{6228229140932198400} f_{n+3} h^2 - \frac{519258865004497711}{108994009966313472000} f_{n+\frac{3}{4}} h^2 \\
&- \frac{22068392467485911}{326982029898940416000} f_{n+\frac{1}{4}} h^2 - \frac{11}{4} y_{n+1} + \frac{94455886236101171707}{72662673310875648000} f_{n+2} h^2 \\
&+ \frac{2059738285640783551}{62282291409321984000} f_{n+1} h^2 + \frac{15}{4} y_{n+2} - \frac{50953913929741617161}{163491014949470208000} f_{n+\frac{9}{4}} h^2 \\
&- \frac{161551311164041023}{235917770489856000} f_{n+\frac{11}{4}} h^2 + \frac{3379185642297713}{4844178229725043200} f_{n+\frac{1}{2}} h^2 \\
&+ \frac{1873614930165917}{594512781634437120000} h^2 f_n - \frac{3283996074259637}{726626733108756480000} f_{n+5} h^2 \\
&- \frac{9774130295446717297}{27248502491578368000} f_{n+\frac{13}{4}} h^2 + \frac{6625704725934894433}{93423437113982976000} f_{n+\frac{9}{2}} h^2 \\
&+ \frac{9597933044690047}{2458511502999552000} f_{n+\frac{19}{4}} h^2 + \frac{68288528065533109}{419207730639667200} f_{n+\frac{7}{4}} h^2
\end{aligned} \tag{24}
$$

Evaluate the first derivative of (5) at t=0 (1/4) 5 and combine with equations (6-24) in order to simultaneously solve second order DDEs.

3. Basic Properties of the Block Methods

3.1. Order and Error Constant of the Block Method

Let the linear Operator defined on the method be $\varsigma[y(x);h]$, where

$$\Delta[y(x);h] = A^{(o)} Y_m^{(i)} - \sum_{i=0}^{k} \frac{jh}{i} y_n^{'} - h^{(2-i)}[d_i f(y_n) + b_i F(y_m)], \tag{25}$$

Expanding the form Y_m and $F(y_m)$ in Taylor Series and comparing coefficients of h, we obtained

$$\Delta[y(x);h] = C_0 y(x) + C_1 h y'(x) + \ldots \; C_p h^p y^p(x) + C_{p+1} h^{p+1} y^{p+1}(x) + \; C_{p+2} h^{p+2} y^{p+2}(x) + \ldots \tag{26}$$

Theorem 1: The linear operator and the associated block method are said to be of order p if $C_0 = C_1 = \ldots C_p = C_{p+1} = 0, C_{p+2} \neq 0 \;\; C_{p+2}$ is called the error constant. It implies that the local truncation error is given by

$$T_{n+k} = C_{p+2} h^{p+2} y^{p+2}(x) + O(h^{p+3}) \tag{27}$$

Expanding the block method (25) in Taylor Series expansion and comparing the coefficients of h, the order of the block is of order $[21,21,21,21,21,21,21,21,21,21,21,21,21,21,21,21,21,21,21,21,21]^T$ with error constant

$$C_{p+2} = \begin{pmatrix} 2.17X10^{-17}, 5.44X10^{-17}, 8.63X10^{-17}, 1.18X10^{-16}, 1.50X10^{-16}, \\ 1.82X10^{-16}, 2.14X10^{-16}, 2.46X10^{-16}, 2.78X10^{-16}, 3.10X10^{-16}, \\ 9.71X10^{-7}, 3.74X10^{-16}, 1.26X10^{-6}, 4.38X10^{-16}, 4.70X10^{-16}, \\ 5.02X10^{-16}, 5.34X10^{-16}, 5.66X10^{-16}, 5.99X10^{-16}, 6.21X10^{-16}, \\ 0 \end{pmatrix}^T$$

3.2. Consistency

Here, the developed method has been examined and found to have order p greater than one and it is also convergence Awoyemi and Kayode [1], Familua et al [6]. Hence, the method satisfies the necessary and sufficient conditions for consistency of a numerical method

3.3. Stability Domain of the Block Methods

Using the approach in Ibijola et al. [18], the region of absolute stability of the five–step fifteen off step points block method is shown below:

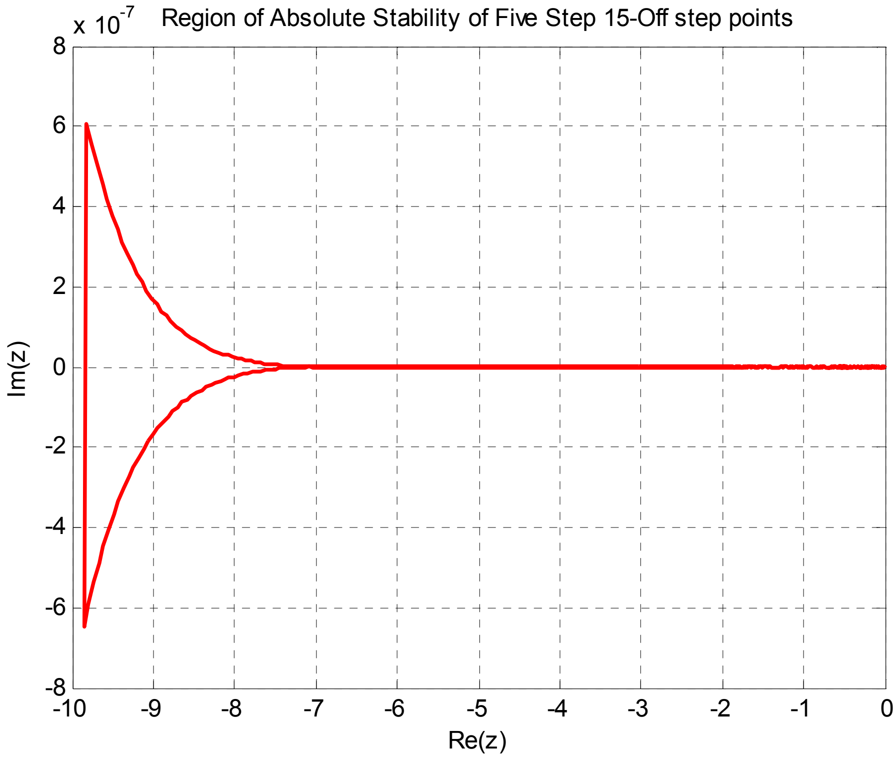

Figure 1. *The Stability domain of Five-step fifteen off-step points.*

4. Implementation of Numerical Examples

The developed methods were adopted on some delay differential equation of special second order to assess the accuracy and efficiency of the methods.

Problem1: Consider the linear delay equation

$$y''(x) = y\left(x-\pi\right),\ x\varepsilon\left[0,\pi\right],$$
$$y(x) = \sin(x),\ y'(x) = -e^{-x},\ -\pi \le x \le 0$$

Exact solution: $y(x) = \sin(x)$

Source: San *et al.* [10]

Problem2: Consider the linear delay equation

$$y''(x) = -\frac{1}{2}y\left(x\right)+\frac{1}{2}y\left(x-\pi\right),\ x\varepsilon\left[0,\pi\right],$$
$$y(x) = 1-\sin(x),\ y'(x) = -e^{-x},\ -\pi \le x \le 0$$

Exact solution: $y(x) = 1-\sin(x)$

Source: Familua *et al.* [6]

Problem 3: Application to Matheiu's Equation, in this section we apply our developed method to solve a well-known equation in engineering, the Matheiu's equation, which defined as follows:

$$y''(t) + (\delta + a\cos t)y(t) + cy^3(t) = by(t - T) \tag{28}$$

Source: Morisson and Rand [9]

which is a nonlinear delay differential equation. where δ, a, b, c and T are parameters. δ is the frequency squared of the simple harmonic oscillator, and a is the amplitude of the parametric resonance, and b is the amplitude of delay which c is the amplitude of the cubic nonlinearity and T is the time delay. Equation (28) is a model for high speed milling, a kind of parametrically interrupted cutting as opposed to the self-interrupted cutting arising in an unstable turning process.

According to Morisson and Rand [9], various special cases of (28) have been studied, depending on which parameters is zero. when δ=a=b=1 and c=0 we obtained the following Linear Matheiu's equation:

$$y''(t) = (1+\cos t)y(t) = y(t-T);\ tE[0,10],\ y(t) = \sin(t),\ y'(t) = \cos(t),\ t < 0 \tag{29}$$

where T=τ=h/10 is the delay term, the exact solution does not exist.

when δ=a=b=c=1 we obtained the following Nonlinear Matheiu's equation:

$$y''(t) = (1+\cos t)y(t) + y^3(t) = y(t-T);\ y(t) = \sin(t),\ y'(t) = \cos(t),\ t < 0 \tag{30}$$

where T=τ=h/10 is the delay term, the exact solution does not exist. Both the linear and nonlinear Matheiu's equations are solved using the developed method and the results are presented in Table 3.

Table 1. *Comparison of the new method with the study of Familua A. B. et al. [6] for Problem 1 using h=0.1.*

X	y-exact	y-computed	Error in new method	Error in the study [6]
0.1	0.900166583353171790	0.900166583353171680	1.11022302e-016	3.33066907e-16
0.2	0.801330669204938780	0.801330669204938340	3.33066907e-016	1.11022302e-16
0.3	0.704479793338660400	0.704479793338660290	1.11022302e-016	2.22044605e-16
0.4	0.610581657691349420	0.610581657691349760	5.55111512e-016	0.00000000e+00
0.5	0.520574461395796990	0.520574461395797440	6.66133815e-016	3.33066907e-16
0.6	0.435357526604964520	0.435357526604965070	7.77156117e-016	2.77555756e-16
0.7	0.355782312762308870	0.355782312762309480	9.43689571e-016	2.22044605e-16
0.8	0.282643909100477210	0.282643909100477760	8.88178420e-016	2.22044605e-16
0.9	0.216673090372516470	0.216673090372517200	1.05471187e-016	3.60822483e-16
1.0	0.158529015192103380	0.158529015192103970	8.04911693e-016	3.33066907e-16

Table 2. *Comparison of the new method with the study of Familua A. B. et al. [6] for Problem 2 using h=0.1.*

X	y-exact	y-computed	Error in new method	Error in the study [6]
0.1	0.099833416646828155	0.099833416646828183	2.77555756e-017	0.00000000e+00
0.2	0.198669330795061220	0.198669330795061130	8.32667268e-017	0.00000000e+00
0.3	0.295520206661339600	0.295520206661339440	1.66533454e-016	0.00000000e+00
0.4	0.389418342308650520	0.389418342308650300	1.66533454e-016	1.11022302e-16
0.5	0.479425538604203010	0.479425538604202560	3.88578059e-016	1.11022302e-16
0.6	0.564642473395035480	0.564642473395035040	3.33066907e-016	0.00000000e+00
0.7	0.644217687237691130	0.644217687237690910	2.22044605e-016	1.11022302e-16
0.8	0.717356090899522790	0.717356090899523010	1.11022302e-016	0.00000000e+00
0.9	0.783326909627483530	0.783326909627483750	2.22044605e-016	0.00000000e+00
1.0	0.841470984807896620	0.841470984807896500	1.11022302e-016	0.00000000e+00

Table 3. *Showing comparison of efficiency of computed results of application problem for Linear and Non linear using h=0.1.*

T	y-computed (Linear problem)	Time	y-computed (Nonlinear problem)	Time
0.1	0.110457480638154760	0.0147	0.237017408334463720	0.0071
0.2	0.212079025783384970	0.0162	0.485358968753846210	0.0079
0.3	0.303638151151402300	0.0176	0.734221665717452070	0.0087
0.4	0.385231777057725430	0.0190	0.984274632632406220	0.0094
0.5	0.458305347148437510	0.0247	1.247690581618721200	0.0102
0.6	0.681323968356713270	0.0259	1.997606110018203700	0.0115
0.7	0.909629165272295200	0.0266	2.804809205657486700	0.0122
0.8	1.129282602498707000	0.0273	3.618927969308167300	0.0129
0.9	1.340951441727805200	0.0280	4.442196732569305700	0.0184
1.0	1.559903778423402200	0.0287	5.329383181271261100	0.0191

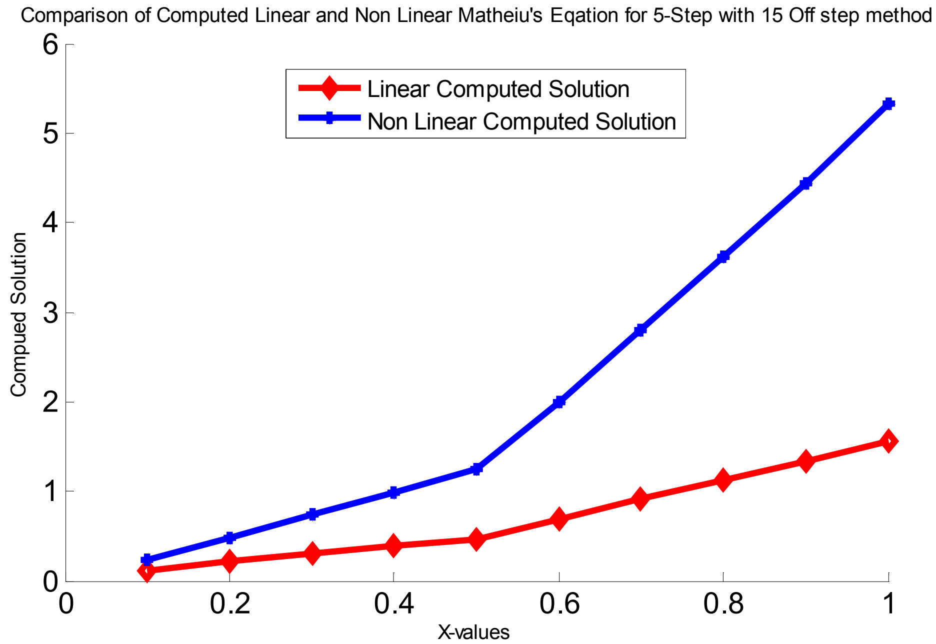

Figure 2. *The efficiency curve for linear and nonlinear in Problem 3 with h=0.1.*

5. Discussion of Results and Conclusion

In this paper, we have derived a new numerical method of order 21 capable of solving special second order DDEs. The method is based on the combination of power series and exponential function; we constructed the method and implemented it such that it can solve special second order DDEs directly. We also examined the basic properties of the method. Numerical results illustrate that the new method is more efficient and converge faster when compared with other methods in the literature. This can be seen in Table 1 – Table 2. We presented the numerical results of problem 3, which is the application problem; the linear and non linear results were compared together in table 3. Figure 1 and figure 2 show the region of absolute stability of the method and the efficiency curve of problem 3 respectively. In conclusion it can be said that, the new method can efficiently solve the special second-order DDEs in terms of accuracy and number of function evaluations.

References

[1] Awoyemi, D. O., and Kayode, S. J. (2005): A maximal order collocation Method for initial value problems of General Second order ordinary differential equation, Proceedings of the Conference organized by the National Mathematical Centre, Abuja, Nigeria.

[2] Evans D. J., and Raslan, K. R., (2005): The Adomian decomposition method for solving delay differential equations. International journal of computer mathematics, vol. 82, no 1, pp 49–54.

[3] Fatunla, S. O (1991): Block method for second order ordinary differential equation. International Journal of computer mathematics, 41 (1 & 2), 55-63.

[4] Kayode S. J. and Obarhua F. O. (2013), Continuous y-function Hybrid Methods for Direct Solutions of Differential equations, International Journal of Differential Equations and Application 12 (1), 37–48.

[5] Kuang Y. (1993): Delay differential equations with applications in population dynamics, vol. 191 of Mathematics in Science and Engineering, Academics Press, Boston, Mass, USA.

[6] Familua A. B., Areo E. A., Olabode B. T., Owolabi M. K. (2019): A class of numerical integrators of order 13 for solving special second order delay differential equations, International Journal of Physics and Mathematics, 1 (2), pp. 01-17.

[7] Zanariah A. M and Hoo Y. S. (2013): Direct method for solving second order delay difference equation, proceedings of the 20th national symposium on mathematical sciences, AIP Conference Preceding. 676-680.

[8] Mechee M., Ismail F., Senu N., and Siri Z., (2013): Directly solving special second order delay differential equations using Runge-kutta-Nystrom method. Journal of Mathematical Problems in Engineering, Article ID 830317, page 1-9.

[9] Morisson T. M. and Rand R. H, (2007): Resonance in the delay nonlinear Mathieu equation, "Nonlinear Dynamics: An international Journal of Nonlinear Dynamics and chaos in Engineering Systems, vol. 50, no 1-2, pp. 341-352.

[10] San, H. C., Majid Z. A., and Othman M. (2011): Solving delay differential equations using coupled block method. In preceding of the 4thk international conference on Modelling, Simulation and Applied Optimization (ICMSAO 11).

[11] Shampine, L. F &Thompson S. T. (2001): Numerical Solution of Delay Differential Equations, Application of Numerical Mathematics, 37, pp 441-458.

[12] Skip Thompson (2007): Delay differential equations-(Scholarpedia, 2 (3): 2367): doi: 10.4249/scholarpediahttp://www.scholarpedia.org/org/article/ Delay-differential_equations.

[13] Suleiman M. B., and Ishak (2010): Numerical solution and stability of multistep method for solving delay differential equations: Japan Journal of Industrial and applied mathematics, vol. 27, pp. 395–410.

[14] Smith H. (2011): An Introduction to delay differential equations with applications to the life sciences, vol. 57, Texts in Applied Mathematics, Springer, New York.

[15] Mohammed S. Mechee, F. Ismail, Z. Siri and N. Senu (2014). A third order direct integrators of Rungekutta type for special third order ordinary and delay differential equations. Asian Journal of Applied Sciences 7 (3): 102–116, ISSN 1996-334/DOI: 10.3923/ajaps.

[16] Baker C. T. H, Paul, C. A. H., and Wille (1994): Issues in the Numerical solution of Evolutionary Delay Differential Equation, Numerical Analysis Report No. 248, University of Manchester, United Kingdom.

[17] Ibijola E. A, Skywame Y, Kumleng G. (2011): Formulation of the continuous multistep collocation. Americal Journal of Scientific and Industrial Research, 2: 161-173.

[18] Ogunfiditimi F. O. (2015), Numerical solution of Delay Differential Equations using Adomain Decomposition method (ADM) The International Journal Of Engineering And Science (IJES). 4 (5): 18-23.

[19] Areo E. A and Omole E. O. (2015): HALF-Step symmetric continuous hybrid block method for the numerical solutions of fourth order ordinary differential equations: Archives of Applied Science Research, 7 (10): 39-49. www.scholarsresearchlibrary.com.

[20] Kayode S. J, Ige S. O, Obarhua F. O and Omole E. O. (2018): An Order Six Stormer- cowell- type Method for Solving Directly Higher Order Ordinary Differential Equations. Asian Research Journal of Mathematics, 11 (3): 1-12. DOI: 10.9734/ARJOM/2018/44676.