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Numerical Simulation of Ocean Currents with Hermite IFinite Elements

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Abstract: The article deals with the numerical simulations for equations of geophysical fluids. These physical phenomena are modeled by the Navier-Stokes equations which describe the motion of the fluid, the ocean currents, the flow of water in a pipe and many other fluid flow phenomenon.These equations are very useful because of there utility. The Navier-Stokes equations for incompressible flow are nonlinear partial differential equations that drive the motion of fluids in the approximation of continuous media. The existence of general solutions to the Navier Stokes equations have already proven but in this paper we have interested to the numerical solution of the incompressible Navier-Stokes equations. We get an optimal discretization of the Navier-Stokes equations and numerical approximations of the solution are also given. The convergence and the stabilityof the approximated system are proven. The numerical resolution is based on Hermite finite elements. The numerical system was expressed in matrix form for computation of velocity and the pression fieldsapproach using MATLAB software. Numerical results for velocity field in two dimensional space of the velocity $\mathbf{u}(x, y)$ and pression $p(x, y,)$ are given. And finally we give physical interpretation of the results obtained.

Keywords: Numerical Simulation, Geophysical Models, Navier-Stokes Equation, Stokes Equations, Hermite Finite Elements, Numerical Simulations

1. Introduction

The solution of partial differential equations using finite elements is an efficient approach since it is applicable to all types of problems. This method consists in replacing the space of infinite dimension by a space of finite dimension where the resolution will be made. The step which allows the implementation of this procedure is finite elements discretization.

The Navier-Stokes equations constitute a remarkable class of partial differential equations. These equations are very useful because of their utility in the fields or meteorology, transport and current modeling and on the other hand by the complexity of their resolution. The Navier-Stokes equations have always been the subject of several studies [16, 17, 22, 23]. Many researchers have conducted large studies on currents going in the direction of their origin to their dynamics, we can cite among others M. Allgeyer et al. [1] and R. Ben Hamouda [13].

The modeling of the currents leads to the resolution of partial differential equations (of the Navier -Stokes) supported by the studies of M. Andrea Doglioli [7], J. L. Lions [17], T. Kheladi [15], H. Fujita and T. Kato [14].

The finite element method [9, 10, 11, 12, 21] constitutes a wide range for modeling and then solving Navier-Stokes equations. This method is frequently used to solve this type of problem. Rectangular mixed finite elements was used by F. Brezzi et al. [5], Fortin [10] uses finite element of mixed type. Araya et al [2, 3] used finite elements stabilized with the wall low to solve the incompressible Navier-Stokes equations.

Despite the varied approaches that they have adopted, their common point lies in the fact that the order of derivability of the solution has not been specified. The Hermite finite elements constitutes a means to get solutions of high order of derivability.

In this paper, we are worked on well-posed Navier-Stokes models in the stationary case in dimension 2. They are valid models for dynamical of oceanic surface waters. We consider universal hypotheses in fluid mechanics to arrive at the following equation

$$\begin{cases} (u\cdot\nabla)u+\frac{1}{\rho_0}\nabla p-\nu\triangle u=f \text{ in } \Omega \\ \nabla\cdot u=0 \text{ in } \Omega \end{cases} \tag{1}$$

where u, ρ_0, p and ν are respectively speed, density, pressure and viscosity for particles of the fluid under consideration. This system of non-linear partial differential equations is a relevant model for dynamics of ocean currents, where $\Omega\subset\mathbb{R}^3$ is a domain of class C^2 with boundary Γ

In 1755, Euler, Swiss mathematician, built equations for a perfect fluid which are used until today. These equations are known as of Euler equations. After, D'Alembert finds a paradoxes to these equations. Therefore, It leaves it to future researchers to remove this paradox.

Between 1820 and 1845, Claude Louis Marie Henry (1785-1836) and George Gabriel Stokes(1819-1903), French and Irish mathematicians establish the Navier equations-Stokes for a viscous fluid. These are nonlinear partial differential equations. They add to Euler's equations a term that dissipates energy and eliminates Alembert paradox. In the literature, there exists, according to the properties of the fluids studied, various types of Navier-Stokes equations. In general, they are written following as

$$\begin{cases} \rho\Big(\frac{\partial}{\partial t}+u\nabla\Big)u-\nu\triangle u=\rho g-\nabla p+f,\ (t,x)\in[0,T)\times\Omega \\ u(0,x)=u_0(x),\ x\in\Omega, \\ \nabla\cdot u=0;\ x\in\Omega. \end{cases} \tag{2}$$

The presence of the coefficient viscous ν shows that the fluid's model is not perfect. Unfortunately, we have the difficulty of obtaining exact analytical solutions. Given the complexity of Navier Stokes equations, several approaches are made in order to model the flow to study.

In the following and according to the hydrostatic hypotheses, we will study our model (1) in two dimensions. Thus, the velocity vector u is written as follows $u^T=\Big(u_1,u_2\Big)$. With homogeneous boundary conditions, the problem (1) becomes

$$\begin{cases} (u\cdot\nabla)u+\frac{1}{\rho_0}\nabla p-\nu\triangle u=f \text{ in } \Omega \\ \nabla\cdot u=0 \text{ in } \Omega \\ u=0 \text{ on } \Gamma \\ \nabla u\cdot n=0 \text{ on } \Gamma, \end{cases} \tag{3}$$

where Γ is the boundary of the domain Ω. The first equation of the system (3) is a vector equation. We get the following system of differential equations for direction j :

$$\begin{cases} (u_j\cdot\nabla)u_j+\frac{1}{\rho_0}\nabla p-\nu\triangle u_j=f_j \text{ in } \Omega \\ \nabla\cdot u_j=0 \text{ in } \Omega \\ u_j=0 \text{ on } \Gamma \\ \nabla u_j\cdot n=0 \text{ on } \Gamma,\ \ j=1,2, \end{cases} \tag{4}$$

where u_1 and u_2 are solutions (4).

In the following we will be interested in the resolution by the Hermite finite elements study of the model illustrated by the system(3).

The aim of this article is to:

1. Study the model, i.e. justify the origin and existence of solutions.
2. Proceed to the discretisation of the problem by the finite element of Hermite in dimension 2 (3) and evaluate its convergence.
3. Proceed to the digital implementation of the model. This approach allows to give the approximate solutions of problem (3).

The application of these different approaches allowed to have the following results.

1. Design a model whose existence and uniqueness is proven in a very particular space. Indeed this space is such that the conditions of Dirichlet and Neuman are implemented there. The model thus discretized by Hermit finite lements is convergent.
2. The digital implementation of the model by Matlab software allowed to have the following results:

A mesh with the finite elements of Hermite with a rather particular number of degrees of freedom. Whith each geometrical element one associated 10 degrees of freedom due to the shape functions and their derivates on each node.

One also has to proceed to the numerical resolution of the model, this is achieved through various tests carried out. Thus the velocity and the pressure are determined then represented in 3D.

2. Models and Existence Results

Let us choose a metric system which allows to write the Navier-Stokes equations according to a single parameter taking into account all the characteristics of the flow (the speed of entry, the size of the domain, the density and the viscosity). A characteristic speed U and a characteristic size L of the domain are chosen for this. It is preferable to choose these two values in the same place of the domain. By dividing the speed by U and the sizes by L we have:

$$u^*=\frac{u}{U},x^*=\frac{x}{L},y^*=\frac{y}{L},p^*=\frac{p}{\rho U^2},$$

We have after calculations and considering that the operators gradient, divergence and Laplacian are now defined with respect to the dimensionless coordinates:

$$\begin{cases} (u^* \cdot \nabla)u^* - \frac{1}{Re}\triangle u^* = -\nabla p^* + f^*, \ \Omega \\ \nabla \cdot u^* = 0, \ \Omega \\ u^* = 0, \ \Gamma \\ \nabla u^* \cdot n = 0 \end{cases} \quad (5)$$

where $Re = \frac{UL}{\nu}$ is Reynolds number.

This number measures the ratio of inertia forces to viscosity forces. We note that our system only depends on the Reynolds number. By ordering the dimensioning calculations differently and we can arrive at the following system

$$\begin{cases} (u^* \cdot \nabla)u^* - \frac{1}{Re}\triangle u^* = -\nabla p^{**} + f^{**}, \ \Omega \\ \nabla \cdot u^* = 0, \ \Omega \\ u^* = 0, \ \Gamma \\ \nabla u^* \cdot n = 0 \end{cases} \quad (6)$$

where

$$p^{**} = LU\rho p^* = \nu Rep^* \quad , f^{**} = \nu Ref^*.$$

or,

$$\begin{cases} (u^* \cdot \nabla)u^* - \frac{1}{Re}\triangle u^* = \frac{1}{\nu Re}(-\nabla p^* + f^*), \ \Omega \\ \nabla \cdot u^* = 0, \ \Omega \\ u^* = 0, \ \Gamma \\ \nabla u^* \cdot n = 0, \quad \Gamma \end{cases} \quad (7)$$

Then the writing becomes simpler

$$\begin{cases} \nu Re(u \cdot \nabla)u - \nu\triangle u = -\nabla p + f, \ \Omega \\ \nabla \cdot u = 0, \ \Omega \\ u = 0, \ \Gamma \\ \nabla u \cdot n = 0, \quad \Gamma \end{cases} \quad (8)$$

In the following we will set $c_o = \nu Re$, which will simplify the calculations.

Further in this section, we will give the main theorem for existence and uniqueness solution in (8). Let :

$$L_0^2(\Omega) = \left\{ v \in L^2(\Omega); \int_\Omega v = 0 \right\} \quad (9)$$

and

$$V = (H_0^1(\Omega))^2 \cap (H_0^2(\Omega))^2 \quad (10)$$

It should be noted that the spaces $H_0^1(\Omega)$ and $H_0^2(\Omega)$ are defined as follows.

$$H_0^1(\Omega) = \left\{ u \in H^1(\Omega), u = 0 \text{ on } \Gamma \right\}$$

$$H_0^2(\Omega) = \left\{ v \in L^2(\Omega); \gamma_0 v = 0 \text{ and } \frac{\partial v}{\partial n} = 0 \right\}$$

We define on $\mathbb{R}$ the scalar product denoted (,) and defined by:

$$\forall\, u, v \in V, \ (u, v) = (u, v) + (\nabla u, \nabla v)$$

where

$$(u, v) = \sum_{i=1}^{2} \int_\Omega u_i, v_i$$

The corresponding norm is

$$\|u\|_V = \|u\|_{L^2} + \|\nabla u\|_{L^2} \quad (11)$$

In the following, for simplicity this norm will be noted $\| \cdot \|$.

Then the variational formula of problem (8) is given by: find $(u, p) \in V \times L_0^2(\Omega)$ such that $\forall\, v \in V$, $\forall q \in L_0^2(\Omega)$

$$a(u, v) + b(u, u, v) + c(v, p) = (f, v) \quad (12)$$

$$c(u, q) = 0 \quad (13)$$

where a, b and c are defined by:

$$b(u, u, v) = c_0 \sum_{i=1}^{n} \sum_{j=1}^{n} \int_\Omega u_j \frac{\partial u_i}{\partial x_j} v_i \quad (14)$$

$$a(u, v) = \nu \sum_{i=1}^{n} \sum_{j=1}^{n} \int_\Omega \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} \quad (15)$$

$$c(p, v) = -\frac{1}{\rho_0} \langle p, div\, v \rangle. \quad (16)$$

Before giving existence and uniqueness of solutions to problems (12)-(13), we determine the associated problem. Then, let's pose

$$\mathcal{V} = \{ v \in V | \nabla v = 0 \}$$

and taking $v \in \mathcal{V}$ then $c(v, p) = 0$. Therefore, we get the following problem:

Find $u \in V$ such that $\forall\, \forall\, v \in V$,

$$a(u, v) + b(u, u, v) = (f, v). \quad (17)$$

We note that any solution to (12)-(13) is solution to (17)

We have also, the following lemma.

Lemma 2.1. The space V defined by (10) is separable.

Proof The space $H^1(\Omega)$ is separable and as $H_0^1(\Omega)$ is a subset of $H^1(\Omega)$ therefore it's separable. Consequently $H_0^1(\Omega)^2$ is separable then is separable.

The following theorem is very useful for studying the existence of solution to the problem (17).

Theorem 2.1. Let E be a Euclidean space and let $P : E \longrightarrow E$ be a continuous function, such that there exists $\rho > 0$: $\forall x \in E$, $\|x\| = \rho$, and $P(x) \cdot x \geq 0$. Then, there exists $x_0 \in E$ such that $\|x_0\| \leq \rho$ and $P(x_0) = 0$.

For more details on the proof, we refer to [8].

Theorem 2.2. Let V be a separable space, then the problem (17) admits at least one solution.

Before establishing the proof of this theorem, we will introduce the trilinear form a_1 defined by:

$$a_1(u, v) = a(u, v) + b(u, u, v) \quad (18)$$

Proof The espace V_h is separable, then there exists a subspaces

$\{V_m\}_m$ dense in V_h. This allows as to define the problem (P_m)

$$(P_m)\begin{cases} a(u_m,v)+b(u_m,u_m,v)+c(v,q)=(f,v), \\ \forall u_m\in V_m,\, p\in L_0^2§\Omega) \\ d(u_m,q)=0\ \forall q\in L_0^2(\Omega). \end{cases} \tag{19}$$

First, let us show the existence of a solution to problem (P_m) :
find $u_m \in V_m$ such that $\forall\, v\in \mathcal{D}(\Omega)^2$

$$(P'_m)\begin{cases} a(u_m,v)+b(u_m,u_m,v)=(f,v) \\ d(u_m,q)=0 \end{cases}$$

Let $P_m : V_m \longrightarrow V_m$ such that $\forall v\in V_m$, $\langle P_m(v),w_i\rangle = a_1(v,w_i)-\langle f,w_i\rangle$ for i=1, …, m.
Taking $w_i = v$, we get:

$$\langle P_m(v),v\rangle = a_1(v,v)-\langle f,v\rangle.$$

The space V is separable therefore, $a_1(v,v)\geq \alpha\|v\|_V^2$, and we get

$$\langle P_m(v),v\rangle \geq \alpha\|v\|_V^2-\langle f,v\rangle \geq \alpha\|v\|_V^2-\sup|<f,v>|.$$

Now, we suppose $\langle l,v\rangle = \langle f,v\rangle$ with $f=l$. We set the norm of l by :

$$\|l\| = sup\frac{|(f,v)|}{\|v\|_V}.$$

Further, we have

$$\langle P_m(v),v\rangle \geq (\alpha\|v\|_V-\|l\|)\|v\|_V \tag{20}$$

Let us choose the real ρ so that $\rho > \frac{\|l\|}{\alpha}$ and assuming $\|v\|=\rho$, we have $\|v\|-\frac{\|l\|}{\alpha}>0$ or $(\|v\|-\frac{\|l\|}{\alpha})\|v\|>0$. Further, we get $\langle P_m(v),v\rangle > 0$.
Because of Theorem 2.1, there exists $u_m \in V_m$ such that $P_m(u_m)=0$.
Let's verify the weak limit u_m of (P_m) is solution to the problem (12) - (13).

As $P_m(u_m)=0$, we get from (20) that

$$\alpha\|u_m\|_V-\|l\|<0$$

giving directly

$$\|u_m\|_V < \frac{\|l\|}{\alpha}.$$

The sequence $(u_m)_{m\in\mathbb{N}}$ is then bounded, then there exists a subsequence of $(u_m)_m$ denoted by (u_m) which converges weakly to u in V. The bilinear form a_1 is weakly continuous then we have,

$$\lim_{p\longrightarrow+\infty} a_1(u_{m_p},w_i)=a_1(u,w_i), \forall w_i\in V_m,$$

Then, we have $a_1(u,w_i)=(f,w_i)$, $\forall v\in V$, $v=\cup_{i=1}^{+\infty}w_i$. By density of w_i, we get: $a_1(u,v)=(f,v)$. Therefore u is solution to (17) or solution of problem (12)-(13).

The uniqueness of pressure p associated to u solution to (17) is giving by condition inf-sup and coercivity of $a_1(u,v)$.

Theorem 2.3. Let a_1 defined by (18) where $a(u,v)$ and $b(u,u,v)$ are given in (15)-(14). Let us suppose that the bilinear form b is continuous. Then, the following conditions are satisfied:

$$a_1(u,u)\geq\alpha\|u\|^2 \tag{21}$$

and

$$inf_{q\in Y}\, sup_{v\in X}\frac{b(v,q)}{\|v\|}\geq\beta, \forall q\in Y. \tag{22}$$

Proof Using Holder's inequality see [], we get from (15) the following inequality

$$a(u,v)\leq\nu(\sum_{i=1}^{n}\sum_{j=1}^{n}\int_\Omega|\frac{\partial u_i}{\partial x_j}|^2)^{\frac12}.(\sum_{i=1}^{n}\sum_{j=1}^{n}\int_\Omega|\frac{\partial v_i}{\partial x_j}|^2)^{\frac12}$$

giving

$$a(u,v)\leq\nu\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}$$

or

$$a(u,v)\leq\nu\|u\|_V\|v\|_V$$

meaning that the form a_1 is continuous.

In an other hand, we have

$$a(v,v)=\nu(\sum_{i=1}^{n}\sum_{j=1}^{n}\int_\Omega|\frac{\partial v_i}{\partial x_j}|^2)=a(v,v)=\nu\sum_{i=1}^{n}\|\nabla v_i\|^2_{L^2(\Omega)}$$

According to the Poincarre's inequality, we can write:

$$\|\nabla v_i\|^2_{L^2(\Omega)}\geq\frac{1}{C_i^2}\|v_i\|^2_{H^1(\Omega)}$$

We thus obtain:

$$a(v,v)\geq\nu\sum_{i=1}^{n}\frac{1}{C_i^2}\|v_i\|_{H^1(\Omega)}$$

which proves the coercivity of a. .
By the relation $b(v,v,v)=0$, we can write

$$a_1(v,v)=a(v,v)\geq a_1(v,v)\geq\nu\sum_{i=1}^{n}\frac{1}{C_i^2}\|v_i\|_{H^1(\Omega)}$$

Introducing the real constant C such that $C=\min_{1\leq i\leq n}C_i$, we have:

$$a_1(v,v)\geq\frac{\nu}{C^2}\|v\|_{H^1(\Omega)},$$

so the form a_1 is coercive.
To prove (22), we use Lemma 2.1([18]). With $c(v,p)=-\frac{1}{\rho_0}\int_\Omega div\, v\cdot$, we deduce that there exists $v\in[H^1(\Omega)]^2$ such that $div\, v=-p$. From this last equality, we get

$$c(v,q)=\frac{1}{\rho_0}\int_\Omega|p|^2dxdy=\frac{1}{\rho_0}\|p\|^2.$$

From this relation, we have

$$\frac{c(v,q)}{\|v\|} \leq \frac{1}{\rho_0}\frac{\|p\|^2}{C\|p\|} = \frac{c(v,q)}{\|v\|} \leq \frac{1}{C\rho_0}\|p\|.$$

Then,

$$sup_{v\in V}\frac{c(v,q)}{\|v\|} \leq \frac{1}{C\rho_0}\|p\|,$$

giving (22).

The existence and uniqueness of pressure is illustrated by the following theorem.

Theorem 2.4. There exists a unique $p \in L_0^2(\Omega)$ such that (u,p) is solution to problem (12)-(13).

For more information for the proof of this theorem, we refer to Theorem 2.1 in [18].

The uniqueness of u is giving by following theorem:

Theorem 2.5. Let $n \leq 4$ and ν large enough and satisfying $\nu^2 > C(n)\|f\|_{V'}$.

Then problem (17) admits a unique solution u.

Proof From problem (17), we get : $a(u,v) + b(u,u,v) = (f,v)$. Taking $u = v$ and the fact that $b(u,u,u) = 0$, we have $a(u,u) = (f,u)$. From this equality we get $\nu\|u\|_V^2 \leq \|f\|_{V'}.\|u\|_{V'}$.

Then, we get :

$$\|u\|_V \leq \frac{1}{\nu}\|f\|_{V'}$$

Let $u_\star$ and $u_{\star\star}$ be two solutions of problem (P').

Then, we have

$$a(u_\star,v) + b(u_\star,u_\star,v) = (f,v),$$

$$a(u_{\star\star},v) + b(u_{\star\star},u_{\star\star},v) = (f,v).$$

Then $u = u_\star - u_{\star\star}$ satisfy $a(u,v) = -b(u,u,v)$ from which we get

$$-\nu\|u\|_V\|v\| \leq C\|u\|_V\|u\|_V\|v\|.$$

Therefore, we have

$$(-\nu - \frac{1}{\nu}C\|f\|_{V'})\|u\|_V \leq 0;\ \forall v \in V.$$

Then we have $\|u\|_V = 0$ and $u_\star = u_{\star\star}$.

Then, because of theorem 2.1 and 2.2, problem (12)-13) admits a unique solution (u,p).

3. Numerical Simulation and Main Results

We use the Hermite finite elements because these types of finite elements have their own characteristics. Most finite elements provide continuous and no-derivable solutions, hermite finite elements have the particularity of providing solutions of class $\mathcal{C}^1$ or more. The Hermite finite elements make it possible to solve the partial differential equations by using a coarse mesh since each node correspond to three degrees of freedom.

In this section, a large part will be devoted to discretization of (12)-(13) by using Hermite finite elements. This discretization will make it possible to transform the problem (12)-(13) into a matrix system. We will use the discrete weak formulation (12)-(13) in finite dimensional spaces V_h and $\mathcal{P}_h$.. The numerical simulation of the results resulting from discretization will end this part.

At first we will proceed a brieve study of Hermite finites elements in dimension 2. A Hermite finite element is the triplet (K, Φ_K, Σ_k) such as:

1. K is a triangle.
2. $\Sigma = \{\delta_{A_i}\}$
3. $\Phi = \mathcal{P}_3(K)$

We denote by ψ_i the functions that form the basis of Φ and by ϕ_i the restriction of ψ_i on some element K wich is denoted reference element (see figure below).

For more information on the expression of functions ϕ_i, we refer to [15].

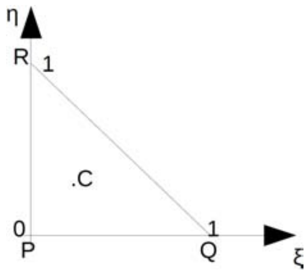

Figure 1. *Reference element.*

3.1. Discretization with Hermite Finite Elements

Our goal is to propose an optimal discretization of system (12)-(13). We are therefore interested to the discretization of the equation presented in the following paragraph. Let $\Omega \subset \mathbb{R}^2$, we start be writing the discrete problem and we prove that it admits a unique solution. We prove stability and convergence. We finally present the necessary tools for the implementation of the discretization method and some numerical results.

3.1.1. Discrete Formulation of the Problem (12)-(13)

We replace the functional spaces respectively V and $L_0^2(\Omega)$ of infinite dimension by finite dimensional spaces V_h and $\mathcal{P}_h$ such as:

$$V_h = \{v_h \in V \text{ such as } v_h/_K \in P_3(K)^2, K \in \mathcal{T}_h\}$$

$$\mathcal{P}_h = \{q_h \in L_0^2(\Omega) \text{ such that } q_h/_K \in P_3(K), K \in \mathcal{T}_h\}$$

The discrete formulation becomes:

Find $(u_h,p_h) \in V_h \times \mathcal{P}_h$ / $\forall v \in V_h; \forall q_h \in \mathcal{P}_h$

$$(P_h)\begin{cases} a_h(u_h,v_h) + b_h(u_h,u_h,v_h) + c_h(v_h,p_h) = (f,v_h) \\ c_h(u_h,q_h) = 0 \end{cases} \tag{23}$$

Let $\psi_i,\ i = 1,\ldots N_h$ the basis functions of V_h, then $\forall u_h \in$

V_h we have

$$u_h(x) = \sum_{i=1}^{N} u_i \psi_i(x), \ \forall x \in \Omega, \tag{24}$$

where $u_i, \ i = 1, \ldots, N_h$ are the components of u_h in the basis (ψ_i), , $i = 1, \ldots, N$. Taking, $v_h = \psi_j, \ j = 1, \ldots, N$, the bilinear forms a_h, the trilinear form b_h and c_h are defined as follows:

$$a_h(u_h, v_h) = \nu \sum_{i=1}^{N_h} (\nabla\psi_i, \nabla\psi_j) u_i \tag{25}$$

$$b_h(u_h, u_h, v_h) = c_0 \sum_{i,j=1}^{N_h} \int_{\Omega_h} (\psi_i, \psi_j) \nabla\psi_i \psi_k . u_i . u_j \tag{26}$$

$$c_h(v, p_h) = \frac{1}{\rho_0} \sum_{i=1}^{N_h} (\psi_i p_i, divv) \tag{27}$$

The functions ψ_i constitute the basic functions of the Hermite element in local coordinates.

The problem (23) is equivalent to the following matrix system.

$$\begin{cases} \nu AU + B(U)U + CP = F \\ C^T U = 0 \end{cases} \tag{28}$$

Where U is the velocity and P the pressure.

Matrix A, B, C and F are defined by:

$$A(i,j) = \int_{\Omega_h} \nabla\psi_i^T \nabla\psi_j$$

$$B(i,j) = \int_{\Omega_h} [\phi_i, \psi_j] \nabla\phi_i \psi_j$$

$$C(i,j) = \frac{1}{\rho_0} \int_{\Omega_h} div\psi_j \psi_i$$

$$F(i) = \int_{\Omega_h} [f_1, f_2][\psi_i, \psi_i]^T$$

3.1.2. Stability of the Problem

The study of the stability of the diagram is done by analyzing the stability of the space V_h.

We introduce the operators g_h and θ_h,called respectively prolongation and restriction operators, by

$$g_h : V_h \longrightarrow V$$

$$u_h \longmapsto g_h u_h$$

and

$$\theta_h : V \longrightarrow V_h$$

$$u \longmapsto \theta_h u$$

These operators can be materialized by the following shema (figure 2).

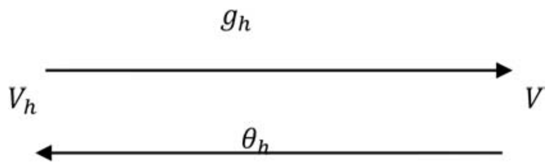

Figure 2. *Operator diagram of g_h and θ_h.*

This following definition gives a necessary and sufficient conditions for an operator to be stable.

Definition 3.1. [22]

The prolongation operators p_h are said to be stable if their norms

$$\|p_h\| = sup_{u_h \in W_h} \|p_h u_h\|_F$$

can be majored independently of h, which implies that the approximation space V_h is stable.

Theorem 3.1. We assume that the following condition is true:

$$\forall\, v \in V, \text{ there exists } \alpha_0 \text{ such that } \|v\|_V \leq \alpha_0.$$

Then the approximation space V_h is stable.

Proof Let w be an element of V such that $g_h u_h = w$.

Thus $\|g_h\| = sup_{u_h \in W_h} \|p_h u_h\|_V$ is equivalent to $\|g_h\| = \|w\|$. So $\|g_h\| \leq \alpha_0$.

The operator g_h is stable and therefore the approximation space V_h is stable.

3.1.3. Convergence

The approximate solutions obtained by solving the approached problem must have the same forms and same graphics representations as the solution of the model problem. To meet these conditions, we will evaluate the behavior of the terms $\|u - u_h\|$ et $\|p - p_h\|$ when h tends to 0.

The following theorem is designed based on the proposition 2.6 [5] et 2.8 [5] gives the markup of the errors.

Theorem 3.2. The solution (u_h, p_h) of the (P_h) satisfies the following inequalities:

$$\|u - u_h\| \leq \alpha_1 inf_{v_h \in V_h} \|u - v_h\|_{V_h} + \alpha_2 inf_{q_h \in \mathcal{P}_h} \|p - q_h\|_{\mathcal{P}_h} \tag{29}$$

$$\|p - p_h\| \leq \beta_1 inf_{v_h \in V_h} \|u - v_h\|_{V_h} + \beta_2 inf_{q_h \in \mathcal{P}_h} \|p - q_h\|_{\mathcal{P}_h} \tag{30}$$

$\forall u_h \in V_h$, $\forall p_h, q_h \in \mathcal{P}_h$, (u,p) satisfies the problem (12)-(13)

Proof We keep the standards defined V and $L_0^2(\Omega)$.

We recall the following relation: $a(u,v) + b(u,u,v) + c(v,q) = f(v)$.

As the approximation is conformal and consistent, we can write: $\forall u \in V$, $u_h \in V_h$

$$a(u-u_h, u-u_h) + b(u-u_h, u-u_h, u-u_h) + c(u-u_h, p) = f(u-u_h)$$

$$a(u-u_h, u-u_h) + c(u-u_h, p) = f(u-u_h)$$

This relation becomes:

$$a(u-u_h, v_h-u_h) + a(u-u_h, u-v_h) + c(u-u_h, p) = f(u-u_h)$$

$$a(u-u_h, v_h-u_h) + a(u-u_h, u-v_h) + c(u-u_h, q_h) + c(u-u_h, p-q_h) = f(u-u_h)$$

By combining these two relations we obtain:

$$a(u-u_h, u-u_h) + c(u-u_h, p) = a(u-u_h, v_h-u_h) + a(u-u_h, u-v_h) + c(u-u_h, p-q_h)$$

The continuity and coercivity of a and c allow to write the following inequalities.

$$a(u-u_h, u-u_h) + c(u-u_h, p) \geq \alpha\|u-u_h\|^2 + \frac{1}{C}\|u-u_h\|^2$$

$$a(u-u_h, v_h-u_h) + a(u-u_h, u-v_h) + c(u-u_h, p-q_h) \leq M\|u-u_h\|.\|u-v_h\| + m\|u-u_h\|.\|p-q_h\|$$

By combining the two inequalities above, we obtain:

$$(\alpha + \frac{1}{C_h})\|u-u_h\|^2 \leq M\|u-u_h\|.\|u-v_h\| + m\|u-u_h\|.\|p-q_h\|$$

$$(\alpha + \frac{1}{C_h})\|u-u_h\| \leq M\|u-v_h\| + m\|p-q_h\|$$

$$\|u-u_h\| \leq \alpha_1\|u-v_h\| + \alpha_2\|p-q_h\|$$

By taking the inf of the terms on the right, we obtain (2.18).
As to the relation (2.19) we adopt the same approach.

$$a(u-u_h, u-u_h) + c(u-u_h, p) = c(u_h-h, p-p_h)$$

$$a(u-u_h, u-u_h) + c(u-u_h, q_h) + c(u-u_h, p-q_h) = c(u_h-u, p-p_h)$$

The use of continuity properties and the lemma 2.3.1 gives the following inequalities:

$$a(u-u_h, u-u_h) + c(u-u_h, p-q_h) \leq M_1\|u-u_h\|^2 + \|u-u_h\|.\|p-q_h\|$$

$$c(u_h-u, p-p_h) \geq \frac{1}{C_h}\|u-u_h\|.\|p-p_h\|$$

These two relations allow to have this inequality

$$\frac{1}{C_h}\|u-u_h\|.\|p-p_h\| \leq M_1\|u-u_h\|^2 + \|u-u_h\|.\|p-q_h\|$$

$$\|p-p_h\| \leq \beta_1\|u-u_h\| + \beta_2\|p-q_h\|$$

As a result we get (2.19).
The convergence of the solution (u_h, p_h) to (u,p) is illustrated by the following theorem.
Theorem 3.3. [6]
We assume the following hypotheses:

$$\|u-u_h\|_V \leq C_1 h^{k+1-m}|u|_{k+1} + C_2 h^{k+1-m}|p|_{k+1} \tag{31}$$

$$\|p-p_h\|_{L^2} \leq K_1 h^{k+1-m}|u|_{k+1} + K_2 h^{k+1-m}|p|_{k+1} \tag{32}$$

Then the discrete solution (u_h, p_h) of (23)converges to (u, p) when h tends to 0.

Table 1. *Constant number.*

Parameters	mesh size	Vertices of the triangle	Viscosity	latitude
Notations	h	$A_i(x_i, y_i)$	ν	ϕ
Some values	$0 < h < 1$	$A_1(0,0)$,$A_2(0,1)$,$A_3(1,0)$	$\nu = k.10^n$	$0 < \phi < \frac{\pi}{2}$

3.1.4. The Mesh

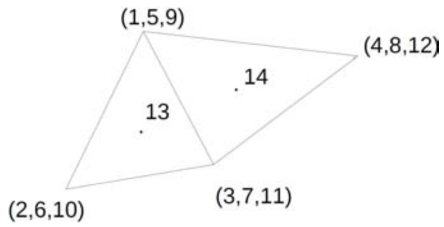

Figure 3. *Conformal mesh with 4 nodes et 14 degrees of freedom.*

The mesh by Hermite finite elements has a big particularity. Firstly with each node, we associate 3 degrees of freedom and on the other hand one degree of freedom on the center of gravity of each element of the mesh.

This particularity is explained by the fact that each node A_k are defined three basic functions ψ_k, $\frac{\partial \psi_k}{\partial x}$ and $\frac{\partial \psi_k}{\partial y}$.

3.1.5. Matrix Structure of System 28

The following figure represents the matrices structure of system (28) resulting from the discretization by the finite elements of Hermite.

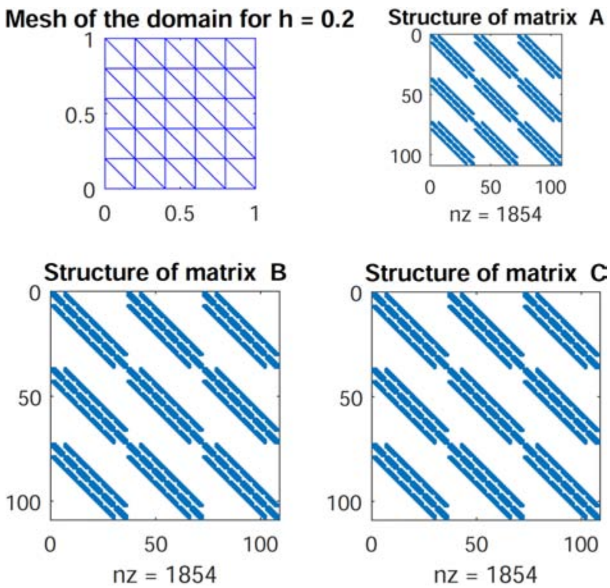

Figure 4. *Structire2 1:Structure of the matrix.*

3.2. Simulation Results

In this part the software used to carry out the simulation is matlab.

To make a good simulation which will allow to obtain a result, it is necessary to vary the values taken by the variables. The following tabular provides some values taken by these variables.

3.2.1. Test 1

The objectif of the test is to see if there is a relation between the solution and the parameter h.

For that we will give some values of h and find the solutions corresponding to each of its values.

Table 2. *Variation of the velocity and the pression depending on the mesh .*

h	$\|u_h\|_{L^2}$	**pressure (Pa)**
$0,05$	294.9927	$3.2829.10^{14}$
$0,06$	225.1051	29.1767
$0,07$	199.4725	922.8613
$0,08$	71.7917	225.5666
$0,09$	87.0559	36.3983
$0,1$	626.7938	$1.8442.10^{14}$

The figure 5 give the representation of the solutions u_h and p_h according to the different values taken by h.

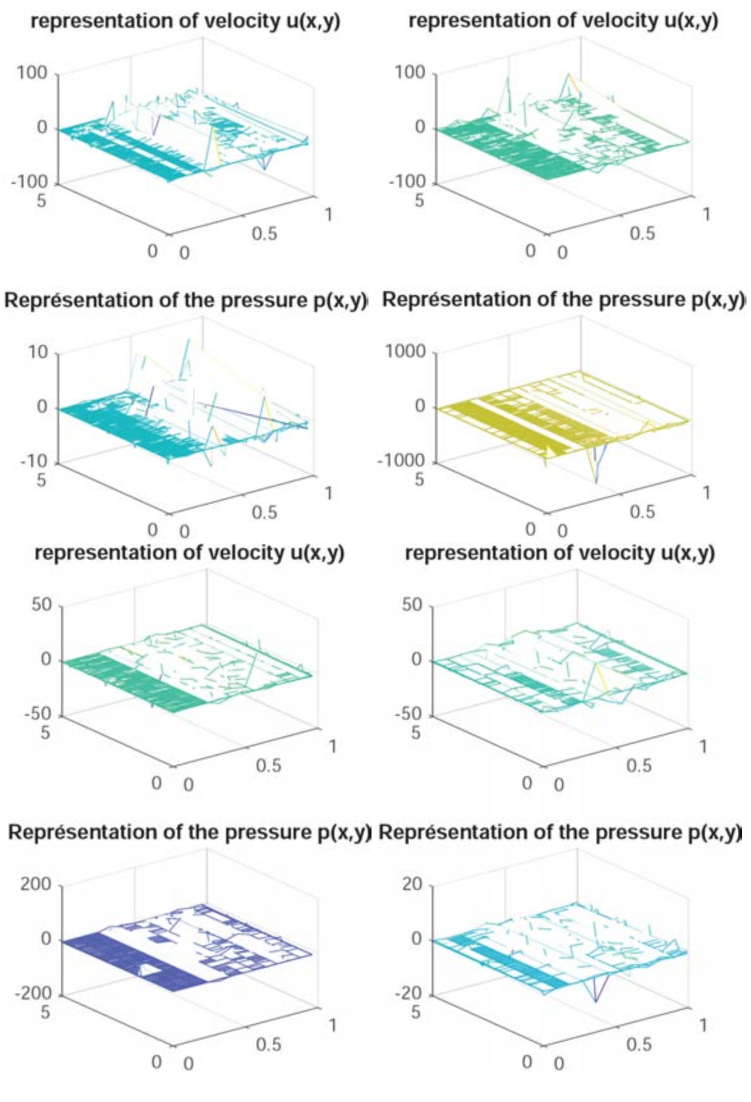

Figure 5. *Structure of solutions for* $h = 0,05$, $h = 0,06$, $h = 0,07$ *and* $h = 0,08$.

3.2.2. Test 2

The objectif of the section is to find if there is, a relation between the evolution of viscosity and the solutions. To achieve this, we will simulate the main.m program for different values of the viscosity coefficient.

The following table summaries the different result obtained after the simulation.

Table 3. *Variation of the velocity and the pression as function of Viscosity.*

viscosity (ν)	$\|u_h\|_{L^2}$	**pressure (Pa)**
10^7	$3,2653.10^{-5}$	$39,2935$
10^6	$3,2643.10^{-4}$	$36,0500$
10^5	$0,0033$	$37,3095$
10^4	$0,0327$	$37,3412$
0.1	$32,6446$	$34,6249$
0.01	$326,3978$	$40,1343$
0.001	$3,2534.10^3$	$42,6934$

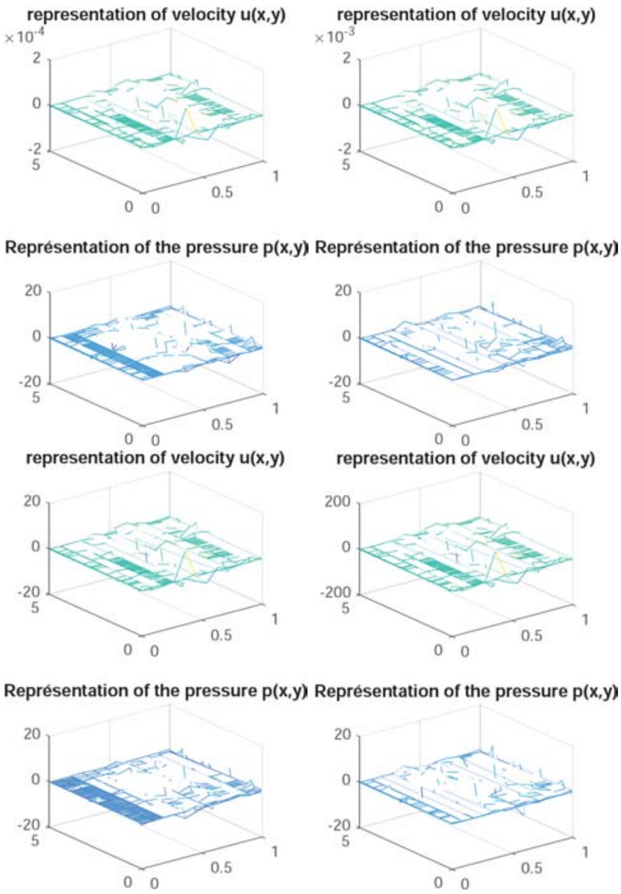

Figure 6. *Structure of solutions for* $\nu = 10^6,\ \nu = 10^5,\ \nu = 0.1$ *and* $\nu = 0.01$.

3.2.3. Test 3: Stokes Problem

In this section, we characterize the influence due to the Reynolds number Re. The Reynolds number represents the ratio between the forces of inertia and the viscous forces. This dimensionless number appears in the dimensionless Navier-Stokes equations. It is defined as follows:

$$Re = \frac{UL_c}{\nu}$$

with U is the characteristic velocity, L_c the characteristic dimension, $\nu = \frac{\mu}{\rho}$ the kinematic viscosity of the fluid, μ the dynamic viscosity of the fluid and ρ, the density of the fluid. We will consider several case for $Re < 1$ corresponding to the Stokes flow. In this case the inertia forces related to the velocity of the fluid being negligible, the viscous forces and the pressure forces are balanced. The objective of this section is to repeat the previous simulations by considering the Reynolds number as variable. The following table provides the different value of the solutions according to the Reynolds number.Two problems will be studied: the Stokes problem and the advection problem.The Reynolds number is the parameter that will be considered as a control variable.

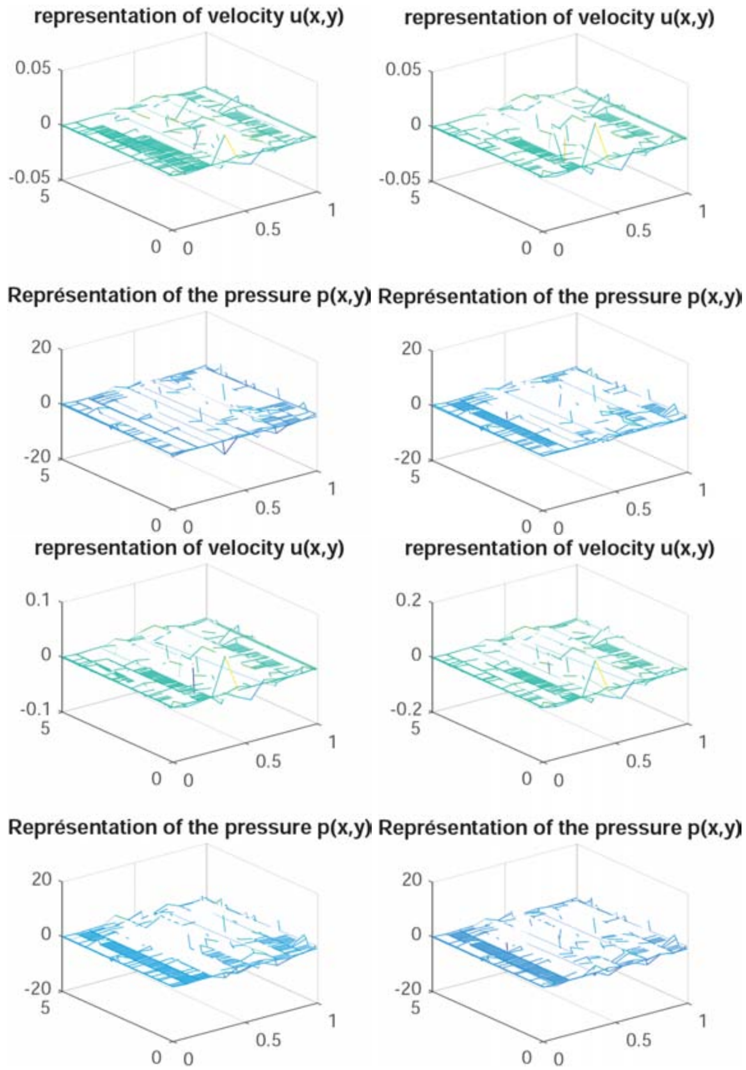

Figure 7. *Structure of solutions for* $Re = 0.001$, $Re = 0.0015$, $Re = 0.0028$ *and* $Re = 0.005$.

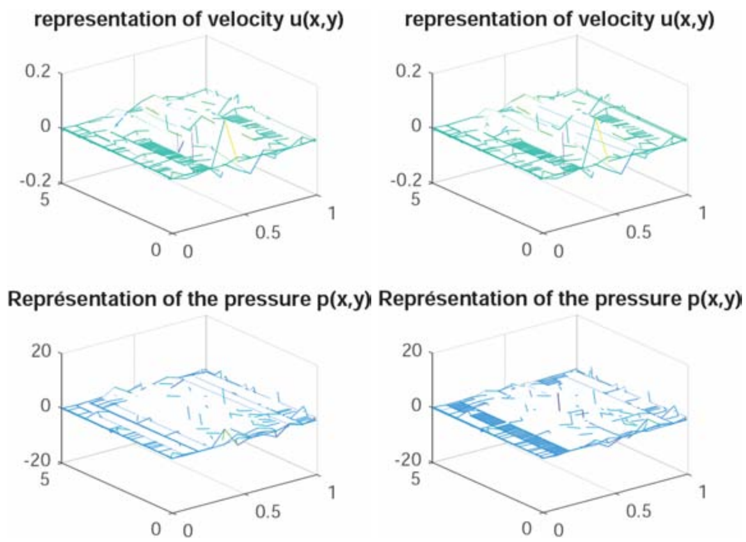

Figure 8. *Structure of solutions for* $Re = 90$ *and* $Re = 150$.

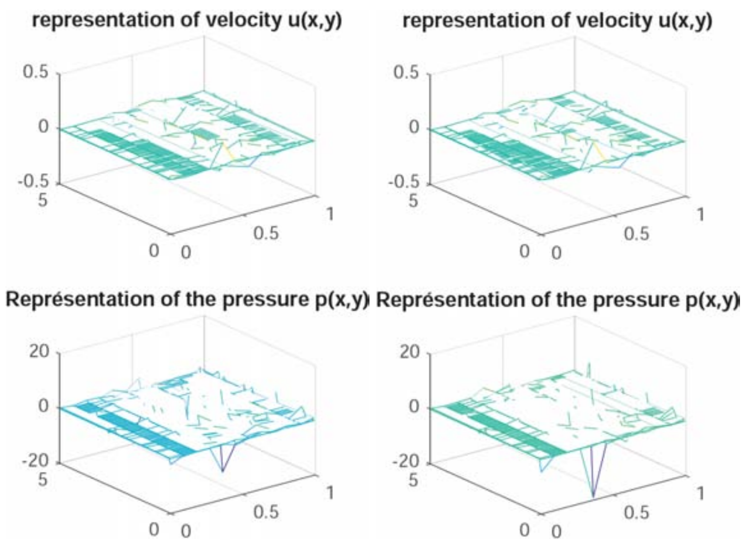

Figure 9. *Structure of solutions for* $Re = 0.0067,\ \ Re = 0.007,\ \ Re = 0.0085$ *and* $Re = 0.009$.

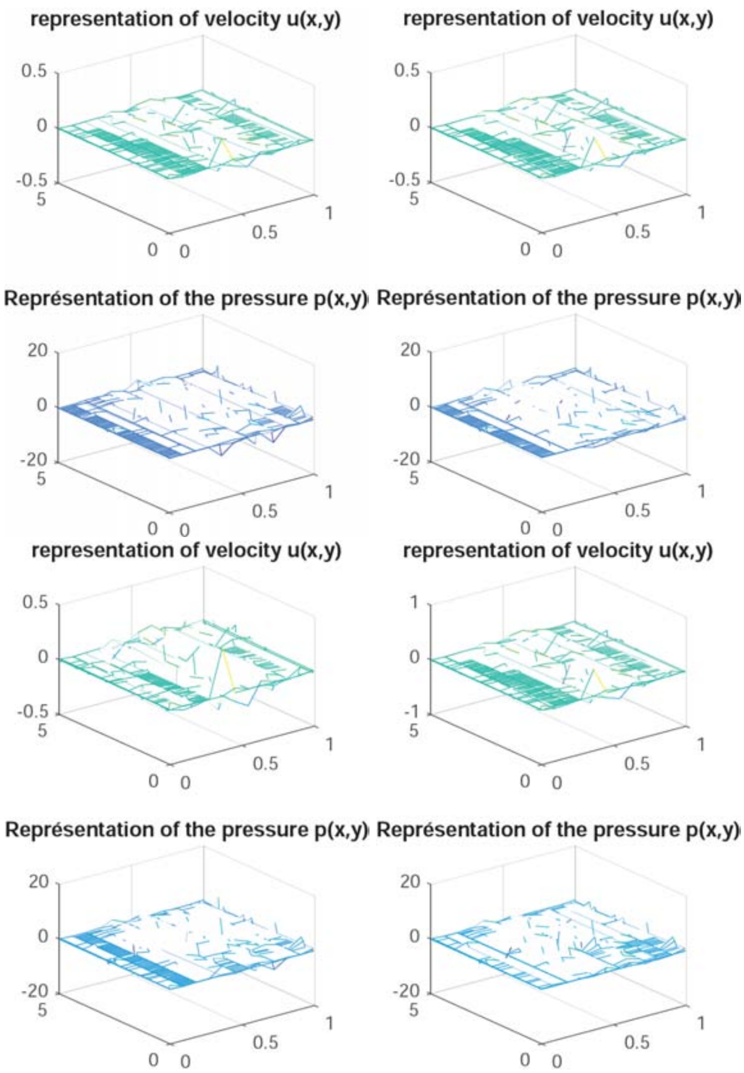

Figure 10. *Structure of solutions for* $Re = 850,\ \ Re = 1000,\ \ Re = 5000$ *and* $Re = 7000$.

Table 4. *Variation of the velocity and the pression as function of Reynolds number.*

Reynolds number (Re)	$\|u_h\|_{L^2}$	**pressure (Pa)**
0.001	0.0653	36.2635
0.0015	0.0979	38.6310
0.0028	0.1828	40.2984
0.005	0.3265	38.4739
0.0067	0.4373	36.3011
0.007	0.4571	35.8914
0.0085	0.5552	37.3622
0.009	0.5877	42.5808
0.0095	0.6203	35.3834
0.01	0.6527	37.1383
0.018	1.1758	40.2984
0.02	1.3057	35.0083

3.2.4. Physical Interpretation of Rsults

The objective of these two tests was to known on which parameters the approximate solutions depend the most.

The results of test 1 do not reveal the evolution of the solutions when the parameters h increases because its direction of evolution varies alternately. Although the intensity of the velocity changes, we can't conclude that the parameter h influences the direction of evolution of the intensity velocity. This fact can be explained by the fact that h is not a physical parameter which characterizes the flow.

With the test 2, when the viscosity takes respectively 10^7, 10^6, 10^5 and 10^4 the velocity takes respectively the following values $3.2643.10^{-5}$, $3.2645.10^{-4}$, 0.0033 and 0.0327. Conversely by making the viscosity small, we observe an increase in velocity. We can say that the viscosity and the velocity vary in the opposite direction, in order worlds if the viscosity increases then the velocity of the fluid decreases. Viscosity is the main factor influencing the velocity,which is logical since the Stokes problem is characterized by a viscosity, which makes the convection term negligible.

In the last test, we consider the variation of the velocity and the pressure of the fluid when the Reynolds number is very low, considering the Stokes problem. We first notice that the velocity of the fluid is very low when the Reynolds number is small. Which corresponds to the Stokes flow: Laminar. When also the Reynolds number is very low, the speed of the fluid and the pressure vary slowly.

4. Conclusion

This document was devoted to the discretization and the numerical simulation of the Navier-Stokes equations in the case of a stationary flow which results in a sationary Stokes problem.

Discretization is fundamental since it made it possible to find a finite dimensional space which contains the solutions, this one allowed to transform relation (3) into a matrix system.

The simulation of the matrix assembly algorithm made it possible to visualize the structure of the matrix system.

The results obtained are very satisfactory because they are in perfect agreement with the theoretical study.

The study of Euler problem and of the unsteady Navier-Stokes equations by Hermite finite elements constitute the chalenges to be met.

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