



# A Complex Variable Circle Theorem for Plane Stokes Flows

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**Abstract:** Two dimensional steady Stokes flow around a circular cylinder is examined in the light of complex variable theory and a circle theorem for the flow, are established. The theorem gives a complex variable expression of the velocity for a Stokes flow external to a circular cylinder, in terms of the same variable expression of the velocity for a slow and steady irrotational flow in unbounded incompressible viscous fluid, and also gives a formula for the stream function for the flow. A few illustrative solutions of Stokes flow around a circular cylinder are presented.

**Keywords:** Two Dimensional Stokes Flow, Complex Variable Theory, Circle Theorem

## 1. Introduction

It is well known (1) that for a two –dimensional uniform streaming motion past a circular cylinder there exists no solution of the governing Stokes equations. It is in great contrast to the corresponding three-dimensional problem of a uniform stream disturbed by a sphere. This situation for the motion past a circular cylinder is known as the Stokes paradox. (2) Recently, Avudainayagam, Jotiram and Ramakrishna (3) have established a necessary condition (called the consistency condition) for the existence of plane Stokes flow past a circular cylinder and the authors have also given, for the first time, an explanation of the Stokes paradox with the aid of the same condition. Sen (4) has given a circle theorem for the stream function for two-dimensional steady Stokes flow past a rigid circular cylinder in terms of the stream function for a slow flow in an unbounded incompressible viscous fluid. Usha and Hemalatha (5) have also established a circle theorem for the stream function for plane Stokes flow past a shear free impermeable circular cylinder.

In the present paper, we have found it much convenient to a study on Stokes flow past a circular cylinder in the light of complex variable theory is considerably convenient here for the formulae for the same flow are mathematically concise, and have “initial conditions” which are very simple. In Section 3, we have given the first theorem for the complex velocity and the stream function for plane Stokes flow external to the circular cylinder, when the primary flow in an

unbounded incompressible viscous fluid is irrotational everywhere, and this theorem corresponds to Milne-Thomson’s circle theorem for potential flow (6) By making use of Taylor’s series which is well known in complex variable theory, the second theorem which is similar to the first one stands for the flow internal to a circular cylinder. We also easily show that the Stokes flow problem only for a two dimensional source or sink, etc. outside a circular cylinder or a two dimensional uniform streaming function past the same cylinder does not exist; but we have found that their combination taken two or more at a time, by choosing their “strengths” and “positions” suitably, gives Stokes flow around a circular cylinder. This phenomenon is illustrated by the exact solutions of a number of the Stokes flow past a circular cylinder with the aid of the “Cauchy Integral formula for multiply-connected region”, well known in complex variable theory; and the same theorem gives the formulae for the velocity and the stream function for the flow. In Section 2 we begin with a discussion of the fundamental singular solutions of the Stokes equations with external force and give the formulation of the velocity and pressure of the two dimensional fundamental singularities, such as Stokeslet, Stoke doublet, rotlet, potential doublet, etc in order to use the theorem in the application of theorem.

## 2. Fundamental Solutions of the Two Dimensional Stokes Equations

The complex variable formed by the two dimensional

Stokes equations for a steady motion in an incompressible viscous fluid can be obtained in a quite easy manner and these are

$$\frac{2}{\rho} \frac{\partial p}{\partial z} = 4\vartheta \frac{\partial^2}{\partial z \partial \bar{z}} (u + iv) + F \quad (1)$$

And,

$$\frac{\partial}{\partial z} (u + iv) + \frac{\partial}{\partial \bar{z}} (u - iv) = 0 \quad (2)$$

Where  $i = \sqrt{-1}$ ,  $u$  and  $v$  are the Cartesian velocity components,  $p$  the pressure,  $\mu$  the constant viscosity coefficient and  $\vartheta$  is kinematic viscosity where  $F = F_1 + iF_2$ ,  $F_1$  and  $F_2$  being the Cartesian components of external force per unit volume. In this note for two dimensional flow in a viscous fluid we denote the combination  $u - iv$  by the symbol  $\bar{v}$  so that.

$$\bar{v} = v(z, \bar{z}) = u - iv \quad (3)$$

Here we shall call  $\vartheta$  the complex velocity after Milne-Thomson (6) who first called  $\vartheta$ , having expression (3) without,  $\bar{z}$ , the complex velocity in connection with potential flow.

We then define the complex conjugate of the complex velocity  $\vartheta$  as

$$\bar{\vartheta} = \bar{\vartheta}(\bar{z}, z) = u + iv \quad (4)$$

Now we need the relation between  $\vartheta$  the complex velocity  $\bar{z}$  and the stream function  $\psi(z, \bar{z})$

For a flow in an incompressible viscous fluid; and since we have, in (6, p.174), the expression.

$$u - iv = -2i \frac{\partial \psi}{\partial z} \quad (5)$$

Next by making use of results (3) and (4) in the Stokes equations (1) and (2), we obtain the new Stokes equations as.

$$\frac{2}{\rho} \frac{\partial p}{\partial z} = 4\vartheta \frac{\partial^2}{\partial z \partial \bar{z}} \bar{v} \quad (6)$$

And

$$\frac{\partial \bar{v}}{\partial z} + \frac{\partial v}{\partial \bar{z}} = 0 \quad (7)$$

Here note that expression (6) is the complex conjugate of equation (1) and further note that  $p = \bar{p}$  since  $p$  is a real scalar function. In this case the vorticity,  $\xi$ , of the fluid motion is given by.

$$\xi = 2i \frac{\partial v}{\partial \bar{z}} \quad (8)$$

We then give below the complex variable forms of the

fundamental solutions to the two dimensional Stokes Equations (6) and (7), which corresponds to the vector forms of the fundamental solutions to the three dimensional Stokes equations in Chwang and Wu (9).

The primary fundamental solution of Equations (6) and (7), is concerned with a singular point force located, say at the origin,

$$F_s = 4\pi\mu\alpha\xi(x)\xi(y) \quad (9)$$

$\alpha$  being a constant complex quantity, and  $\xi(x)$  and  $\xi(y)$  one-dimensional Dirrac delta functions.  $\alpha$  characterizes its strength (in magnitude  $|\alpha|$  and direction  $\arg \alpha$ ).

In fact, expression (9) is the two-dimensional Stokeslet in (7, expression (18)) by treating the constant vector  $\hat{\alpha}$ . The solution of equations (6) and (7) for the Stokeslet  $F = F_s$  can be obtained in quite a straightforward manner. Thus the complex velocity  $v_s$  and pressure  $P_s$  of a Stokeslet of strength  $\alpha$  at the origin are given by.

$$v_s(z, \bar{z}; \alpha) = -\frac{1}{2}\bar{\alpha} \log z\bar{z} + \frac{1}{2} \frac{(\alpha\bar{z})}{\alpha z} \quad (10)$$

And

$$P_s(z, \bar{z}; \alpha) = \mu \left( \frac{\alpha}{z} + \frac{\bar{\alpha}}{\bar{z}} \right) \quad (11)$$

Clearly, a derivative of any order of  $v_s$  and  $P_s$  is also a solution of equation (2.7), the corresponding.

$\bar{F}$  being the derivative of the same order of the conjugate of the Stokeslet. We may now introduce two-dimensional potential doublet, potential quadrapole, rotlet, stresslet etc. as follows. A two- dimensional potential doublet corresponding to its three dimensional as analogue in (9) has the simple complex velocity representation.

$$v_D(z, \bar{z}; \alpha) = \alpha \frac{1}{z^2} \quad (12)$$

Where  $\alpha$  (a constant complex quantity) is the doublet strength. It is of interest to note that a potential doublet is related to the Stokeslet by.

$$v_D(z, \bar{z}; \alpha) = -\frac{1}{2} \nabla^2 v_s(z, \bar{z}; \alpha) \quad (13)$$

Where

$$\nabla^2 = \frac{4\partial^2}{\partial z \partial \bar{z}}. \text{ The corresponding pressure is given by.}$$

$$P_D(z, \bar{z}; \alpha) = -\frac{1}{2} \nabla^2 P_s(z, \bar{z}; \alpha) = 0 \quad (14)$$

The potential quadrapole and potential octupole may be introduced respectively as the complex velocities.

$$v_{D4}(z, \bar{z}; \alpha, \beta) = -\left(\beta \frac{\partial}{\partial z} + \bar{\beta} \frac{\partial}{\partial \bar{z}}\right) v_D(z, \bar{z}; \alpha) \\ = 2\alpha\beta \frac{1}{z^3} \quad (15)$$

And.

$$v_{D8}(z, \bar{z}; \alpha, \beta, \gamma) = \left(\gamma \frac{\partial}{\partial z} + \bar{\gamma} \frac{\partial}{\partial \bar{z}}\right) \left(\beta \frac{\partial}{\partial z} + \bar{\beta} \frac{\partial}{\partial \bar{z}}\right) v_D(z, \bar{z}; \alpha) \\ = -6\alpha\beta\gamma \frac{1}{z^4} \quad (16)$$

And their corresponding pressures are zero  $\beta$  and  $\gamma$ , being constant complex quantities. Similarly, the Stokes doublet, Stokes quadrupole, etc. may be introduced as follows.

$$v_{SD}(z, \bar{z}; \alpha, \beta) = -\left(\beta \frac{\partial}{\partial z} + \bar{\beta} \frac{\partial}{\partial \bar{z}}\right) v_S(z, \bar{z}; \alpha) \\ = \frac{1}{2} \left( (\bar{\alpha}\beta - \alpha\bar{\beta}) \frac{1}{z} + \left( \bar{\alpha}\beta \frac{1}{z} + \alpha\bar{\beta} \frac{\bar{z}}{z^2} \right) \right) \quad (17)$$

$$P_{SD}(z, \bar{z}; \alpha, \beta) = -\left(\beta \frac{\partial}{\partial z} + \bar{\beta} \frac{\partial}{\partial \bar{z}}\right) P_S(z, \bar{z}; \alpha) \\ = \mu \left( \alpha\beta \frac{1}{z^2} + \bar{\alpha}\bar{\beta} \frac{1}{\bar{z}^2} \right) \quad (18)$$

$$v_{S4}(z, \bar{z}; \alpha, \beta, \gamma) = \left(\gamma \frac{\partial}{\partial z} + \bar{\gamma} \frac{\partial}{\partial \bar{z}}\right) \left(\beta \frac{\partial}{\partial z} + \bar{\beta} \frac{\partial}{\partial \bar{z}}\right) v_S(z, \bar{z}; \alpha) \\ = \frac{1}{2} \left[ \left( \gamma(\bar{\alpha}\beta - \alpha\bar{\beta}) + \bar{\gamma}\alpha\beta \right) \frac{1}{z^2} - \left( \bar{\alpha}\bar{\beta}\gamma \frac{1}{z^2} + 2\alpha\beta\gamma \frac{\bar{z}}{z^3} \right) \right] \quad (19)$$

$$P_{S4}(z, \bar{z}; \alpha, \beta, \gamma) = \left(\gamma \frac{\partial}{\partial z} + \bar{\gamma} \frac{\partial}{\partial \bar{z}}\right) \left(\beta \frac{\partial}{\partial z} + \bar{\beta} \frac{\partial}{\partial \bar{z}}\right) P_S(z, \bar{z}; \alpha) \\ = 2\mu \left( \alpha\beta\gamma \frac{1}{z^3} + \bar{\alpha}\bar{\beta}\bar{\gamma} \frac{1}{\bar{z}^3} \right) \quad (20)$$

We then recognize the anti-symmetric component (with respect to interchange of the complex quantities  $\alpha$  and  $\beta$ ) of the Stokes doublet (17) gives a fundamental singularity; it is called a rotlet by Chwang and Wu (8), and also a couplet by Batchelor (11). Its complex velocity and pressure have the following simple representations.

$$v_R(z, \bar{z}; ik) = \frac{1}{2} \left[ \left( \alpha \frac{\partial}{\partial z} + \bar{\alpha} \frac{\partial}{\partial \bar{z}} \right) v_S(z, \bar{z}; \beta) - \left( \beta \frac{\partial}{\partial z} + \bar{\beta} \frac{\partial}{\partial \bar{z}} \right) v_S(z, \bar{z}; \alpha) \right] \\ = ik \frac{1}{z} \quad (21)$$

And.

$$P_R(z, \bar{z}; ik) = \frac{1}{2} \left[ \left( \alpha \frac{\partial}{\partial z} + \bar{\alpha} \frac{\partial}{\partial \bar{z}} \right) P_S(z, \bar{z}; \beta) - \left( \beta \frac{\partial}{\partial z} + \bar{\beta} \frac{\partial}{\partial \bar{z}} \right) P_S(z, \bar{z}; \alpha) \right] = 0 \quad (22)$$

Where  $ik = \frac{1}{2}(\bar{\alpha}\beta - \alpha\bar{\beta})$  is a pure complex number.

Again the symmetric component (with respect to an interchange of the complex quantities  $\alpha$  and  $\beta$ ) of the Stokes doublet is itself a physical quantity called a stresslet after Batchelor (11). Its complex velocity and pressure are respectively.

$$v_{SS}(z, \bar{z}; \alpha, \beta) = \frac{1}{2} \left( \bar{\alpha}\bar{\beta} \frac{1}{z} + \alpha\beta \frac{\bar{z}}{z^2} \right) \quad (23)$$

And.

$$P_{SS}(z, \bar{z}; \alpha, \beta) = \mu \left( \alpha\beta \frac{1}{z^2} + \bar{\alpha}\bar{\beta} \frac{1}{\bar{z}^2} \right) \quad (24)$$

Finally, note that the term of  $\frac{1}{z^2}$  in the Stokes quadrupole (19) is a fundamental singularity, i.e., a potential doublet; and so the remaining terms of  $\frac{1}{z^2}$  and  $\frac{\bar{z}}{z^3}$ , as a whole, in the

same quadrupole constitute another fundamental singularity which is symmetric with respect to interchange of any two of the complex quantities  $\alpha, \beta, \gamma$ . We may call it the Stokes quadrupole and its velocity and pressure are as follows.

$$v_{S4S}(z, \bar{z}; \alpha, \beta, \gamma) = -\frac{1}{2} \left( \bar{\alpha}\bar{\beta}\bar{\gamma} \frac{1}{z^2} + 2\alpha\beta\gamma \frac{\bar{z}}{z^3} \right) \quad (25)$$

And.

$$P_{S4S}(z, \bar{z}; \alpha, \beta, \gamma) = 2\mu \left( \alpha\beta\gamma \frac{1}{z^3} + \bar{\alpha}\bar{\beta}\bar{\gamma} \frac{1}{\bar{z}^3} \right) \quad (26)$$

### 3. The Circle Theorems

In the case of conservative forces the Stokes equation (6) reduces to the form.

$$\frac{\partial}{\partial z} (p + \Omega) = 2 \frac{\partial^2}{\partial z \partial \bar{z}} v \quad (27)$$

Where  $\Omega$  is the potential function due to the forces. The

complex conjugate form of this equation is.

$$\frac{\partial}{\partial \bar{z}}(p + \Omega) = 2 \frac{\partial^2}{\partial z \partial \bar{z}} \bar{v} \quad (28)$$

Wherein we note  $p = \bar{p}$  and  $\Omega = \bar{\Omega}$ , since  $p$  and  $\Omega$  are real (scalar) functions.

Now a differential equation satisfied by  $(p + \Omega)$  without the complex velocity  $v$  can be obtained from equations (27) and (28) with the aid of the mass conservation equation (7) as.

$$\frac{\partial^2}{\partial z \partial \bar{z}}(p + \Omega) = 0 \quad (29)$$

And therefore by using this result in (27) we have the following equation for the complex velocity.

$$\frac{\partial^3}{\partial z \partial \bar{z}^2} v = 0 \quad (30)$$

Finally, substituting (2.6) in this expression, gives the equation for steady Stokes flow, satisfied by the stream function  $\Psi(z, \bar{z})$  as.

$$\frac{\partial^4}{\partial z^2 \partial \bar{z}^2} (2i\Psi) = 0 \quad (31)$$

This equation for Stokes flow is in agreement with that in Milne-Thomson (6, p.683), and the solution of it for  $2i\Psi$ , due to him is stated here in a slight different form for future reference as follows.

$$2i\Psi = \bar{z}W(z) - z\bar{W}(\bar{z}) + \int w(z)dz - \int \bar{w}(\bar{z})d\bar{z} \quad (32)$$

Where  $w(z)$  and  $\bar{w}(\bar{z})$  are arbitrary complex functions. By applying formula (5) to this expression, we then obtain the general complex velocity for two-dimensional steady Stokes flow as follows.

$$v(z, \bar{z}) = \bar{W}'(\bar{z}) - \bar{z}W'(z) - w(z) \quad (33)$$

Where the prime in second term of the R.H.S of (33) denotes differentiation with respect to  $z$ . The expression for  $(p + \Omega)$  corresponding to the complex velocity (33) is given by.

$$p + \Omega = -2\mu \left( W'(z) + \bar{W}'(\bar{z}) \right) + p_0 \quad (34)$$

Where  $p_0$  is an arbitrary real constant.

First we present relatively simple expressions for the complex velocity and the stream function for a two-dimensional steady Stokes flow external (or internal) to a circular cylinder in terms of the complex velocity for a slow irrotational flow in an incompressible viscous fluid with no

rigid boundaries.

**Circle Theorem:** Let there be steady, slow and two-dimensional irrotational flow in incompressible viscous fluid with no rigid boundaries, in the  $z$ -plane. Let the flow be characterized by the complex velocity  $v_0 = v_0(z)$ , whose singularities are all at a distance greater than  $a$  from the origin, and let  $v_0 \approx O(z^k)$  for  $k \geq 1$ , at the origin. If a circular cylinder of radius  $a$  (whose intersection with the  $z$ -plane is the circle  $\Gamma: |z| = a$ ), be introduced into the flow, the complex velocity and the stream function for the Stokes flow past the circular cylinder become respectively.

$$v(z, \bar{z}) = v_0 + v_0^* = v_0(z) - v_0\left(\frac{a^2}{z}\right) + \left(\bar{z} - \frac{a^2}{z}\right) \bar{v}_0'\left(\frac{a^2}{z}\right) \quad (35)$$

And

$$\begin{aligned} \Psi = & \frac{1}{2}i \left[ \bar{z} \bar{v}_0\left(\frac{a^2}{z}\right) - z v_0\left(\frac{a^2}{z}\right) + \int v_0(z) dz \right. \\ & \left. - \int \bar{v}_0(\bar{z}) d\bar{z} + \int \frac{a^2}{z} v_0'\left(\frac{a^2}{z}\right) d\bar{z} - \int \frac{a^2}{z} \bar{v}_0'\left(\frac{a^2}{z}\right) dz \right] \end{aligned} \quad (36)$$

$$\text{where } v_0^* = -v_0\left(\frac{a^2}{z}\right) + \left(\bar{z} - \frac{a^2}{z}\right) \bar{v}_0'\left(\frac{a^2}{z}\right) \quad (37)$$

is the perturbation complex velocity and where the prime denotes differentiation with respect to  $z$ .

**Proof** The proof essentially consists in showing that the complex velocity (35) must satisfy the following four conditions, and in deriving the stream function (36) out of the complex velocity.

- (i) Expression (35) must be obtained from the general complex velocity (33)
- (ii) On  $\Gamma: |z| = a$ ,  $v(z, \bar{z}) = 0$  (implying that the radial and azimuthal (or tangential) components of the velocity of the fluid motion on  $\Gamma$  vanish).
- (iii)  $v_0^*$  must introduce no singularities outside  $\Gamma$ .
- (iv) The perturbation velocity, i.e.,  $|v_0^*|$  must tend to vanish as  $|z| \rightarrow \infty$ .

First, if we assign to the (arbitrary) complex functions  $W(z)$  and  $w(z)$  referred to the general complex velocity (3.7) the expressions  $W(z) = -\bar{v}_0\left(\frac{a^2}{z}\right)$  and  $w(z) = -v_0(z) + \frac{a^2}{z} \bar{v}_0'\left(\frac{a^2}{z}\right)$  the complex velocity (35) is obviously obtained. Thus condition (i) is satisfied.

Since on the circle  $\Gamma, z\bar{z} = a^2$ , it is then simple to see that on the same circle  $v(z, \bar{z}) = 0$ . Therefore condition (ii) is satisfied.

Next, to verify condition (iii) we note that  $z$  and  $\frac{a^2}{z}$  are inverse points with respect to the circle  $\Gamma$ . So if the point  $z$

is outside  $\Gamma$ , the point  $\frac{a^2}{z}$  is inside  $\Gamma$  and vice versa. Thus the singularities of  $v_0(z)$  f, by hypothesis, being at distances greater than  $a$  from the origin (i.e. outside  $\Gamma$ ), those of  $v_0(\frac{a^2}{z})$  and  $\bar{v}_0'(\frac{a^2}{z})$  and therefore of  $v_0^*$  are at distances less than  $a$  from the origin (i.e., inside the circle) Therefore  $v_0^*$  introduces no singularities outside so that condition (iii) is satisfied.

Again since  $v_0 \approx O(z^k)$  for  $k \geq 1$  at the origin, it is simple to calculate that  $|v_0^*| \approx O\left(\frac{1}{|z|^k}\right)$  for large  $|z|$  so that the perturbation velocity  $|v_0^*|$  tends to zero as  $|z| \rightarrow \infty$ . This shows that the remaining condition (iv) is satisfied.

Finally, substitution of the complex functions  $W_z = -\bar{v}_0\left(\frac{a^2}{z}\right)$  and  $w(z) = -v_0(z) + \frac{a^2}{z}\bar{v}_0'$  in formula (32) yields easily required stream function (3.10) for the Stokes flow past the circular cylinder  $|z| = a$ . (Here note that by making use the complex velocity (35) in formula (5), we can also obtain the stream function (36). This completes the proof of the theorem.

[Remarks: When the singularities of an arbitrary complex velocity  $v_0(z)$  are all outside the circle  $\Gamma: |z| = a$ ,  $v_0(z)$  is analytic inside  $\Gamma$ ; and therefore at each point  $z$  inside, has in general, a Taylor's series, well-known in complex variable theory, about the origin (the centre of  $\Gamma$ ), of the form.

$$v_0(z) = b_0 + b_1 z + b_2 z^2 + \dots \quad (38)$$

Where  $b_0, b_1, b_2, \dots$  are all constants.

It is now simple to see  $v_0(z) \approx O(b_0)$ ,  $b_0 \neq 0$  at the origin, and therefore expression (3.11) referred to in the circle theorem 1 yields non-zero perturbation velocity, i.e.,  $|v_0^*| \approx O(|b_0|)$  as  $|z| \rightarrow \infty$ . Hence for a primary flow characterized by the complex velocity  $v_0 = v_0(z)$  whose singularities are all at distances greater than  $a$  from the origin

and  $v_0(z) \approx O(b_0)$ ,  $b_0 \neq 0$ , at the origin the Circle Theorem 1 does not give the Stokes flow past the circular cylinder  $|z| = a$ . For example, the primary complex velocities of a uniform stream in the positive direction of the x-axis, a simple source of strength  $m_1$  at the point  $z = z_1$ , and a simple sink of strength  $m_2$  at the point  $z = z_2$ , in an incompressible viscous fluid, are respectively, say  $v_0^{(1)}(z) = v$ ,  $v_0^{(2)}(z) = \frac{m_1}{(z - z_1)}$  and  $v_0^{(3)}(z) = \frac{-m_2}{z - z_2}$ , where

the points  $z_1$  and  $z_2$  are outside the circle  $\Gamma$ . It is then easy to calculate that each of these complex velocities follows the condition  $v_0(z) \approx O(b_0)$ ,  $b_0 \neq 0$ , at the origin, referred above (instead of the condition  $v_0(z) \approx O(z^k)$ ,  $k \geq 1$ , at the origin, referred to in the Circle Theorem 1). Thus by the Circle Theorem 1 the Stokes flow problem for a uniform stream or a simple source or sink outside a circular cylinder does not exist but after taking the suitable combination of them taken two or more at a time, that the same theorem gives the Stokes flow past a circular cylinder is illustrated below].

We now show how Circle Theorem 1 presents, in a relatively simple way, exact solutions to a number of stokes flow problems (a), (b), (c), (d), and (e) referred below. The last two ones have been previously obtained by different methods. (a) a source superimposed on a uniform stream past a circular cylinder.

Let there be a simple source of strength  $m$  at the point on the x-axis of the positive direction and a uniform stream  $\frac{m}{f}$  in the same direction of the same axis, in an incompressible viscous fluid, where  $f \in (a, \infty)$  They constitute a basic flow whose complex velocity is given by.

$$v_0(z) = \frac{m}{z - f} + \frac{m}{f} \quad (39)$$

We then easily see  $v_0 \approx O(z)$  at the origin. Therefore when the circular cylinder  $|z| = a$  is introduced into the flow, the circle theorem 1 yields the following expression for the complex velocity for the Stokes flow past the cylinder.

$$v(z, \bar{z}) = \frac{m}{z - f} + \frac{m}{f} + \frac{ma^2}{f^2} \left( \frac{1}{\bar{z} - a^2/f} + \frac{\bar{z} - a^2/f}{\left(z - \frac{a^2}{f}\right)^2} \right) - \frac{m}{z} + \frac{m}{z - \frac{a^2}{f}} - \frac{ma^2(f^2 - a^2)}{f^3} \frac{1}{\left(\frac{z - a^2}{f}\right)^2} \quad (40)$$

Where the last five terms constitute the image system within the circular cylinder, which thus consists of.

- (i) A stresslet of strength  $\frac{2ma^2}{f^2}$  formed by the combination of the third and fourth terms, at the point

$$z = \frac{a^2}{f}.$$

- (ii) A sink of strength  $m$  at the origin.

- (iii) A source of strength  $m$  at the point  $z = \frac{a^2}{f}$ , and,

(iv) A doublet of strength  $\frac{ma^2(f^2 - a^2)}{f^3}$  at the point

$z = \frac{a^2}{f}$  with its axis in the negative direction of the x-axis.

(v) Here we are interested in seeing the form of the stream

function  $\Psi$  in terms of the polar co-ordinates  $(r, \theta)$ .

Substitution of expression (39) in formula (36) yields the stream function  $\Psi$  in terms of  $z$  and  $\bar{z}$  for the Stokes flow which on the transformation to the polar co-ordinates  $(r, \theta)$ , takes after reduction the form.

$$\begin{aligned} \Psi = & -m \tan^{-1} \left( \frac{r \sin \theta}{r \cos \theta - f} \right) - \frac{m}{f} r \sin \theta - \frac{ma^2}{f^2} \frac{r \sin 2\theta - 2r \left( \frac{a^2}{f} \right) \sin \theta}{r^2 - 2r \left( \frac{a^2}{f} \right) \cos \theta + \frac{a^4}{f^2}} \\ & + m\theta - m \tan^{-1} \left( \frac{r \sin \theta}{r \cos \theta - \frac{a^2}{f}} \right) + \frac{ma^2(f^2 - a^2)}{f^3} \frac{r \sin \theta}{r^2 - 2r \left( \frac{a^2}{f} \right) \cos \theta + \frac{a^4}{f^2}}. \end{aligned} \quad (41)$$

Here it is of interest to note that the expression  $-m \tan^{-1} \left( \frac{r \sin \theta}{r \cos \theta - f} \right), -\frac{m}{f} r \sin \theta, etc$  of this result are respectively the stream functions of the (flow) singularities  $\frac{m}{(z-f)}, \frac{m}{f}$  etc, referred to in the complex velocity (3.14).

(b) A source and a sink outside a circular cylinder.

Consider a source of strength  $\frac{1}{2}m$  at the point  $z = f$  and a sink of strength  $m$  at the point  $z = 2f$ . On the x-axis of the positive direction, in a viscous fluid, where  $f \in (a, \infty)$ . The complex potential of the primary flow in this case may be

given by

$$v_0(z) = \frac{\frac{1}{2}m}{(z-f)} - \frac{m}{(z-2f)} \quad (42)$$

Next it is simple to calculate  $v_0(z) \approx O(z)$  at the origin. Therefore, the circle theorem 1 is applied here; and thus from (35) the complex velocity for the Stokes flow past the circular cylinder  $|z| = a$ , on omitting the details of the algebraic reduction, becomes.

$$\begin{aligned} v(z, \bar{z}) = & \frac{m}{2(z-f)} - \frac{m}{z-2f} + \frac{ma^2}{2f^2} \left( \frac{1}{z - \frac{a^2}{f}} + \frac{\bar{z} - \frac{a^2}{f}}{(z - \frac{a^2}{f})^2} \right) \\ & - \frac{ma^2}{4f^2} \left( \frac{1}{z - \frac{a^2}{2f}} + \frac{\bar{z} - \frac{a^2}{2f}}{\left( \frac{z - \frac{a^2}{2f}}{2f} \right)^2} + \frac{1}{2} m \frac{1}{z} \right) \\ & + \frac{1}{2} \frac{m}{\left( z - \frac{a^2}{f} \right)} - \frac{m}{\left( z - \frac{a^2}{2f} \right)} - \frac{ma^2(f^2 - a^2)}{2f^3} \frac{1}{\left( z - \frac{a^2}{2f} \right)^2} + \frac{ma^2(4f^2 - a^2)}{8f^3} \frac{1}{\left( z - \frac{a^2}{2f} \right)^2} \end{aligned} \quad (43)$$

Where the last nine terms constitute the image system within the circular cylinder in the following manner.

A stresslet of strength  $\frac{ma^2}{f^2}$ , consisting of the third and fourth terms, at the point  $z = \frac{a^2}{f}$ .

Another stresslet of strength  $-\frac{ma^2}{2f^2}$ , consisting of the fifth

and sixth terms, at the point  $z = \frac{a^2}{2f}$ .

(i) A source of strength  $\frac{1}{2}m$  at the origin.

(ii) Another source of strength  $\frac{1}{2}m$  at the point  $z = \frac{a^2}{f}$ .

(iii) A sink of strength  $m$  at the point  $z = \frac{a^2}{2f}$ .

(iv) A doublet of strength  $ma^2 \frac{(f^2 - a^2)}{2f^3}$  at the point

$z = \frac{a^2}{2f}$  in the negative direction of the x-axis.

(v) Another doublet of strength  $ma^2 \frac{(4f^2 - a^2)}{af^3}$  at the

point  $z = \frac{a^2}{2f}$  in the positive direction of the x-axis.

(vi) The corresponding stream function  $\Psi$  for the Stokes flow, in the same way of getting expression (41) appear as

$$\begin{aligned} \Psi = & \frac{1}{2} m \tan^{-1} \frac{r \sin \theta}{(r \cos \theta - f)} + m \tan^{-1} \frac{r \sin \theta}{(r \cos \theta - 2f)} - \frac{ma^2}{2f^2} \frac{r^2 \sin 2\theta - 2r \left(\frac{a^2}{f}\right) \sin \theta}{r^2 - 2r \left(\frac{a^2}{f}\right) \cos \theta + \frac{a^4}{f^2}} \\ & + \frac{ma^2}{4f^2} \frac{r^2 \sin 2\theta - r \left(\frac{a^2}{f}\right) \sin \theta}{r^2 - r \left(\frac{a^2}{f}\right) \cos \theta + \frac{a^4}{4f^2}} - \frac{1}{2} m \theta - \frac{1}{2} m \tan^{-1} \frac{r \sin \theta}{\left(r \cos \theta - \frac{a^2}{f}\right)} + m \tan^{-1} \left( \frac{r \sin \theta}{r \cos \theta - \frac{a^2}{2f}} \right) \\ & + \frac{ma^2(f^2 - a^2)}{2f^3} \frac{r \sin \theta}{r^2 - 2r \left(\frac{a^2}{f}\right) \cos \theta + \frac{a^4}{f^2}} - \frac{ma^2(4f^2 - a^2)}{8f^3} \frac{r \sin \theta}{r^2 - r \left(\frac{a^2}{f}\right) \cos \theta + \frac{a^4}{4f^2}} \end{aligned}$$

## 4. Conclusion

In the present paper, we have found it much convenient to a study on Stokes flow past a circular cylinder in the light of complex variable theory and it is considerably convenient here. That is because the formulae for the same flow are mathematically concise and have “initial conditions” which are very simple. Therefore, it has been possible to establish a circle theorem for the flow with respect to the complex variable theory.

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