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Strict descriptions of some typical 2×2 games and negative games

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Abstract: In this paper, we shall give the strict description of some typical 2×2 games such as stag hunt, game of chicken, battle of the sexes, prisoner's dilemma (briefly PD) and Rasumsen's boxed pigs and we shall invent three new game stories, game of musicians, game of ruffians and negative Rasumsen boxed pigs. Every game is not necessarily symmetrical. We shall divide battle of the sexes into two forms and introduce the concept of weak PD. A game is said to be negative game of another one if the sum of their payoff functions is equal zero. The following main results will be obtained. (1) A negative game of chicken is a stage hunt. (2) A negative non-weak PD is a non-weak PD, where defection and cooperation of the original non-weak PD are corresponding to cooperation and defection of the negative non-weak PD respectively. The negative game of a weak PD is not a PD. (3) The negative first battle of the sexes is the game of musicians and the negative second one is the game of ruffians. (4) The negative Rasmusen boxed pigs is a weak PD and so is asymmetrical. (5) The guilt game as a negative Rasmusen boxed pigs is not a non-weak PD; it is a weak PD if and only if the small pig is conscienceless.

Keywords: Stag Hunt, Game of Chicken, Battle of the Sexes, Prisoner's Dilemma, Rasumsen's Boxed Pigs, Game of Musicians, Game of Ruffians, Negative Game

1. Introduction

The prisoner's dilemma, stag hunt, game of chicken, battle of the sexes, prisoner's dilemma (briefly PD) and Rasumsen's boxed pigs are the most famous examples of 2×2 bi-matrix games so that almost every book on game theory involves some of them.

Rousseau(1755) wrote a story about a group of hunters who wish to catch a stag. They will succeed if everyone remains sufficiently attentive, however everybody is tempted to desert his post and catch a hare. In order to simplify the multi-person game as the simplest form, i.e., a 2×2 game, it was simplified as follows. Two hunters can either jointly hunt a stag (an adult deer and rather large meal) or individually hunt a rabbit (tasty, but substantially less filling). Hunting stags is quite challenging and requires mutual cooperation. If either hunts a stag alone, the chance of success is minimal. Hunting stags is most beneficial for society but requires a lot of trust among its members. Game of chicken is also called hawk-dove game or snowdrift, its earliest presentation was given by Smith and Price (1973). Two drivers drive towards each other on a collision course: one must swerve, or both may die in the crash, but if one driver swerves and the other does not, the one who swerved will be called a "chicken," meaning a coward; this terminology is most prevalent in economics. Bach or Stravinsky (or Battle of the sexes, both are called BOS, an acronym) was given by Luce and Raiffa (1957), pp.90-91. Two people wish to go out together. Two concerts are available: one of music by Bach, and one of music by Stravinsky. One person prefers Bach and the other prefers Stravinsky. If they go to different concerts, each of them is equally unhappy listening to the music of other composer. As everybody knows, Dresher and Flood found a 2×2 bi-matrix game in which a pair of actions is strictly Parato-better than a unique strictly pure Nash equilibrium. In May of 1950, when Alber Tucker devised to introduce game theory to psychology department of Stanford University, he invented a most vivid story to describe Dresher and Flood's discovery. This story is called prisoner's dilemma (or briefly, PD) and is widely circulated. About these histories, we can see Rasmusen (2001, p61). Adam Smith's famous "invisible hand" said that under certain conditions independent rational choice will lead to a "good" allocation. The First Theorem of Welfare Economics proves the conjecture embedded in Adam

Smith's that. Thus it is the most important theorem in all of the social science. However the prisoner's dilemma demonstrates the existence of cases in which independent rational choice leads to a Parato- inefficient outcome. Rapoport & Chammah (1965)， Rapoport, Guyer and Gordon (1976) gave a generalization of the original one-shot Prisoner's dilemma. Ahn, Ostrom, Schmidt, Shupp and Walker.(2001), Ahn, Ostrom, and Walker.(2003), and Ahn, Lee, Ruttan and Walker (2007) researched the prisoner's dilemmas in which a player takes on an issue of conscience due to his defection when the another player takes cooperation. Rasmusen (1989) invented a story in his book on game theory called Boxed Pigs. Two pigs are put in a Skinner box with a special panel at one end and a food dispenser at the other. When a pig presses the panel at a utility cost of 2 units, 10 units of food are dispensed. One pig is "dominant" (let us assume he is bigger), and if he gets to the dispenser first, the other pig will only get his leavings, worth 1 unit. If, instead, the small pig arrives first, he eats 4 units, and even if they arrive at the same time, the small pig gets 3 units. In order to promote these to a systematic and scientific theory, an axiomatic theory, called L-system of boxed pigs, is established and some special subsystems are deduced from it by Jiang（2010-2015）et al. In chapter 11 of Jiang (2015), the author discussed the relation amount the game with two same pigs, prisoner's dilemma, and chicken game. In section 1.5 of Jiang (2015), with the title "Preliminary negative simple 0-peace system: the first open problem", the prototype of negative game was given.

In this paper, we shall give the strict descriptions of stag hunt, game of chicken, battle of the sexes, prisoner's dilemma (briefly PD) and Rasumsen's boxed pigs and shall invent three new game stories, game of musicians, game of ruffians and negative Rasumsen boxed pigs. We do not assume one of them is symmetrical. We shall divide battle of the sexes into two forms and shall introduce the concept of weak PD. A game is said to be negative game of another one if the sum of their payoff functions is equal zero. We shall obtain the following results. (1) A negative game of chicken is a stage hunt. (2) A negative non-weak PD is a non-weak PD, where defection and cooperation of the original non-weak PD are corresponding to cooperation and defection of the negative non-weak PD respectively. The negative game of a weak PD is not a PD. (3) The negative first battle of the sexes is the game of musicians and the negative second one is the game of ruffians. (4) The negative Rasmusen boxed pigs is a weak PD and so is asymmetrical. (5) The guilt game as a negative Rasmusen boxed pigs is not a non-weak PD; it is a weak PD if and only if the small pig is conscienceless.

2. Stag Hunt

Our one of duty is to generalize every typical 2×2 game. Specially, we need improve their symmetry as the more general form. Now let us replace "rabbit" by "non-stag" which can be one of wolf, fox, roe deer, hare and so on. The payoff matrix is rewritten as

$$\begin{array}{ccc} & & \text{Hunter 2} \\ & & \begin{array}{cc} \text{stag} & \text{non-stag} \end{array} \\ \text{Hunter 1} & \begin{array}{c} \text{stag} \\ \text{non-stag} \end{array} & \begin{bmatrix} (s_1, s_2) & (z_1, ns'_2) \\ (ns'_1, z_2) & (ns''_1, ns''_2) \end{bmatrix} \end{array}.$$

Where ns'_i stands for the hunter i's benefit when one hunts non-stag(s) and ns''_i when both do and $s_i \ge ns'_i$, $s_i > ns''_i > z_i$, $i = 1, 2$. Then one of the following four cases holds,

$$\begin{array}{cc} & \begin{array}{cc} \text{stag} & \text{non-stag} \end{array} \\ \begin{array}{c} \text{stag} \\ \text{non-stag} \end{array} & \begin{bmatrix} (\underline{s_1}, \underline{s_2}) & (z_1, \underline{ns'_2}) \\ (\underline{ns'_1}, z_2) & (\underline{ns''_1}, \underline{ns''_2}) \end{bmatrix} \end{array},$$

$$\begin{array}{cc} & \begin{array}{cc} \text{stag} & \text{non-stag} \end{array} \\ \begin{array}{c} \text{stag} \\ \text{non-stag} \end{array} & \begin{bmatrix} (\underline{s_1}, \underline{s_2}) & (z_1, ns'_2) \\ (\underline{ns'_1}, z_2) & (\underline{ns''_1}, \underline{ns''_2}) \end{bmatrix} \end{array},$$

$$\begin{array}{cc} & \begin{array}{cc} \text{stag} & \text{non-stag} \end{array} \\ \begin{array}{c} \text{stag} \\ \text{non-stag} \end{array} & \begin{bmatrix} (\underline{s_1}, \underline{s_2}) & (z_1, \underline{ns'_2}) \\ (ns'_1, z_2) & (\underline{ns''_1}, \underline{ns''_2}) \end{bmatrix} \end{array}, \text{ and}$$

$$\begin{array}{cc} & \begin{array}{cc} \text{stag} & \text{non-stag} \end{array} \\ \begin{array}{c} \text{stag} \\ \text{non-stag} \end{array} & \begin{bmatrix} (\underline{s_1}, \underline{s_2}) & (z_1, ns'_2) \\ (ns'_1, z_2) & (\underline{ns''_1}, \underline{ns''_2}) \end{bmatrix} \end{array}.$$

It can be clearly known that there are two pure Nash equilibria for every case-------either they cooperate to hunt a stag or they separate or cooperate to hunt non-stags. However the first choose is Pareto better than the second.

Assume $s_i < ns'_i$ for a hunter i. When the hunter j wants to hunt a stag, cooperation is the hunter i's bad choice. Thus $s_i \ge ns'_i$, $i = 1, 2$ is a necessary condition for they cooperate to hunt a stag.

Since $s_i \ge ns'_i$, $s_i > ns''_i > z_i$, let us divide the discussion into the following three cases according to the size of ns'_i .

Case (1). The "non-stag" stands for an animal can be easily hunted, such as hares. Since both hunting hares results in competitiveness, one hunter's income when he hunts hares alone is not less than his income when both do. That is $s_i \ge ns'_i \ge ns''_i$, $i = 1, 2$.

Subcase (1.1). $s_i = ns'_i > ns''_i$, $i = 1, 2$. The hares are so more that each hunter hunting hares alone can be equivalent to their cooperation to hunt a slag. But each hunting hares alone is better than both hunting hares.

Subcase(1.2). $s_i = ns'_i > ns''_i , s_j > ns'_j > ns''_j , i, j = 1, 2$. The number of hares is moderate. One hunter hunting hares alone can be equivalent to their cooperation to hunt a slag. Another hunting hares alone is not as good as their cooperation to hunt a slag. One hunting hares alone is better than both hunt hares.

Subcase (1.3). $s_i > ns'_i > ns''_i , i = 1, 2$. The hares are so less that one hunting hares alone is not as good as their cooperation to hunt a slag. One hunting hares alone is better than both hunt hares.

Subcase (1.4). $s_i = ns'_i = ns''_i$, $i = 1,2$. The hares are so more that each hunter hunting hares alone is equivalent to their cooperation to hunt a slag. And each hunter hunting hares alone is equivalent to both doing.

Subcase (1.5). $s_i = ns'_i = ns''_i$, $s_j > ns'_j = ns''_j$, $i, j = 1,2$. The number of hares is moderate. Either hunter hunting hares alone can be equivalent to their cooperation to hunt a slag. Another hunting hares alone is not as good as their cooperation to hunt a slag. Each hunter hunting hares alone can be equivalent to both hunting hares.

Subcase (1.6). $s_i > ns'_i = ns''_i$, $i = 1,2$. The hares are so less that each hunter hunting hares alone is not as good as their cooperation to hunt a slag. But one hunter hunting hares alone can be equivalent to both doing.

Case (2). The "non-slag" stands for sly animals such as foxes and so on. Each hunter hunting non-slag alone is not as good as their cooperation to hunt non-slag but is better than hunting a slag alone by him, i.e., $ns''_i > ns'_i > z_i$, $i = 1,2$.

Case (3). The "non-slag" stands for fierce animals. Hunting it or them by one is dangerous, i.e., $z_i \ge ns'_i$, $i = 1,2$.

Subcase (3.1). $z_i = ns'_i$, $i = 1,2$. The animals have a low danger. Two hunters hunting them separately is equivalent to hunting a slag by one.

Subcase (3.2). $z_i = ns'_i$, $z_j > ns'_j$, $i, j = 1,2$. The animals have a moderate danger. One hunting them alone is equivalent to hunting a slag alone by him and another one hunting them alone is not as good as hunting a slog alone by him (maybe he would be hurt).

Subcase (3.3). $z_i > ns'_i$, $i = 1,2$. The animals have a so great danger that anyone hunting them alone is not as good as hunting a slog alone by him (maybe he would be hurt).

3. Game of Chicken

A car driver, called player 1, and a motorcycle, player driver, 2, drive towards each other on a collision course. If only the motorcycle driver swerves, then the car driver is said to be a warrior which is equivalent to the payoff T_1, and the motorcycle driver is said to a coward which is equivalent to the payoff P_2. If only the car driver swerves, then the motorcycle driver is said to be a warrior which is equivalent to the payoff T_2, and the car driver is said to a coward which is equivalent to the payoff P_1. If both collide, the car driver's payoff is S_1 and the motorcycle driver's is S_2. However if both swerve, the car driver's payoff is R_1 and motorcycle driver's that is R_2. Then the payoff matrix of the game can be written as

$$\begin{array}{ccc} & & \text{Motorcycle} \\ & & \begin{array}{cc}\text{swerve} & \text{straight}\end{array} \\ \text{Car} & \begin{array}{c}\text{swerve}\\ \text{straight}\end{array} & \begin{bmatrix} (R_1, R_2) & (P_1, T_2) \\ (T_1, P_2) & (S_1, S_2) \end{bmatrix}, \end{array}$$

where $T_i > R_i > P_i > S_i$, $i = 1,2$. In particular, if $T_i = T$, $R_i = R$, $P_i = P$ and $S_i = S$, $i = 1,2$, the model degenerates into symmetrical form which is the traditional game of chicken. In this case, the motorcycle driver is a car driver, too.

4. Battle of the Sexes

According to the apart giving their different preference, we will divide Battle of the Sexes into two forms.

4.1. First Battle of the Sexes

A husband and wife agree to meet this evening, but cannot recall if they will be attending the opera or a football match. He prefers the football match and she prefers the opera, though both prefer being together to being apart (even if apart, each persists to end).

Theorem 4.1 The payoff matrix of the first battle of the sexes is

$$\begin{array}{ccc} & & \text{Wafe} \\ & & \begin{array}{cc}\text{football} & \text{opera}\end{array} \\ \text{Husband} & \begin{array}{c}\text{football}\\ \text{opera}\end{array} & \begin{bmatrix} (T_1, R_2) & (P_1, P_2) \\ (S_1, S_2) & (R_1, T_2) \end{bmatrix}, \end{array}$$

where $T_i > \max\{S_i, R_i\}$, $R_i > P_i$, $i = 1,2$; $P_1 > S_1$, $S_2 > P_2$.

Proof: Since both prefer being together to being apart, it can be obtained that $T_i > S_i$, and $R_i > P_i$, $i = 1,2$. Let us assume they have been at the same place. Then it implies the following cases. (1) he prefers the football match and she prefers the opera shows that he prefer being together at the football field to the opera house, (2) she prefer being together at the opera house to the football field. Formally, $T_i > R_i$, $i = 1,2$. Thus we obtain $T_i > \max\{S_i, R_i\}$ and $R_i > P_i$, $i = 1,2$. Now let them be apart, then one of the pairs (P_1, P_2), preferred by him, and (S_1, S_2), preferred by her, appears. We so obtain $P_1 > S_1$ and $S_2 > P_2$. Conversely, it can be proved that the inequalities can guarantee that the game is a battle of the sexes. Q.E.D.

4.2. Second Battle of the Sexes

A husband and wife agree to meet this evening, but cannot recall if they will be attending the opera or a football match. Both prefer being together and they think that apart is the worst result and each leaves. If they are together, then he prefers the football match and she prefers the opera.

Note Battles of the Sexes in most literatures are symmetrical and are written as the simplest form

$$\begin{bmatrix} (2,1) & (0,0) \\ (0,0) & (1,2) \end{bmatrix}.$$

Theorem 4.2 The payoff matrix of the second battle of the sexes is

$$\begin{array}{ccc} & & \text{Wafe} & \\ & & \text{football} & \text{opera} \\ \text{Husband} & \begin{array}{c}\text{football}\\ \text{opera}\end{array} & \multicolumn{2}{c}{\begin{bmatrix} (T_1,R_2) & (0,0) \\ (0,0) & (R_1,T_2) \end{bmatrix}} \end{array},$$

$$T_i > R_i > 0 \,, i=1,2 \,.$$

Proof: Since they think that apart is the worst result, we can write $P_i = S_i = 0$, $i=1,2$. By imitating proof of the first one, the proof can be obtained. Q.E.D.

5. Game of Musicians and Game of Ruffians

In this section, we will invent two new stories to describe two 2×2 games.

5.1. Game of Musicians

Two musicians Smith and Jones, who can only play a violin and a piccolo, were independently invited to play a Beethoven's music together. Each was allowed to play one musical instrument and they can not discuss each other before playing. Each would get better evaluation if he plays a musical instrument differing from another. If both play the same musical instruments, then Smith would get worse evaluation and Jones better when they do violins instead of piccolos. If both play the different musical instruments, then the piccolo player would get better evaluation than the violin player.

Theorem 5.1 The musician game can be denoted as

$$\begin{array}{ccc} & & \text{Jones} & \\ & & \text{violin} & \text{piccolo} \\ \text{Smith} & \begin{array}{c}\text{violin}\\ \text{piccolo}\end{array} & \multicolumn{2}{c}{\begin{bmatrix} (t_1,r_2) & (\underline{p_1},\underline{p_2}) \\ (\underline{s_1},\underline{s_2}) & (r_1,t_2) \end{bmatrix}} \end{array}, t_i < \min\{s_i, r_i\}\,, r_i < p_i\,,$$

$$i=1,2\,;\ \ p_1 < s_1 , s_2 < p_2 \,.$$

Proof: Smith and Jones are said to be player 1 and player 2 respectively. It can be expanded to describe as follows that each would get better evaluation if he plays a musical instrument differing from another. (1) When Jones plays his violin, Smith playing piccolo is better than violin, i.e., $t_1 < s_1$. (2) When Jones plays his piccolo, Smith playing violin is better than piccolo, i.e., $r_1 < p_1$. Similarly, we can obtain that $t_2 < s_2$ and $r_2 < p_2$. We have known that if both play the same musical instruments, then Smith would get worse evaluation and Jones better when they do violins instead of piccolos. Thus we obtain that $t_i < r_i$, $i=1,2$. Finally, it implies the inequalities $p_1 < s_1$ and $s_2 < p_2$ that if both play the different musical instruments, then the piccolo player would get better evaluation than the violin player. Q.E.D.

Nash equilibrium for this game is that they will play the different musical instruments.

If the conditions $p_1 < s_1$ and $s_2 < p_2$ are changed into $p_i = s_i = 0$, $i=1,2$, then the game of musicians can be explained as the following game of ruffians.

5.2. Game of Ruffians

Two gangs of local ruffians, Jones's and Smith's, want to have a fight and the two battle sites, A and B, can be chosen. However the site A is favorable to Jones and B to Smith. If they choose the same place, then both sides suffer; if not, each gains nothing. It is clear that the game can be written as

$$\begin{array}{ccc} & & \text{Jones} & \\ & & \text{A} & \text{B} \\ \text{Smith} & \begin{array}{c}\text{A}\\ \text{B}\end{array} & \multicolumn{2}{c}{\begin{bmatrix} (t_1,r_2) & (\underline{0},\underline{0}) \\ (\underline{0},\underline{0}) & (r_1,t_2) \end{bmatrix}} \end{array}, \ t_i < r_i < 0 \,, i=1,2 \,.$$

Nash equilibrium for this game is that they will choose the different places.

6. Prisoner's Dilemma

In this section, based on Predecessors' results, we shall give out the concept of weak Prisoner's Dilemma. We shall also introduce Prisoner's Dilemma with guilt and research the relation among them. In order to distinguish the different types of these PD, we want to use the first letters of the relative authors' names to name these PDs.

6.1. RGG Prisoner's Dilemma

According to Rapoport, Guyer, & Gordon (1976) p.62, a prisoner's dilemma can be written as the following form

$$\begin{array}{cc} & \text{defection} \quad \text{operation} \\ \begin{array}{c}\text{defection}\\ \text{operation}\end{array} & \begin{bmatrix} (\underline{P},\underline{P}) & (\underline{T},S) \\ (S,\underline{T}) & (R,R) \end{bmatrix} \end{array} . \tag{6-1}$$

Where T means Temptation and it stands for a defector's payment due to betraying the cooperator; R implies Revolt and it stands for each player's payment when both cooperate to revolt against the temptation; P is for Punishment, i.e., the payment received by one of a pair of defectors; and S is the sucker's payoff received by a cooperator paired with a defector. They require to satisfy the relation *T> R > P> S*. We call this game to be a RGG prisoner's dilemma. It is clear that a RGG prisoner's dilemma is symmetrical.

6.2. RC Prisoner's Dilemma

Rapoport & Chammah (1965) gave a generalization of a non-symmetrical prisoner's dilemma.

$$\begin{array}{cc} & \text{defection} \quad \text{cooperation} \\ \begin{array}{c}\text{defection}\\ \text{cooperation}\end{array} & \begin{bmatrix} (\underline{P_1},\underline{P_2}) & (\underline{T_1},S_2) \\ (S_1,\underline{T_2}) & (R_1,R_2) \end{bmatrix} \end{array} . \tag{6-2}$$

Where T_i , R_i , P_i and S_i with $S_i < P_i < R_i < T_i$ stand for the four objective payoffs for player i whose meanings are same to that of RGG prisoner's dilemma. We call it to be a RC prisoner's dilemma. In Raporport and Chammah (1965), Ahn, Ostrom, Schmidt, Shupp and Walker (2001), and Ahn, Ostrom, and Walker (2003), the three differences $R_i - P_i$, $T_i - R_i$ and $P_i - S_i$ are said to be the cooperator *i*'s gain, greed and fear, respectively.

It is obvious that RC prisoner's dilemma is degenerated into RGG prisoner's dilemma when

$T_1 = T_2 = T$, $R_1 = R_2 = R$, $P_1 = P_2 = P$, and $S_1 = S_2 = S$.

6.3. Weak Prisoner's Dilemma

Theorem 6.1 The game to satisfy the condition $R_i > P_i > S_i$, $i = 1,2$

$$\begin{array}{cc} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} (P_1, P_2) & (T_1, S_2) \\ (S_1, T_2) & (R_1, R_2) \end{bmatrix} \end{array}, \quad (6\text{-}3)$$

either is in a dilemma or is a stag hunt, and it is a Stag Hunt if and only if the condition $T_i \le R_i, i = 1,2$ is satisfied.

Proof: Necessity Suppose it is false that $T_i \le R_i$, $i = 1,2$. Let us divide the discussion into the following cases:

Case 1. If $T_i > R_i$, $i = 1,2$. By Raporport & Chammah (1965), this game is a prisoner's dilemma.

Case 2. Let $T_2 > R_2$ and $T_1 \le R_1$. (1) When the player 1 judges that the player 2 will use the action 0, the player 1 will use his or her action 0 because $P_1 > S_1$. (2) When the player 1 judges that the player 2 will use the action 1, the player 1 will use the action 1 because $T_1 \le R_1$. In other words, the two players will use their actions with the same number.

However what action will the player 2 use? (1) When the player 2 judges that the player 1 will use the action 0, he or she will use his or her action 0 because $P_2 > S_2$. (2) When the player 2 judges that the player 1 will use the action 1, he or she will use the action 0 because $T_2 > R_2$. In a nutshell, the player 2 will use the action 0 whether the player 2 judges that the player 1 will use the action 0 or 1.

Since the two players will use their actions with the same number, the player 2 will use the action 0 as well. It is obvious that the pure situation (1,1) is strictly Parato-better than the pure situation (0,0), thus the two players will be in a dilemma.

Case 3. Let $T_1 > R_1$ and $T_2 \le R_2$. By imitating the case 2, the two players will be in a dilemma as well.

Sufficiency. Let $T_i \le R_i$, $i = 1,2$ and let us write $R_i = s_i$, $T_i = ns'_i$, $P_i = ns''_i$, and $S_i = z_i$, $i = 1,2$. Since $R_i > P_i > S_i$, $i = 1,2$, we have $s_i \ge ns'_i$, $s_i > ns''_i > z_i$, $i = 1,2$. This shows that (6-3) is a Stag Hunt. Q.E.D.

By this theorem, we can divide a one-shot prisoner's dilemma into the two types.

The first type (RC prisoner's dilemma): It satisfies the condition $T_i > R_i > P_i > S_i$, $i = 1,2$.

The second type (weak prisoner's dilemma): It satisfies the condition $R_i > P_i > S_i$, $i = 1,2$; $T_i > R_i$, $T_j \le R_j$, $i, j = 1,2$.

It is clear that RC prisoner's dilemma and weak prisoner's dilemma are incompatible. However every one of them is generally said to be a prisoner's dilemma.

Theorem 6.2 For a 2×2 bi-matrix game (6-3), we have the conclusions: (1) It is a weak prisoner's dilemma if and only if it satisfies the three conditions: (1.1) there exists one and only one strictly pure Nash equilibrium, without lose of generality, let it be (0,0); (1.2) the situation (1,1) is strictly better than (0,0); and (1.3) there exists one player $i \in \{1,2\}$ such that $T_i > R_i$. (2) It is a RC prisoner's dilemma if and only if it satisfies the following three conditions: (2.1) there exists one and only one strictly pure Nash equilibrium, without lose of generality, let it be (0,0); (2.2) the situation (1,1) is strictly better than (0,0); and (2.3) $T_i > R_i$, $i = 1,2$.

Proof: We can sufficiently prove that the conditions (1.1) and (1.2) of this theorem are equivalent to the inequalities $R_i > P_i > S_i$, $i = 1,2$.

Suppose the conditions (1.1) and (1.2) of this theorem are true. By (1.1), the situation (0,0) is a unique strictly pure Nash equilibrium. When the player 2 judges that the player 1 will use his action 0, the player 2 had better to use his action 0, i.e., $P_2 > S_2$. Similarly, $P_1 > S_1$. By (1.2), the situation (1,1) is strictly better than (0,0), we obtain that $R_i > P_i$, $i = 1,2$.

Conversely, let $R_i > P_i > S_i$, $i = 1,2$. By that $P_i > S_i$, $i = 1,2$, the situation (0,0) is a unique strictly pure Nash equilibrium. This proves (1.1). By that $R_i > P_i$, $i = 1,2$, the situation (1,1) is strictly better than (0,0). This proves (1.2). Q.E.D.

In this theorem, the actions 0 are called defection and 1 are called cooperation.

Theorem 6.3 A weak prisoner's dilemma is asymmetrical.

Proof: Without lose of generality, let us suppose $T_1 > R_1$ and $T_2 \le R_2$. Assume that the prisoner's dilemma (6-3) is symmetrical. We can obtain that $T_1 > R_1 = R = R_2 \ge T_2$, a contradiction. Q.E.D.

6.4. Prisoner's Dilemma with Guilt

Ahn, Lee, Ruttan,and Walker (2007) supposed payers experience guilt when defecting on someone who cooperates. In other words, when a player defects a cooperator, he or she will feel guilt, which is an issue of conscience.

Let $g_i (0 \le g_i < T_i - R_i)$ denote the magnitude of guilt (payoff lose) the player i incurs when the player j takes cooperation, which can be understood as a "cost". In the game (6-3), let us replace the player i's temptation T_i by his or her "profit" $T_i - g_i$ and so obtain a new prisoner's dilemma with guilt

$$\begin{array}{cc} & \begin{array}{cc} \text{defection} & \text{cooperation} \end{array} \\ \begin{array}{c} \text{defection} \\ \text{cooperation} \end{array} & \begin{bmatrix} (P_1, P_2) & (T_1 - g_1, S_2) \\ (S_1, T_2 - g_2) & (R_1, R_2) \end{bmatrix} \end{array}. \quad (6\text{-}4)$$

We call (6-4) to be an ALRW guilt game of the prisoner's dilemma (6-3). Ahn, Lee, Ruttan, and Walker (2007) regarded a player without guilt as an egoist.

Because $R_i < T_i - g_i$ if and only if $0 \le g_i < T_i - R_i$, we obtain easily the theorem.

Theorem 6.4 For an ALRW guilt game, we have that (1) it is a RC prisoner's dilemma if and only if $0 \le g_i < T_i - R_i, i = 1,2$; *(2) it is a weak prisoner's dilemma if and only if* $0 \le g_i < T_i - R_i$ *and* $g_j \ge T_j - R_j$; *and (3) it is not a stag hunt if and only if* $g_i \ge T_i - R_i$, $i = 1,2$.

Theorem 6.4 tells us that an ALRW guilt game is a stronger prisoner's dilemma if and only if the two players' defecting guilt degrees are lower; it is weak prisoner's dilemma if and only if one of the two players has lower defecting guilt degree; it is a stag hunt if and only if each player has higher defecting guilt degree. Thus an ALRW guilt game is a prisoner's dilemma if and only if at least one player has lower defecting guilt degree.

7. Rasmusen Boxed Pigs

We have given Rasmusen's story about boxed pigs in introduction. Now let us define a generalization of the story as follows.

Two pigs, one big pig and one small pig, are put in a box with a special panel at one end and a food dispenser at the other. When a pig presses the panel at a utility cost of c units, q units of food are dispensed. If the big pig gets to the dispenser first, he will eat s units. If, instead, the small pig arrives first, the big pig can eat b units, and even if they arrive at the same time, the big pig gets t units. Where the parameters satisfy the inequalities

$$q - s < c < b < t < \min\{s, t + c\} < q\,. \qquad (7\text{-}1)$$

The game can be written as

$$\begin{array}{ccc} & & \text{Small} \quad\quad \text{Pig} \\ & & \begin{array}{cc} \text{Press} & \text{Wait} \end{array} \\ \begin{array}{c}\text{Big} \\ \text{Pig}\end{array} & \begin{array}{c}\text{Press} \\ \text{Wait}\end{array} & \begin{bmatrix} (t-c, q-t-c) & (b-c, q-b) \\ (s, q-s-c) & (0,0) \end{bmatrix} \end{array}. \qquad (7\text{-}2)$$

In Rasmusen's story, we have that $c = 2$, $b = 6$, $t = 7$, $s = 9$, and $q = 10$. They satisfy obviously the inequalities (7-1).

8. Negative Games of the above Games

Definition 8.1 A formal game $-\Gamma \equiv [N, (A_i), (-u_i)]$ is called the negative game of the formal game $\Gamma \equiv [N, (A_i), (u_i)]$.

8.1. The Negative Game of Chicken

Theorem 8.1 A negative game of chicken is a stage hunt.

Proof: The payoff matrix of a negative game of chicken is

$$\begin{bmatrix} (-R_1, -R_2) & (-P_1, -T_2) \\ (-T_1, -P_2) & (-S_1, -S_2) \end{bmatrix}.$$

Let $z_i = -T_i$, $r_i{''} = -R_i$, $r_i{'} = -P_i$ and $s_i = -S_i$. By substituting them into the above matrix and commuting the rows and columns, we can obtain that

$$\begin{bmatrix} (\underline{s_1}, \underline{s_2}) & (z_1, r'_2) \\ (r'_1, z_2) & (\underline{r''_1}, \underline{r''_2}) \end{bmatrix}, s_i > r'_i > r''_i > z_i, i = 1,2\,.$$

It is exactly a stag hunt. Q.E.D.

8.2. The Negative Game of a Prisoner's Dilemma

Theorem 8.2 A negative RC prisoner's dilemma is a RC prisoner's dilemma, where defection and cooperation of the original RCPD are corresponding to cooperation and defection of the negative RCPD respectively. The negative game of a weak prisoner's dilemma is not a prisoner's dilemma.

Proof: The negative game of the RCPD

$$\begin{array}{cc} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} (P_1, P_2) & (T_1, S_2) \\ (S_1, T_2) & (R_1, R_2) \end{bmatrix} \end{array},$$

where $T_i > R_i > P_i > S_i, i = 1,2$ and 0 stands for defection and 1 stands for cooperation, is

$$\begin{array}{cc} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} (-P_1, -P_2) & (-T_1, -S_2) \\ (-S_1, -T_2) & (-R_1, -R_2) \end{bmatrix} \end{array}.$$

Let $s_i = -T_i$, $p_i = -R_i$, $r_i = -P_i$, and $t_i = -S_i$. By substituting them into the above matrix and commuting their rows and columns, we can obtain an isomorphic game

$$\begin{array}{cc} & \begin{array}{cc} 1 & 0 \end{array} \\ \begin{array}{c} 1 \\ 0 \end{array} & \begin{bmatrix} (p_1, p_2) & (t_1, s_2) \\ (s_1, t_2) & (r_1, r_2) \end{bmatrix} \end{array}, \text{where } t_i > r_i > p_i > s_i, i = 1,2\,.$$

In this case, 1 stands for defection and 0 stands for cooperation.

For a weak PD, we have

$$\begin{array}{cc} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} (P_1, P_2) & (T_1, S_2) \\ (S_1, T_2) & (R_1, R_2) \end{bmatrix} \end{array}, R_i > P_i > S_i, i = 1,2; T_i > R_i, T_j \le R_j,$$
$$i, j = 1,2\,.$$

Without loss of generality, let $R_i > P_i > S_i, i = 1,2$; $T_1 > R_1$ and $T_2 \le R_2$. If let $z_i = -T_i$, $r_i{''} = -R_i$, $r_i{'} = -P_i$ and $s_i = -S_i$, then the negative game of the above game can be written as

$$\begin{array}{cc} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} (p_1,p_2) & (t_1,s_2) \\ (s_1,t_2) & (r_1,r_2) \end{bmatrix} \end{array}, \; r_i < p_i < s_i, i=1,2;\, t_1 < r_1, t_2 \geq r_2 .$$

By scribing method, we obtain either

$$\begin{array}{cc} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} (p_1,p_2) & (t_1,\underline{s_2}) \\ (\underline{s_1},\underline{t_2}) & (\underline{r_1},r_2) \end{bmatrix} \end{array}, \text{ or } \begin{array}{cc} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} (p_1,p_2) & (t_1,\underline{s_2}) \\ (\underline{s_1},t_2) & (\underline{r_1},\underline{r_2}) \end{bmatrix} \end{array}.$$

The former has a unique pure Nash equilibrium (1,0) which is not Pareto better than (0,1) because $t_1 < r_1 < s_1$. This shows that the negative game is not a prisoner's dilemma. The latter is not a PD neither because it has no pure Nash equilibrium. Q.E.D.

It should be pointed out that it is not necessary that the negative game of a RC prisoner's dilemma can be explained as a story about two prisoners though its name is a prisoner's dilemma.

For example, when $T_i = 0$, $R_i = -1$, $P_i = -8$ and $S_i = -10$, the following game is a traditional prisoner's dilemma

$$\begin{array}{cc} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} (\underline{-8},\underline{-8}) & (\underline{0},-10) \\ (-10,\underline{0}) & (-1,-1) \end{bmatrix} \end{array}.$$

However it obvious that the negative game of the prisoner's dilemma

$$\begin{array}{cc} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} (8,8) & (0,\underline{10}) \\ (\underline{10},0) & (\underline{1},\underline{1}) \end{bmatrix} \end{array}$$

can not be explained by a story about two prisoners.

By imitating Roger(2004), we can write the following story to explain it.

The two tobacco companies Fumco and Tabacs are facing two strategy choices: advertising and not doing. The advertising cost is 7 units. If neither of them advertises, each can get 8 units of profit. If both do, each gets 1 unit. If one of them does, then the company to do gets 10 units and the other one has no profit. Where 1 stands for advertisement and 0 stands for not. The pure Nash equilibrium is that each doing is Pareto bad than neither doing.

8.3. Negative Games of two Battles of the Sexes

Theorem 8.3 The negative game of First Battle of the sexes is the Game of Musicians and the negative game of Second Battle of the sexes is the Game of Ruffians.

Proof: Let $-T_i = t_i$, $-R_i = r_i$, $-P_i = p_i$ and $-S_i = s_i$, $i = 1,2$. Then the negative games of first and second Battles of the Sexes are

$$\begin{bmatrix} (t_1,r_2) & (\underline{p_1},\underline{p_2}) \\ (\underline{s_1},\underline{s_2}) & (r_1,t_2) \end{bmatrix}, \; t_i < \min\{s_i, r_i\}, r_i < p_i, i=1,2;$$

$$p_1 < s_1, s_2 < p_2 .$$

$$\begin{bmatrix} (t_1,r_2) & (\underline{0},\underline{0}) \\ (\underline{0},\underline{0}) & (r_1,t_2) \end{bmatrix}, \; t_i < r_i < 0, i=1,2,$$

respectively, as desired. Q.E.D.

8.4. Negative Game of Rasmusen Boxed Pigs

Model of negative Rasmusen Boxed Pigs: A big pig and a small pig are put in separate mesh cages. There is a trough in each cage, and c units of pig food are in each trough. The two cages are in a closed room. Two spray nozzles are installed on the two walls which are close to the cages. The third spray nozzle is installed at the center of the room. When the small pig eats his food, q units of poisonous gas erupt from the spray nozzle close to the big pig. When the big pig eats his food, q units of poisonous gas erupt from the spray nozzle close to the small pig. When the two pigs eat their food at the same time, q units of poisonous gas erupt from the spray nozzle at the center of the room. The quantity of the poisonous gas inhaled by the big pig are b units, t units, and s units, respectively, when the big eats his food alone, the two pigs eat their food at the same time, and the small pig eats his food alone. We assume units of pig food and units of the poisonous gas are converted into an appropriate equivalent and $q - s < c < b < t < \min\{s, t + c\} < q$. Then payoff matrix of the game is

$$\begin{array}{ccc} & & \text{Small Pig} \\ & & \begin{array}{cc} \text{Press} & \text{Wait} \end{array} \\ \text{Big Pig} & \begin{array}{c} \text{Press} \\ \text{Wait} \end{array} & \begin{bmatrix} (c-t, c-q+t) & (c-b, -q+b) \\ (-s, c-q+s) & (0,0) \end{bmatrix} \end{array}. \quad (8\text{-}1)$$

It is clear that the game (8-1) is a negative game of the game (7-2). This is reason of the name Negative Rasmusen Boxed Pigs.

Theorem 8.4 The negative Rasmusen boxed pigs is a weak prisoner's dilemma and so it is an asymmetrical prisoner's dilemma.

Proof: In the negative Rasmusen boxed pigs, let

$$T_1 = c - b,\ R_1 = 0,\ P_1 = c - t,\ S_1 = -s;$$

$$T_2 = s + c - q,\ R_2 = 0,\ P_2 = t + c - q, \text{and } S_2 = b - q.$$

By (8-1), it can be verified easily that $R_i > P_i > S_i$, $i = 1,2$ and $T_1 < R_1$ and $T_2 > R_2$. This proves that the Rasmusen boxed pigs is a weak prisoner's dilemma. Q.E.D.

Theorem 8.5 The ALRW guilt game as a negative Rasmusen boxed pigs is not a RC prisoner's dilemma; it is a weak prisoner's dilemma if and only if $0 \leq g_2 < s + c - q$.

Proof: Let us make an ALRW guilt game of the Rasmusen boxed pigs (7-2) as follows.

$$\begin{array}{cc} & \text{Small Pig} \\ & \begin{array}{cc} \text{Press} & \text{Wait} \end{array} \\ \begin{array}{c} \text{Big} \\ \text{Pig} \end{array} \begin{array}{c} \text{Press} \\ \text{Wait} \end{array} & \begin{bmatrix} (c-t, c-q+t) & (c-b-g_1, -q+b) \\ (-s, c-q+s-g_2) & (0,0) \end{bmatrix} \end{array}. \quad (8\text{-}2)$$

By proof of theorem 8.4, we have $R_i > P_i > S_i$, $i = 1, 2$. By $c < b$ we obtain $g_1 \geq 0 > (c-b) \quad -0 = T_1 - R_1$, i.e., $T_1 - g_1 < R_1$. This proves that the ALRW guilt game of the Rasmusen boxed pigs is not a RC prisoner's dilemma.

Since $c > q - s$ is equivalent to $s + c - q > 0$, the ALRW guilt game of the Rasmusen boxed pigs is a weak prisoner's dilemma if and only if $R_2 < T_2 - g_2$, i.e., $0 \leq g_2 < T_2 - R_2 = s + c - q$. Q.E.D.

Theorem 8.5 tells us that it is dependent on the small pig whether the ALRW guilt game of given negative Rasumsen boxed pigs is a prisoner's dilemma or not. If the small pig is conscienceless, i.e., $g_2 < s + c - q$, then the ALRW guilt game is in a dilemma; but if the small pig has his or her conscience, i.e., $g_2 \geq s + c - q$, then the ALRW guilt game should not be in a dilemma, i.e., the two pigs should take cooperation.

Example There are two vegetable families, a strong one and a weak one, in a village that is famous for its pollution-free vegetable. If one of them secretly uses chemical fertilizers or pesticides to increase production further, then this family should increase 2 units of profits，but the village will lose 10 units because of the damage of the reputation. If only the strong one secretly does, then it should lose 6 units; if only the weak does, then it should lose 1 unit; if both, then the strong one should lose 7 units. (1) What is the result of the game if conscience is not considered? (2) What is the result of the game if conscience is considered?

Solution: (1) If conscience is not considered, by that $c = 2$, $b = 6$, $t = 7$, $s = 9$ and $q = 10$, the condition (7-1) is satisfied. This shows that the game is a negative Rasmusen boxed pigs. By theorem 8.4, the game is a weak prisoner's dilemma. In other words, each family secretly uses chemical fertilizers or pesticides. The result is that the reputation of the village should be destroyed by their own.

(2) If conscience is considered, by theorem 8.5, the ALRW guilt game is weak prisoner's dilemma if and only if the weak one's defecting guilt degree $g_2 < s + c - q = 1$. In other words, it is dependent on the weak one's conscience idea whether the game is a prisoner's dilemma or not. If its conscience idea is weaker than 1 unit, the reputation of the village should be destroyed; if not, the reputation of the village should be kept.

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