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Some Bounds of the Largest *H*-eigenvalue of *R*-uniform Hypergraphs

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Abstract: The spectral theory of graphs and hypergraphs is an active and important research field in graph and hypergraph theory. And it has extensive applications in the fields of computer science, communication networks, information science, statistical mechanics and quantum chemistry, etc. The *H*-eigenvalues of a hypergraph are its *H*-eigenvalues of adjacent tensor. This paper presents some upper and lower bounds on the largest *H*-eigenvalue of *r*-hypergraphs.

Keywords: *H*-eigenvalue, Hypergraph, Adjacency Tensor, Bounds

1. Introduction

A hypergraph H is an ordered pair (V, E), where V is a set of elements referred as vertices and E is a family of subsets of V called edges. Hypergraph H is said to be *r*-uniform for an integer $r \geq 2$ if $|e| = r$ for all $e \in E(H)$. For convenience, refer *r*-uniform hypergraphs to *r*-graphs, and thus a 2-graph means a graph.

The spectral theory of graphs and hypergraphs is an active and important research field in graph and hypergraph theory. And it has extensive applications in the fields of computer science, communication networks, information science, statistical mechanics and quantum chemistry, etc. It studies graphs by using algebraic properties of associated matrices [1, 2, 4, 5], such as the adjacency matrix and the Laplacian matrix, and has become one of the most active branches in the graph theory. In order to generalize spectral techniques to hypergraphs, an important tool–tensor [8-10] is used, which could reveal more higher order structures than matrices. There have been attempts in the literature to define eigenvalues for hypergraphs and study their properties [6-8, 11, 14, 16].

Most of this work concerns generalizations of the adjacency, Laplacian, or signless Laplacian spectrum of graphs. J. Cooper and A. Dutle [3] obtain a number of results closely paralleling results from classical spectral graph theory, including bounds on the largest eigenvalue, a spectral bound on the chromatic number, a sub-hypergraph counting description of the coefficients of the characteristic polynomial, and the spectrum for some natural hypergraph classes and operations.

This paper presents some upper and lower bounds on the largest *H*-eigenvalue of *r*-hypergraphs.

2. Preliminaries

In this section, the definition of eigenvalues of tensors and some basic concepts associated uniform hypergraphs are presented. Denote the set $\{1, \cdots, n\}$ by $[n]$.

Definition 2.1 [3]. A tensor $\mathcal{A}$ over a set $\mathbb{S}$ of dimension n and order r is a collection of n^r elements $a_{i_1 i_2 \cdots i_r} \in \mathbb{S}$, where $i_j \in [n]$.

Let $\mathbb{S} = \mathbb{C}$ for the following discussion.

Definition 2.2 [3]. A tensor is said to be symmetric if entries which use the same index sets are the same. That is, $\mathcal{A}$ is symmetric if $a_{i_1 i_2 \cdots i_r} = a_{i_{\sigma(1)} i_{\sigma(2)} \cdots i_{\sigma(r)}}$ for all $\sigma \in \mathcal{G}_r$, where $\mathcal{G}_r$ is the symmetric group on $[r]$.

In the case of graphs, i.e., $r = 2$, tensors are simply square matrices, and symmetric tensors are just symmetric matrices.

A real order r dimensional n tensor $\mathcal{A}$ uniquely defines homogeneous degree r polynomial in n variables by

$$F_{\mathcal{A}}(\mathrm{x}) = \sum_{i_1,\, i_2,\cdots,i_r=1}^{n} a_{i_1 i_2 \cdots i_r} x_{i_1} x_{i_2} \cdots x_{i_r}. \quad (1)$$

Definition 2.3 [8]. Call $\lambda \in \mathbb{C}$ an eigenvalue of $\mathcal{A}$ if there is a non-zero vector $\mathrm{x} \in \mathbb{C}^n$, which is an eigenvector, satisfying

$$\sum_{i_2,\cdots,i_r=1}^{n} a_{j i_2 \cdots i_r} x_{i_2} \cdots x_{i_r} = \lambda x_j^{r-1}, \quad (2)$$

for all $j \in [n]$. When λ and x are all real, then x is called a H-eigenvector associated with the H-eigenvalue λ.

Assume x^r is the order r dimension n tensor with entry $x_{i_1} x_{i_2} \cdots x_{i_r}$ and $\mathrm{x}^{[r]}$ is the vector with $i-$th entry x_i^r. Then the expressions above can be written rather succinctly. Equation (1) is equivalent to

$$F_{\mathcal{A}}(\mathrm{x}) = \mathcal{A}\mathrm{x}^r, \quad (3)$$

where multiplication is taken to be tensor contraction over all indices. Similarly, the eigenvalue equations (2) can be written as

$$\mathcal{A}\mathrm{x}^{r-1} = \lambda \mathrm{x}^{r-1}, \quad (4)$$

where contraction is taken over all but the first index of $\mathcal{A}$.

Definition 2.4 [3]. For an r-graph H on n labeled vertices, the adjacency tensor $\mathcal{A}_H$ is the order r dimension n tensor with entries

$$a_{i_1 i_2 \cdots i_r} = \begin{cases} \dfrac{1}{(r-1)!} \; if \; \{i_1,\, i_2, \cdots, i_r\} \in E(G), \\ 0 \text{ otherwise.} \end{cases}$$

Denote the monomial $x_{i_1} x_{i_2} \cdots x_{i_r}$ by x^e, where $e = \{i_{1,} i_2 \cdots, i_r\}$ is an edge of an r-graph. Recall that the link of a vertex i in H, denoted $H(i)$, is the $(r-1)$-graph whose edges are obtained by removing vertex i from each edge of H containing i. That is, $E(H(i)) = \{e \backslash \{i\} | i \in e \in E(H)\}$ and $V(H(i)) = \cup E(H(i))$. Then

$$F_H(\mathrm{x}) = \sum_{e \in H} r x^e, \quad (5)$$

and the eigenvalue equations (2) become

$$\sum_{e \in H(i)} x^e = \lambda x_i^{r-1}, \quad (6)$$

for all $i \in V(H)$.

Definition 2.5 [11]. Let $G = (V, E)$ be an r-uniform hypergraph. If there is a disjoint partition of the vertex set V as $V = V_0 \cup V_1 \cup \cdots V_d$ such that $|V_0| = 1$ and $|V_1| = \cdots = |V_d| = r - 1$, and $E = \{V_0 \cup V_i | i \in [d]\}$, then G is called a hyperstar. The degree d of the vertex in V_0, which is called the heart, is the size of the hyperstar. The edges of G are leaves, and the vertices other than the heart are vertices of leaves.

Definition 2.6 [12]. Let $\mathcal{A}$ and $\mathcal{B}$ be two order m dimension n tensors. $\mathcal{A}$ and $\mathcal{B}$ are diagonal similar, if there exists some invertible diagonal matrix D of order n such that $\mathcal{B} = D^{-(m-1)} \mathcal{A} D$.

Theorem 2.7 [12]. If the two order m dimension n tensors $\mathcal{A}$ and $\mathcal{B}$ are diagonal similar, Then $F_{\mathcal{B}}(\lambda) = F_{\mathcal{A}}(\lambda)$.

3. Main Results

The main result in this section is as follow.

Theorem 3.1. Let $G = (V, E)$ be a connected $r-$graph with vertex set $[n]$. Let $\mathcal{A}$ be its adjacency tensor and λ_1 be the largest H-eigenvalue of $\mathcal{A}(G)$. Then

$$\lambda_1 \le \max_{\{i_1,\, i_2, \cdots, i_r\} \in E(G)} \sqrt[r]{d_{i1} d_{i2} \cdots d_{ir}} \quad (7)$$

Proof. Let $\mathrm{x} = (x_1, x_2, \cdots, x_n)$ be a positive eigenvector of eigenvalue λ_1. Suppose $\{1,2,\cdots,r\}$ is the edge with the largest product in edge set $E(G)$. Then by (6),

$$\lambda_1 x_1^r \le d_1 x_1 x_2 \cdots x_r,$$

$$\lambda_1 x_2^r \le d_2 x_1 x_2 \cdots x_r,$$

$$\lambda_1 x_r^r \le d_r x_1 x_2 \cdots x_r.$$

Multiplying the left and the right sides of above inequalities, respectively. So

$$\lambda_1^r x_1^r x_2^r \cdots x_r^r \le d_1 d_2 \cdots d_r (x_1 x_2 \cdots x_r)^r. \quad (8)$$

That is

$$\lambda_1 \le \sqrt[r]{d_1 d_2 \cdots d_r},$$

$$\lambda_1 \le \max_{\{i_1,\, i_2, \cdots, i_r\} \in E(G)} \sqrt[r]{d_{i1} d_{i2} \cdots d_{ir}} \quad (9)$$

In [13], authors give another upper bound on the largest H-eigenvalue of r-uniform hypergraphs as follow.

Theorem 3.2 [13]. Let $G = (V, E)$ be a connected $r-$graph with vertex set $[n]$. Let $\mathcal{A}$ be its adjacency tensor and λ_1 be the largest H-eigenvalue of $\mathcal{A}(G)$. Then

$$\lambda_1 \le \max_{\{i_1,\, i_2, \cdots, i_r\} \in E(G)} \sqrt[2(r-1)]{d_{i1}^{r-1} d_{i2} \cdots d_{ir}} \quad (10)$$

Comparing with Theorem 3.1, it is easy to find that the bound of the largest H-eigenvalue in Theorem 3.1 is tighter than that of Theorem 3.2 in the case of hyper-stars with the condition $d_{i1} \ge d_{i2} \ge \cdots d_{ir}$, for any edge $\{i_1,\, i_2, \cdots, i_r\} \in E(G)$.

Example.

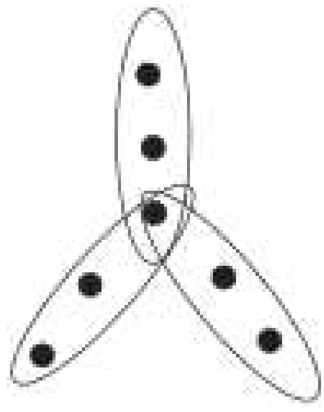

Figure 1. *Hyper-Star* G_1.

From Figure 1, $\lambda_1 \le \sqrt[3]{3}$ and $\lambda_1 \le \sqrt{3}$ hold by Theorem 1 and Theorem 2, respectively. Next, two lower bounds of the largest *H*-eigenvalue of connected $r-$graphs will be given.

Theorem 3.3. Let $G = (V, E)$ be a connected r−graph with vertex set $[n]$. Let $\mathcal{A}$ be its adjacency tensor and λ_1 be the largest *H*-eigenvalue of $\mathcal{A}$(G). Then

$$\lambda_1 \ge \min_{\{i_1,\, i_2, \cdots, i_r\} \in E(G)} \sqrt[2(r-1)]{d_{i1}^{r-1} d_{i2} \cdots d_{ir}} \quad (11)$$

Proof. Let $x = (x_1, x_2, \cdots, x_n)$ be a positive eigenvector of λ_1. Suppose $\{1,2, \cdots, r\}$ is the edge with the smallest product in edge set $E(G)$. Let x_1 be the least entry among $(x_1, x_2, \cdots, x_r)$. Then by (3),

$$d_1 x_2 x_3 \cdots x_r \le \sum_{e \in G(1)} x^e = \lambda_1 x_1^{r-1},$$

$$d_2 x_1^{r-1} \le \sum_{e \in G(2)} x^e = \lambda_1 x_2^{r-1},$$

$$d_3 x_1^{r-1} \le \sum_{e \in G(3)} x^e = \lambda_1 x_3^{r-1},$$

$$d_r x_1^{r-1} \le \sum_{e \in G(r)} x^e = \lambda_1 x_r^{r-1}.$$

From above second inequality to $\mathrm{r-th}$ inequality, multiply the left and the right sides of them, respectively. So

$$d_2 d_3 \cdots d_r x_1^{(r-1)^2} \le \lambda_1^{r-1} (x_2 x_3 \cdots x_r)^{r-1}. \quad (12)$$

For the first inequality, take $r-1$ powers of two sides. So

$$d_1^{r-1} (x_2 x_3 \cdots x_r)^{r-1} \le \lambda_1^{r-1} x_1^{(r-1)^2}. \quad (13)$$

By (12) and (13), it arrives

$$\sqrt[2(r-1)]{d_1^{r-1} d_2 d_3 \cdots d_r} \le \lambda_1, \quad (14)$$

and

$$\lambda_1 \ge \min_{\{i_1,\, i_2, \cdots, i_r\} \in E(G)} \sqrt[2(r-1)]{d_{i1}^{r-1} d_{i2} \cdots d_{ir}} \quad (15)$$

Theorem 3.4. Let $G = (V, E)$ be a connected r-graph with vertex set $[n]$. Let $\mathcal{A}$ be its adjacency tensor and λ_1 be the largest *H*-eigenvalue of $\mathcal{A}$(G). Then

$$\lambda_1 \ge \min_{\{i\}} \frac{\sum_{\{i_1, i_2, \cdots i_r\} \in E(G)} d_{i2} \cdots d_{ir}}{d_i^{r-1}} \quad (16)$$

Proof. Let D be a $n \times n$ diagonal matrix with all degrees of vertices of G as its diagonal elements. That is

$$\begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & d_n \end{pmatrix}$$

Let $\mathcal{B} = D^{-(r-1)} \mathcal{A} D$. That is $\mathcal{A}$ and $\mathcal{B}$ are diagonal similar. By the definition of tensor product in [13], it has

$$\mathcal{B}_{i_1 i_2 \cdots i_r} = \begin{cases} d_{i_1}^{-(r-1)} a_{i_1 i_2 \cdots i_r} d_{i_2} \cdots d_{i_r} \; if \; \{i_1,\, i_2, \cdots, i_r\} \in E(G), \\ \qquad 0 \text{ otherwise.} \end{cases}$$

Obviously, $\mathcal{A}$ and $\mathcal{B}$ have the same eigenvalues by Theorem 2.7. Let x be a positive eigenvector of the largest *H*-eigenvalue λ_1 of $\mathcal{B}$. Let x_j be the least entry of x. Then

$$d_j^{-(r-1)} \left(\sum_{\{j, i_2, \cdots i_r\} \in E(G)} d_{i_2} \cdots d_{i_r} \right) x_j^{r-1} \le d_j^{-(r-1)} \sum_{\{j, i_2, \cdots i_r\} \in E(G)} (d_{i_2} \cdots d_{i_r} x_{i_2} \cdots x_{i_r}) = \lambda_1 x_j^{r-1} \quad (17)$$

So

$$d_j^{-(r-1)} \left(\sum_{\{j, i_2, \cdots i_r\} \in E(G)} d_{i_2} \cdots d_{i_r} \right) \le \lambda_1, \quad (18)$$

and

$$\lambda_1 \ge \min_{\{i\}} \frac{\sum_{\{i, i_2, \cdots i_r\} \in E(G)} d_{i_2} \cdots d_{i_r}}{d_i^{r-1}} \quad (19)$$

4. Conclusion

In this paper, with using degrees of hypergraphs, several upper bounds and lower bounds are given. By the proofs of above results, the relations between degrees of hypergraphs and their characteristic equations should be known clearly. Moreover, other invariants such as their diameters, matching numbers of hypergraphs may be also play a role in estimating their eigenvalues. Thus, the further research on this topic can be done along these directions.

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References

[1] N. L. Biggs, Algebraic Graph Theory (2nd ed.), Cambridge University Press, Cambridge, 1993.

[2] F. R. K. Chung, Spectral Graph Theory, Regional Conference Series in Mathematics 92, Amer. Math. Soc., 1997.

[3] J. Cooper, A. Dutle, Spectra of uniform hypergraphs, Linear Algebra Appl., 436(2012)3268-3292.

[4] D. M. Cvetković, M. Doob and H. Sachs, Spectra of Graphs, Theory and Application, Academic Press, 1980.

[5] K. Feng, W. Li, Spectra of hypergraphs and applications, J. Number Theory, 60(1996) no. 1, 1-22.

[6] S. Hu, L. Qi, Algebraic connectivity of an even uniform hypergraph, J. Comb. Optim, 24(2011) 564-579.

[7] S. Hu, L. Qi, The eigenvectors of the zero laplacian and signless laplacian eigenvalues of a uniform hypergraph, Discrete Appl Math, 169(2014) 140–151.

[8] L. Qi, Eigenvalues of a real supersymmetric tensor, J. Symb. Compute., 40 (2005) 1302-1324.

[9] L. Zhang, L. Qi, Linear convergence of an algorithm for computing the largest eigenvalue of a nonnegative tensor, Numerical Linear Algebra with Applications, 19 (2012) 830-841.

[10] L. Qi, Symmetric nonnegative tensors and coposotive tensors, Linear Algebra Appl., 439(2013)228-238.

[11] S. Hu, L. Qi, J. Xie, The largest Laplacian and Signless Laplacian H-Eigenvalues of a Uniform Hypergraph, Linear Algebra Appl., 469(2015) 1–27.

[12] J. Y. Shao, A general product of tensors with applications, Linear Algebra Appl., 439(2013)2350-2366.

[13] G. Yi, Properties of spectrum of tensors on uniform hypergraphs, Master thesis of Fuzhou university, 2014.

[14] J. Xie, L. Qi, Spectral directed hypergraph theoy via tensors, Linear and Multilinear Algebra, 64 (2016) 780-794.

[15] C. Xu, Z. Luo, L. Qi and Z. Chen, {0, 1} Completely Positive Tensors and Multi-Hypergraphs, Linear Algebra Appl., 510 (2016) 110-123.