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Some Forbidden Subgraphs of Trees Being Opposition Graphs

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Abstract: In this paper, we use the number of vertices with degree greater than or equal to 3 as a criterion for trees being opposition graphs. Finally, we prove some families of graphs such as the complement of P_n, C_n with n≥3 and n = 4k, for k∈ℕ, are opposition graphs and some families of graphs such as the complement of T_n, C_n with n≥3 and n ≠ 4k, for k∈ℕ, are not opposition graphs.

Keywords: Trees, Orientations, Opposition Graphs

1. Introduction

From the book [1] and papers [2, 3], they introduce many containment relationships between classes of perfect graphs. For example, it mentions the relations between opposition graphs and threshold graphs, and the relations between opposition graphs and perfect graphs. There are also papers about perfectly orderable graphs [4, 5, 6, 7, 8], and papers about Welsh–Powell opposition graphs [9, 10].

Now we put our attention on the necessary and sufficient conditions of trees being opposition graphs. A graph G is called an *opposition graph* if there exists an orientation such that every induced P_4: *abcd*, $a \rightarrow b$ if and only if $d \rightarrow c$. We call such an orientation *oppositional orientation*.

2. Some Opposition Graphs

In this paper, we will discuss relations between opposition graphs and trees. Let T be a tree and let $R(T) = \{x \in V(T) \mid deg(x) \geqq 3\}$, we have the following four cases:

Case 1: $|R(T)| = 0$.
Case 2: $|R(T)| = 1$.
Case 3: $|R(T)| = 2$.
Case 4: $|R(T)| \geqq 3$.

In this section, we focus on these cases and provide some discussion and examples after each case.

Case 1: $|R(T)| = 0$.

In this case, we discuss the case $|R(T)| = 0$. Every vertex in the tree T has only degree 1 or 2, so T is a path P_n.

Theorem 2.1 The path P_n is an opposition graph.

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of P_n. We can give an orientation of P_n as follows:

a. $v_i \rightarrow v_{i+1}$ for all i = 4k, 4k+1, where k∈ℕ and $i < n$.
b. $v_{i+1} \rightarrow v_i$ for all i = 4k+2, 4k+3, where k ∈ ℕ and $i < n$.

Then P_n is an opposition graph shown as Figure 1

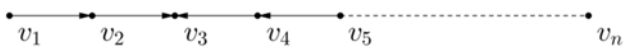

Figure 1. *P_n is opposition.*

Theorem 2.2 There are only four oppositional orientations of P_n.

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of P_n.

Case 1: If the direction between v_1 and v_2 is $v_1 \rightarrow v_2$, then we must have the following directions:

a. $v_i \rightarrow v_{i+1}$ for all i = 4k+1, where k∈ℕ and $i < n$.
b. $v_{i+1} \rightarrow v_i$ for all i = 4k+3, where k∈ℕ and $i < n$.

Then we have two subcases:

Subcase 1: The direction between v_2 and v_3 is $v_2 \rightarrow v_3$, then we have the following directions:

i. vi → vi+1 for all i = 4k+2, where k∈ℕ *and i < n..*
ii. vi+1 → vi for all i = 4k+4, where k∈ℕ *and i < n.*

Subcase 2: The direction between v_2 and v_3 is $v_3 \rightarrow v_2$, then we have the following directions:

i. vi → vi+1 for all i = 4k+4, where k∈ℕ *and i < n.*

ii. vi+1 → vi for all i = 4k+2, where k∈ℕ *and i < n.*

Case 2 If the direction between v_1 and v_2 is $v_2 \rightarrow v_1$, then we must have the following directions:

a. vi → vi+1 for all i = 4k+3, where k∈ℕ *and i < n.*

b. vi+1 → vi for all i = 4k+1, where k∈ℕ *and i < n..*

Then we have two subcases:

Subcase 1: The direction between v_2 and v_3 is $v_3 \rightarrow v_2$, then we have the following directions:

a. vi → vi+1 for all i = 4k+4, where k∈ℕ *and i < n.*

b. vi+1 → vi for all i = 4k+2, where k∈ℕ *and i < n.*

Subcase 2: The direction between v_2 and v_3 is $v_2 \rightarrow v_3$, then we have the following directions:

a. vi → vi+1 for all i = 4k+2, where k∈ℕ *and i < n.*

b. vi+1 → vi for all i = 4k+4, where k∈ℕ *and i < n.*

Theorem 2.2 told us that there are only four oppositional orientations D_1, D_2, D_3 and D_4 for a path. We can choose any one of these four oppositional orientations to give an orientation for a path.

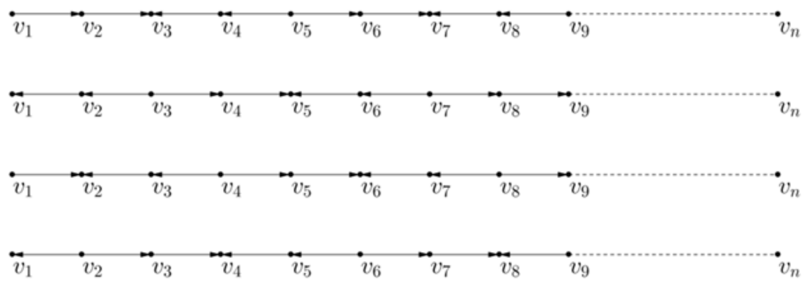

Figure 2. *The orientation of P_n.*

Case 2: |R(T)| = 1.

If there is only one vertex *u* in *R(T)*, then *T* must be the tree shown as Figure 3, we call it *sunshine graph*. We will discuss whether *T* is an opposition graph.

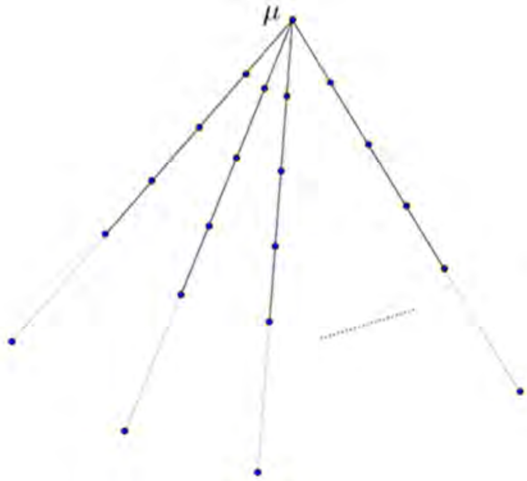

Figure 3. *Sunshine graph.*

Theorem 2.3 If *T* is a sunshine graph, then *T* is an opposition graph.

Proof. Let $u \in R(T)$ be the root of *T*. We can give an orientation for the edges of *T* as follows:

a. Level *i* → level *i+1* for all *i* = 4k, 4k+1, where k∈ℕ and *i* < *l*, where *l* is the height of *T*.
b. Level *i+1* → level *i* for all *i* = 4k+2, 4k+3, where k∈ℕ and *i* < *l*, *l* is the height of, where k∈ℕ and *i* < *l*, where *l* is the height of *T*.

Then *T* is an opposition graph shown as Figure 4.

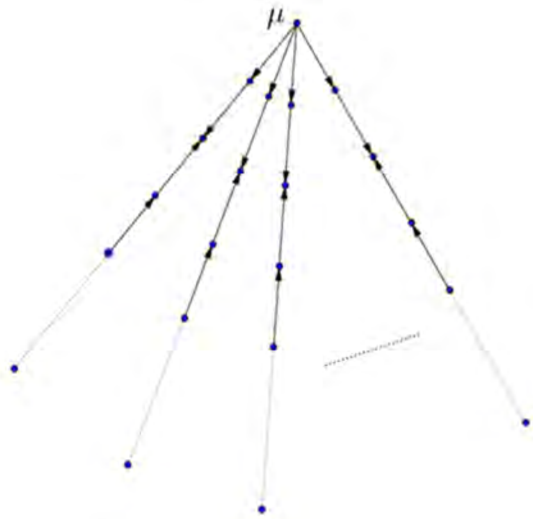

Figure 4. *Sunshine graph is an opposition graph.*

Theorem 2.4 For a sunshine graph *T*. Let *u* be the root of *T*. If there are at least two vertices in level 2, then there are only two oppositional orientations for a sunshine graph *T*.

Proof. Let *T* be a sunshine graph. Let $u \in R(T)$ be the root of the tree *T*. There are n paths from *u* to leaves $Q_1, Q_2, \ldots, Q_n$. By Theorem 2.2, there are only four oppositional orientations for a path:

Case 1: If the orientation of Q_1 is D_1, then the orientation of $Q_2, \ldots, Q_n$ must be D_1. Hence, the orientation of *T* is level *i* → level *i+1* for all *i* = 4k, 4k+1 and level *i+1* → level *i* for all *i* = 4k+2, 4k+3, where k∈ℕ and i < *l*, where *l* is the height of *T*.

Case 2: If the orientation of Q_1 is D_2, then the orientation of

Q2, …, Qn must be D_2. Hence, the orientation of T is level $i+1$ → level i for all i = 4k, 4k+1, and level i → level $i+1$ for all i = 4k+2, 4k+3, where k∈ℕ and $i < l$, where l is the height of T.

Suppose the vertices of level 1 in Q_1, Q_2, Q_3 are v_{11}, v_{12}, v_{13}, and suppose the vertices of level 2 in Q_1, Q_2 are v_{21}, v_{22}.

Case 3: If the orientation of Q_1 is D_3, then the directions of T must be $v_{12} \rightarrow u$, $v_{12} \rightarrow v_{22}$, $v_{13} \rightarrow u$. Hence, the orientation of the path $v_{13}\ u\ v_{12}\ v_{22}$ gives us a contradiction.

Case 4: If the orientation of Q_1 is D_4, then the directions of T must be $u \rightarrow v_{12}$, $v_{22} \rightarrow v_{12}$, $u \rightarrow v_{13}$. Hence, the orientation of the path $v_{13}\ u\ v_{12}\ v_{12}$ gives us a contradiction.

So there are only two oppositional orientations for a sunshine graph T.

By Theorem 2.4, we can give another orientation of edges of T as follows:

a. Level i ← level $i+1$ for all i = 4k, 4k+1, where k∈ℕ and $i < l$, where l is the height of T.

b. Level $i+1$ ← level i for all i = 4k+2, 4k+3, where k∈ℕ and $i < l$, where l is the height of T.

Then T is an opposition graph shown as Figure 5.

Corollary 2.5 For a sunshine graph T. Let u be the root of T. If there are at least two vertices in level 2, then the orientation of T must be given as follows:

a. Level i → level $i+1$ for all i = 4k, 4k+1, where k∈ℕ and $i < l$, where l is the height of T.

b. Level $i+1$ → level i for all i = 4k+2, 4k+3, where k∈ℕ and $i < l$, where l is the height of T.

Proof. By Theorem 2.4, there are two orientations for T, these two orientations are symmetric, so we can use case1 to give the orientation for T.

Figure 5. *An sunshine graph is an opposition graph.*

Theorem 2.6 For a tree T. Let u be the root of T. If there are at least two vertices in level two and T is opposition, then the orientation of T must be given as follows:

a. Level i → level $i+1$ for all i = 4k, 4k+1, where k∈ℕ and $i < l$, where l is the height of T.

b. Level $i+1$ → level i for all i = 4k+2, 4k+3, where k∈ℕ and $i < l$, where l is the height of T.

Proof. Let T be a tree. Suppose $R(T) = \{u, u_1, u_2, \ldots, u_n\}$. There is a maximal subtree T_1 containing u which is a sunshine graph. Then T can be decomposed into T_1 and some paths Q_1, Q_2, …, Q_k with one of endpoints in $R(T)$.

Because T_1 is a sunshine graph, the orientation is given by Corollary 2.5. Now we add all paths Q_i into T_1. Suppose u_j is an endpoint of Q_i. Then uu_j U Q_i is a path, the orientation of this path is given by case 1 of Theorem 2.2.

Hence, the orientation of T must be given as follows:

a. Level i → level $i+1$ for all i = 4k, 4k+1, where k∈ℕ and $i < l$, where l is the height of T.

b. Level $i+1$ → level i for all i = 4k+2, 4k+3, where k∈ℕ and $i < l$, where l is the height of T.

Now, by Theorem 2.6, when we want to determine if a tree T is an opposition graph, we can give the orientation by only one way: Let $u \in R(T)$ be the root.

Level i → level $i+1$ for all i = 4k, 4k+1 and level $i+1$ → level i for all i = 4k+2, 4k+3, where k∈ℕ and $i < l$, where l is the height of T. When the orientation is given as above, if some induced P_4 doesn't satisfy the definition of opposition graphs, then T is not an opposition graph.

Case 3: |R(T)| = 2.

If there are exactly two vertices u and v in $R(T)$, then T must be the tree shown as Figure 6, we call it *wing graph*. We will discuss whether T is an opposition graph.

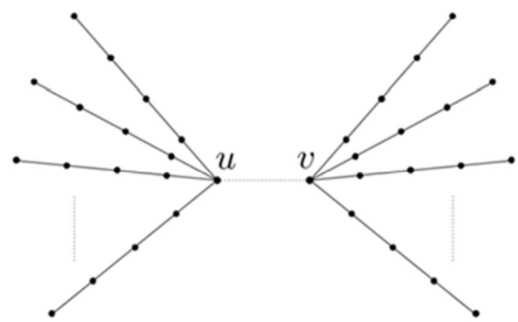

Figure 6. *T is a wing graph.*

Now, if we delete all the vertices between u and v, then we can get two subtrees containing u and v, we call them T_1 and T_2. Observably, the degrees of u and v are greater than or equal to 2. The trees T_1 and T_2 are paths or sunshine graphs because the degrees of every vertices are less than 3 except u and v.

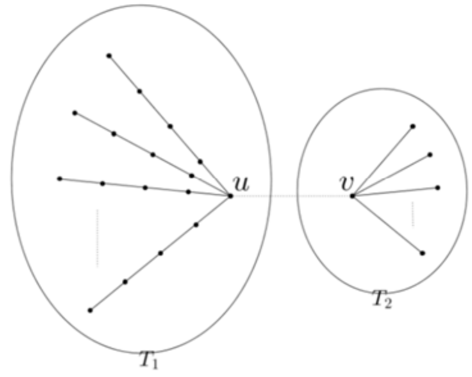

Figure 7. *The graph of Theorem 2.7.*

Theorem 2.7 Let T be a tree with exactly two vertices u, v in $R(T)$. Let T_1 and T_2 be the subtrees from deleting the vertices between u and v. If at least one of T_1 and T_2 does not contain P_4, then T is an opposition graph.

Proof. Suppose T_2 does not contain P_4 and v is in T_2. Let u be the root of the tree T. We can give an orientation of edges of T as follows:

a. Level $i \rightarrow$ level $i+1$ for all i = 4k, 4k+1, where k$\in\mathbb{N}$ and $i < l$, where l is the height of T.

b. Level $i+1 \rightarrow$ level i for all i = 4k+2, 4k+3, k$\in\mathbb{N}$ and $i < l$, where l is the height of T.

Then T is an opposition graph shown as Figure 8.

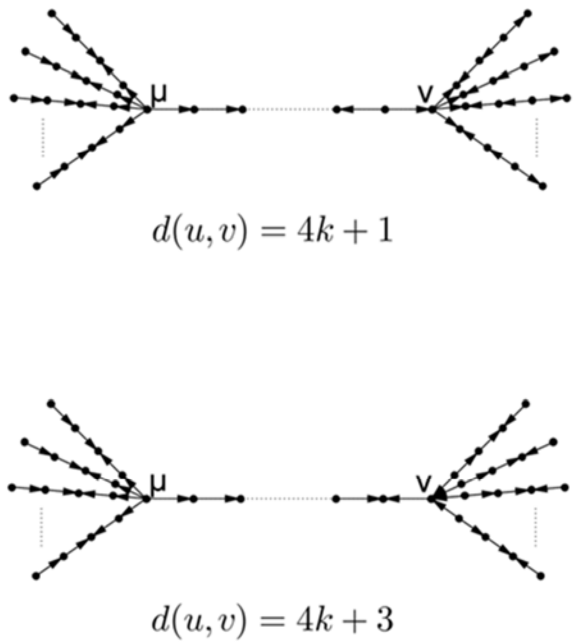

Figure 8. *The orientation of Theorem 2.7.*

Theorem 2.8 Let T be a tree and $v \in R(T)$. There are n paths $Q_1, Q_2, \ldots, Q_n$ with endpoint v. Let $v_{11} \in Q_1$, $v_{12} \in Q_2$, …, $v_{1n} \in Q_n$ be the vertices whose distance from v is 1. Let $v_{21} \in Q_1$, $v_{22} \in Q_2$, …, $v_{2n} \in Q_n$ be some vertices whose distance from v is 2. If T is an opposition graph, then the directions of the edges uv_{1i} and $v_{1i}v_{2i}$ must be as follows:

Case 1: The directions are $v \rightarrow v_{1i}$ for all i = 1, …, n and $v_{1i} \rightarrow v_{2i}$ for all i = 1, …, n.

Case 2: The directions are $v_{1i} \rightarrow v$ for all i = 1, …, n and $v_{2i} \rightarrow v_{1i}$ for all i = 1, …, n.

Proof. T is a tree. Let $u \in R(T)$ be the root of T. Suppose the path Q_1 is between u and v.

By Theorem 2.6, we give an orientation for T, there are two cases in the edge between v_{11} and v_{21}:

Case 1: If we give the direction $v_{11} \rightarrow v_{2i}$, then the directions of the edges uv_{1i} and $v_{1i}v_{2i}$ is $v \rightarrow v_{1i}$ for all i = 2, …, n, $v_{1i} \rightarrow v_{2i}$ for some i = 2, …, n, and $v \rightarrow v_{11}$.

Case 2: If we give the direction $v_{2i} \rightarrow v_{11}$, then the directions of the edges uv_{1i} and $v_{1i}v_{2i}$ is $v_{1i} \rightarrow v$ for all i = 2, …, n, $v_{2i} \rightarrow v_{1i}$ for some i = 2, …, n, and $v_{11} \rightarrow v$.

So there are only two cases for the directions of the edges uv_{1i} and $v_{1i}v_{2i}$.

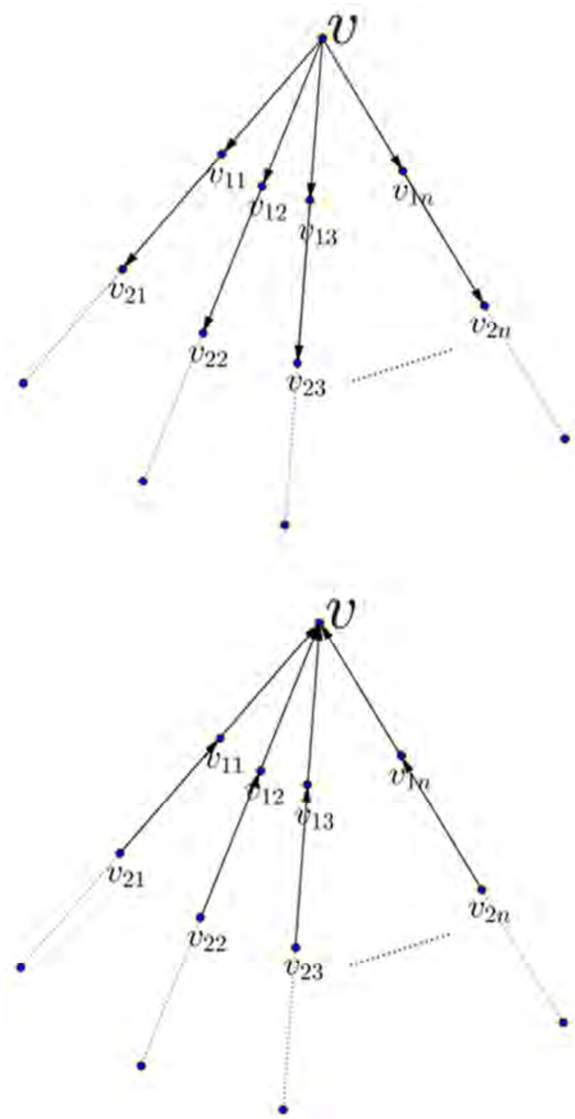

Figure 9. *The orientation of Theorem 2.8.*

Theorem 2.8 can give us a way to determine if T is an opposition graph. For a tree T, by Theorem 2.6, we can give an orientation, then the orientation of every vertex u in $R(T)$ must satisfy Theorem 2.8 2.8. If the orientation of any vertex u in $R(T)$ doesn't satisfy Theorem 2.8, then T is not an opposition graph.

Then we will discuss that both T_1 and T_2 contain P_4. Let dist(u,v) be the distance between node u and node v. Then we have the following two cases:

Case 1: If dist(u,v) is odd.

Case 2: If dist(u,v) is even.

Theorem 2.9 Let T be a tree with exactly two vertices u, v in $R(T)$. Let T_1 and T_2 be the subtrees from deleting the vertices between u and v. If both T_1 and T_2 contain P_4 and dist(u,v) is odd, then T is not an opposition graph.

Proof. Suppose u is in T_1 and v is in T_2. Let u be the root of the tree T. We can give an orientation of edges of T by Corollary 2.5. Then the orientation of T is shown as Figure 10. The orientation of T_2 doesn't satisfy Theorem 2.8, so T is not an opposition graph.

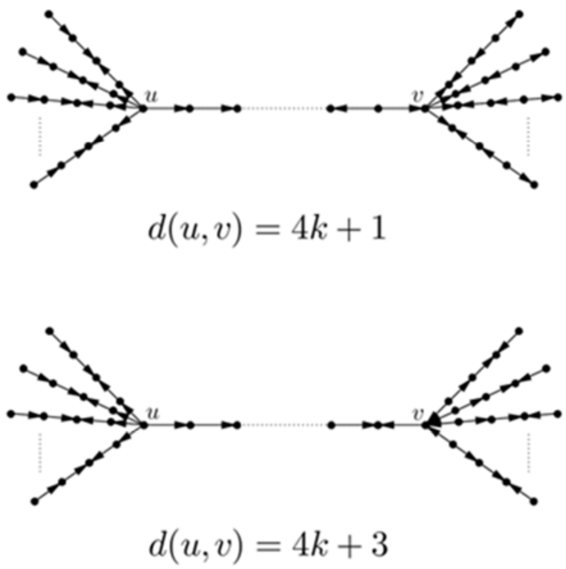

Figure 10. *The orientation of Theorem 2.9.*

Theorem 2.10 Let T be a tree with exactly two vertices u, v in $R(T)$. Let T_1 and T_2 be the subtrees from deleting the vertices between u and v. If both T_1 and T_2 contain P_4 and dist(u,v) is even, then T is an opposition graph.

Proof. Let u be the root of the tree T. We can give an orientation of edges of T by Corollary 2.5. Then the orientation of T is shown as Figure 11, so T is an opposition graph.

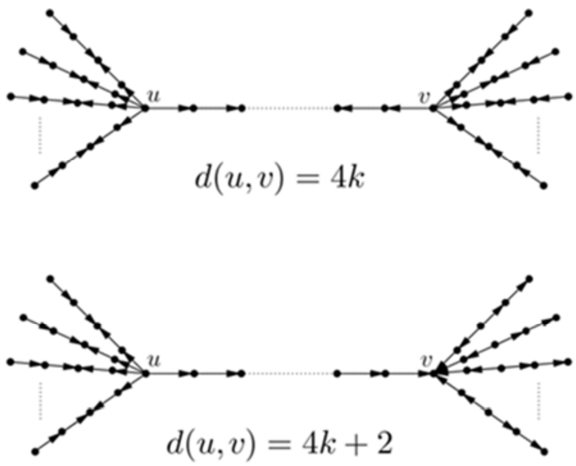

Figure 11. *The orientation of Theorem 2.10.*

Case 4: $|R(T)| \geqq 3$

Theorem 2.11 Let T be a tree. Let $R(T) = \{v_1, v_2, ..., v_n\}$ be the set of vertices in T whose degree is greater than or equal to 3. If d(v_i, v_{i+1}) is even for all i = 1, …, n, then T is an opposition graph.

Proof. We use the induction on $R(T)$ to prove the statement. Let T be a tree and $R(T) = \{v_1, v_2, ..., v_n\}$ be the set of vertices in T which degree is greater than or equal to 3.

Basic step: Suppose n = 2. By Theorem 2.10, T is an opposition graph.

Induction step: Suppose n > 2. Let v_1 be the root of the tree T. Suppose dist(v_i, v_1) ≦ dist(v_j, v_1) for all $i < j$. Let T_n be the subtree of T whose vertex set $V(T_n)$ are v_n and all of its descendant. Let T' be the subtree of T whose vertex set $V(T')$ are $\{v_n\} \cup V(T) - V(T_n)$.

Now, $|R(T')|$ = n - 1, so T' is an opposition graph by induction hypothesis.

Let v_1 be the root of T'. We can give an orientation to T':

a. Level i → level $i+1$ for all i = 4k, 4k+1, where k∈ℕ and $i < l$, where l is the height of T.
b. Level $i+1$ → level i for all i = 4k+2, 4k+3, where k∈ℕ and $i < l$, where l is the height of T.

Then we give the orientation for T_n and add T_n to T'. Let v_n be the root of T_n. There are two cases in T_n:

Case 1: If dist(v_1, v_n) = 4k, then level i → level $i+1$ for all i = 4k, 4k+1 and level $i+1$ → level i for all i = 4k+2, 4k+3, where k∈ℕ and $i < l$, where l is the height of T.

Case 2: If dist(v_1, v_n) = 4k+2, then level $i+1$ → level i for all i = 4k, 4k+1 and level i → level $i+1$ for all i = 4k+2, 4k+3, where k∈ℕ and $i < l$, where l is the height of T.

Hence, T is an opposition graph for n > 2.

Definition Let the path $u_1u_2u_3u_4$ and $v_1v_2v_3v_4$ be two P_4. We add an odd path between u_2 and v_2, the graph is called *H graph* shown as Figure 12.

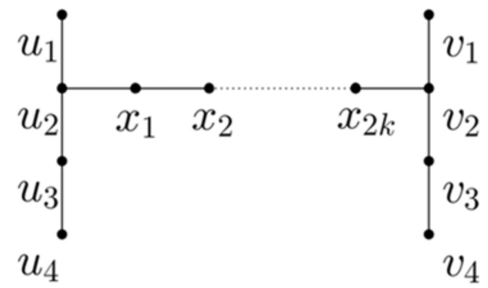

Figure 12. *H graph.*

Theorem 2.12 If T be an H graph, then T is a minimal obstruction for the class of opposition graphs.

Proof. If we remove u_1, then there is only one vertex v_2 which degree is greater than or equal to 3, by Theorem 2.3, T is an opposition graph.

If we remove u_4, the path $u_1u_2u_3$ is a P_3, then by Theorem 2.7, T is an opposition graph.

Similar for the vertices v_1 and v_4.

3. Conclusion

By using the size of the set of vertices whose degree greater or equal to three, we state some conditions of trees being opposition graphs.

There are four cases studied in this paper. In first case, there are only four different kinds of oppositional orientations in P_n. Sunshine graphs are considered in second case and the number of oppositional orientation can be determined by theorem we provided. Wing graphs play an important role in third case as well as sunshine graphs in previous case. We state and prove the last case by mathematical induction.

4. Open Problems and Further Directions of Studies

In this paper, we show some sufficient condition for trees being opposition graphs. For the future direction of research, we would like to study the necessary and sufficient for trees or other category of graphs being opposition graph.

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