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Some Aspects of Certain Form of Near Perfect Numbers

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Abstract: It is well known that a positive integer n is said to be near perfect number, if $\sigma(n) = 2n+d$ where d is a proper divisor of n and function $\sigma(n)$ is the sum of all positive divisors of n In this paper, we discuss some results concerning with near perfect numbers from known near perfect numbers.

Keywords: Divisor Function, Mersenne Prime, Fermat Prime, Perfect Number, Near Perfect Number

1. Introduction

A positive integer n is called perfect number if the sum of all proper divisors of n is equal to n. Proper divisors of n are positive divisors of n other than n itself. The smallest perfect number is 6 since $6 = 1 + 2 + 3$. First four perfect numbers are: 6, 28, 496, 8128; which are known from ancient time. Well known divisor function $\sigma(n)$ is the sum of all positive divisors of n. Divisor function $\sigma(n)$ is a multiplicative function i.e., $\sigma(n)$ satisfies the functional condition $\sigma(mn) = \sigma(m)\sigma(n)$ if g.c.d $(m, n) = 1$. For any perfect number n, $\sigma(n) = 2n$. All known perfect numbers are even. Euler [9] proved that all known perfect numbers are of the form $n = 2^{p-1}\ M_P$, where both p and $M_P = 2^p - 1$ are primes. The primes of the form $= 2^p - 1$ are called Mersenne primes. Up to now (March, 2017), only 49 Mersenne prime numbers are known which means there are only 49 even perfect numbers that are discovered [14]. Multi-perfect numbers are natural extension of classical perfect numbers. A positive integer n is called k −perfect number [3, 4](multi perfect of abundancy k or generalized perfect number) if $\sigma(n) = kn$, where $k > 2$. All known $k -$ perfect numbers are even. There are only finite number of multi-perfect numbers are discovered. No single example for an odd perfect number and odd multi perfect number has been found nor has a proof for their non-existence been established, although many people have been working on this problem for centuries.

In recent time, other possible generalized perfect numbers are hyper perfect numbers [13], near perfect numbers [10], near and deficient hyperperfect numbers [5], near k −perfect numbers. In 2012, P. Pollack and V. Shevelev [10] introduced the notion of near perfect numbers. It is well known that a positive integer n is called near perfect number, if n is the sum of all of its proper divisors, except for one of them, which is termed as redundant divisor. For example, proper divisors of 12 are 1, 2, 3, 4 and 6. We can write $12 = 1 + 2 + 3 + 6$, which shows that 12 is a near perfect with redundant divisor 4. Moreover, using the definition of divisor function $\sigma(n)$ one can say: a positive integer n is near perfect number with redundant divisor d if and only if d is a proper divisor of n and $\sigma(n) = 2n + d$. Some near perfect numbers are associated with classical perfect numbers. If $m = 2^{p-1}\ M_P$ is an even perfect number, then $n = 2m$, $n = 2^p m$ and $n = (2^p - 1)m$ are near perfect numbers with two distinct prime factors. There are infinitely many near perfect numbers other than above shapes. In [10], P. Pollack and V. Shevelev have also defined k −near perfect number. A positive integer n is called k near perfect number [10], if n is the sum of all of its proper divisors, except k numbers of proper divisors. If n is a k near perfect number with redundant divisors $d_1, d_2, d_3, \ldots, d_k$ then we can write

$$\sigma(n) = 2n + d_1 + d_2 + d_3 + \ldots + d_k .$$

Near perfect numbers are 1 near perfect number.

From the definition of $k -$ perfect number, it is clear that there are infinite numbers of $k -$ near perfect number.

Another special kind of prime number is Fermat Prime. Odd Prime number of the form $F_n = 2^m + 1$ is called Fermat prime, where m must be some power of 2 [2]. There are only five known Fermat prime numbers: $F_0 = 3$, $F_1 = 5$, $F_2 = 17$,

$F_3 = 257$ and $F_4 = 65537$.

2. Main Results

We have obtained the following results from known near perfect numbers.

Proposition 2.1. For any integer $k \geq 1$, there is no near perfect number of the form $n = 2^k$.

Proof. If $n = 2^k$ is a near perfect number with redundant divisor 2^a, where $0 < a < k$, then from definition of near perfect numbers $\sigma(n) = 2^{k+1} + 2^a$. But the equation $2^{k+1} - 1 = 2^{k+1} + 2^a$ strictly implies that $2^a + 1 = 0$, which is a contradiction. Hence there is no near perfect number of the form $n = 2^k$, where $k \geq 1$.

Proposition 2.2. For any integer $a \geq 1$, there is no near perfect number of the form $n = 2^a p$ with redundant divisor p, where p is an odd prime.

Proof. Suppose $n = 2^a p$ is a near perfect number with redundant divisor p, then from definition of near perfect numbers $\sigma(n) - 2n = p$. Therefore the equation $(2^{a+1} - 1)(p + 1) - 2^{a+1}p = p$, strictly implies that $2^{a+1} - 1 = 2p$. But the expression $2^{a+1} - 1$ is always odd for any integer $a \geq 1$. Therefore there is no near perfect number of the form $n = 2^a p$ with redundant divisor p.

Proposition 2.3. If p is an odd prime, then any number of the form $n = 2p^2$ is a near perfect number if and only if $p = 3$.

Proposition 2.4. If p is an odd prime, then any number of the form $n = 2^2 p^2$ is a near perfect number if and only if $p = 7$.

Proposition 2.5. Let $F_m = 2^m + 1$ is a Fermat prime. If $n = 2^2 F_m$ is a near perfect number, then $n = 12$ or $n = 20$.

Proof. If $n = 2^2 F_m$ is a near perfect number with redundant divisor R, then we have $\sigma(n) = 2^3 F_m + R$ and therefore $7(F_m + 1) - 8F_m = R$. This equation strictly implies that $7 - F_m = R$. Since R is a positive proper divisor of n and F_m is a Fermat prime, so the last equation strictly implies that $F_m = 3$ and $F_m = 5$.

Proposition 2.6. Let $F_m \geq 17$ is a Fermat prime and p is an odd prime relatively prime to F_m. If $n = 2^3 p F_m$ is a near perfect number with redundant divisor $2^3 F_m$, then the prime p must be of the form $p = \dfrac{7F_m + 15}{F_m - 15}$.

Proof. If $n = 2^3 p F_m$ is a near perfect number with redundant divisor $2^3 F_m$, then $\sigma(n) - 2n = 8F_m$. This equation implies that $15(F_m + 1)(p + 1) - 16pF_m = 8F_m$. Solving this equation one can obtain the form of p.

Example 2.1.

Suppose $n = 2^3.17.p$ is a near perfect number with redundant divisor $2^3.17$.

Here $F_m = 17$, then $p = \dfrac{7F_m + 15}{F_m - 15} = 6$, i.e., $n = 2^3.17.67$ is a near perfect number.

Proposition 2.7. Let $F_m \geq 17$ is a Fermat prime and p is an odd prime relatively prime to F_m. If $n = 2^3 p F_m$ is a near perfect number with redundant divisor 2^3, then the prime p must be of the form $p = \dfrac{15F_m + 7}{F_m - 15}$.

Proof. Since $n = 2^3 p F_m$ is a near perfect number with redundant divisor 2^3, so we can write $\sigma(n) - 2n = 8$. This equation implies that $15(F_m + 1)(p + 1) - 16pF_m = 8$. Solving this equation one can get the form of p.

Example 2.2. $n = 17816 = 2^3.17.131$ is a near perfect number with redundant divisor 2^3.

Here $F_m = 17$ and therefore $p = \dfrac{15F_m + 7}{F_m - 15} = 131$.

Proposition 2.8. There is no near perfect number of the form $n = p_1 p_2$, where p_1 and p_2 are distinct odd primes.

Proof. Suppose $n = p_1 p_2$ is a near perfect number with redundant divisor R, then either $R = p_1$ or $R = p_2$. If $= p_1$, then $\sigma(n) = 2p_1 p_2 + R$. Therefore $(p_1 + 1)(p_2 + 1) = 2p_1 p_2 + p_1$, which implies that $p_2(p_1 - 1) = 1$. This equation has solution only for $p_1 = 2$ and $p_2 = 1$, which contradict the facts that p_1 and p_2 are odd primes.

Proposition 2.9. There is no near perfect number of the form $n = 2p_1 p_2$, where p_1 and p_2 are distinct odd primes.

Proof. Suppose $n = 2p_1 p_2$ is a near perfect number with redundant divisor R, then $\sigma(n) = 4p_1 p_2 + R$, where possible values of R are $R = p_1$ or p_2 or $2p_1$ or $2p_2$ or $p_1 p_2$.

Case I. Suppose $R = p_1$, then $\sigma(n) = 4p_1 p_2 + p_1$, which gives $3(p_2 + 1) = p_1(p_2 - 2)$. From the last equation we obtain $p_1 = p_2 + 1$ and $p_2 - 2 = 3$. Solving these two equations, we obtain $p_1 = 6$ and $p_2 = 5$, which contradict that p_1 is an odd prime.

Case II. Suppose $= 2p_1$, then $\sigma(n) = 4p_1 p_2 + 2p_1$, which gives $3(p_2 + 1) = p_1(p_2 - 1)$. From the last equation we get $p_1 = p_2 + 1$ and $p_2 - 1 = 3$. Solving we obtain $p_1 = 5$ and $p_2 = 4$, which also contradict that p_2 is an odd prime.

Case III. Suppose $R = p_1 p_2$, then $\sigma(n) = 4p_1 p_2 + p_1 p_2$, which gives $3(p_1 + p_2 + 1) = 2p_1 p_2$. Since the expression $p_1 + p_2 + 1$ is always odd, therefore the last equation has no solution for any odd primes p_1 and p_2.

Hence the above three cases strictly imply that there is no near perfect number of the form $n = 2p_1 p_2$, where p_1 and p_2 are distinct odd primes.

Proposition 2.10. If $= 2^a p_1 p_2$, where $a > 1$ and p_1 and p_2 are distinct odd primes, then n is not a near perfect number with redundant divisor $p_1 p_2$.

Proof. If $n = 2^a p_1 p_2$, then

$$\sigma(n) - 2n = \sigma(2^a)\,\sigma(p_1)\sigma(p_2) - 2^{a+1} p_1 p_2$$
$$= (2^{a+1} - 1)(p_1 + 1)(p_2 + 1) - 2^{a+1} p_1 p_2$$

$$= (2^{a+1}-1)(p_1+p_2+1)-p_1p_2$$

Suppose n is a near perfect number with redundant divisor p_1p_2, then $(2^{a+1}-1)(p_1+p_2+1) = 2p_1p_2$.

Since the numbers $2^{a+1}-1$ and p_1+p_2+1 are odd numbers, so L. H. S. of the last equation is an odd number and therefore $(2^{a+1}-1)(p_1+p_2+1) \equiv 1 \pmod 2$, but R. H. S. is even and divisible by 2. The parity of L. H. S. and R. H. S. of the equation do not match, which is a contradiction.

Proposition 2.11. Let p_1 and p_2 are distinct odd primes with $p_1 < p_2$. If $n = 2p_1^2p_2$ is a near perfect number with redundant divisor 2, then $n = 650$.

Proof. If $n = 2p_1^2p_2$ is a near perfect number with redundant divisor 2, then

$$2 = \sigma(n) - 2n = 3(p_1^2+p_1+1)(1+p_2) - 4p_1^2p_2$$
$$= 3p_1^2+3p_1+3p_2+3p_1p_2-p_1^2p_2+3$$

After simple simplification we get

$$(p_1+1)(2p_1+3p_2+1) = p_1^2(p_2-1).$$

We solve this equation in terms of p_1 and p_2.

From the last equation we obtain the following three possibilities:

Case I. If $p_1+1 = p_2-1$ and $2p_1+3p_2+1 = p_1^2$ then $p_2 = p_1+2$ an d $p_1(p_1-5) = 7$. But the equation $p_1(p_1-5) = 7$ has no solution.

Case II. If $p_1+1 = \frac{p_2-1}{2}$ and $2p_1+3p_2+1 = 2p_1^2$ then $p_2 = 2p_1+3$ and $p_1^2-4p_1-5 = 0$. Solving these two equations, we get $p_1 = 5$, $p_2 = 13$ and therefore $n = 650$.

Proposition 2.12. If $n = p_1p_2...p_m$, where $p_1 < p_2 < \ldots < p_m$ are odd primes, then n is not a near perfect number with redundant divisor p_m.

Proof. Suppose $n = p_1p_2...p_m$ is a near perfect number with redundant divisor p_m, then $\sigma(n) = 2n + p_m$ and therefore $p_m \mid \sigma(n)$, then $p_m \mid (1+p_1)(1+p_2)...(1+p_m)$.

Since $p_1 < p_2 < \ldots < p_m$, so it is clearly $\frac{p_i+1}{2} < p_m$ for all prime factors p_i of n. Therefore

$p_m \nmid (1+p_1)(1+p_2)...(1+p_m)$, which is a contradiction.

Proposition 2.13. If 2^k-3 is an odd prime, then $n = 2^{k-1}(2^k-3)$ is a near perfect number with redundant divisors 2.

Proof. We have

$$\sigma(n) - 2n = (2^k-1)(2^k-3+1) - 2^k(2^k-3)$$
$$= 2^k(2^k-3) + 2^k - (2^k-2) - 2^k(2^k-3) = 2.$$

Proposition 2.14. If 2^k-5 is an odd prime, then $n = 2^{k-1}(2^k-5)$ is a near perfect number with redundant divisors 4.

Proposition 2.15. Let $F_m = 2^m+1$ is a Fermat prime. For $m < k$, if $2^k - F_m$ is an odd prime, then $n = 2^{k-1}(2^k - F_m)$ is a near perfect number with redundant divisor 2^m.

Proof. We have

$$\sigma(n) - 2n = (2^k-1)\left(2^k - F_m + 1\right) - 2^k(2^k - F_m)$$
$$= 2^k(2^k - F_m) + 2^k - \left(2^k - F_m + 1\right) - 2^k(2^k - F_m)$$
$$= 2^m.$$

Proposition 2.16. If J and $F_m = 2^m+1$ are respectively perfect number and Fermat prime with g.c.d $\left(J, F_m\right) = 1$, then $n = J\,F_m$ is not a near perfect number.

Proof. We have $\sigma(n) - 2n = 2J(F_m+1) - 2JF_m = 2J$.

But $2J$ is not a divisor of n.

Proposition 2.17. If $M = 2^p - 1$ is a Mersenne prime, then $n = 2^{p-1}M^3$ is 2 near perfect number with redundant divisors M and M^2.

Proof. We have

$$\sigma(n) - 2n = (2^p-1)(M^3+M^2+M+1) - 2^pM^3$$
$$= M^2 + M.$$

In general we obtain the following proposition

Proposition 2.18. If $M = 2^p - 1$ is a Mersenne prime, then for any $k \geq 2$, $n = 2^{p-1}M^{k+1}$ is a k near perfect number with redundant divisors M^i, where $i = 1, 2, ..., k$.

Proposition 2.19. For each $i = 1, 2$; if $M_i = 2^{p_i}-1$ is a Mersenne prime, then $n = 2^{p_1-1}M_1M_2$ is not a near perfect number.

Proof. We have $\sigma(n) - 2n = M_1 + M_1^2$.

But M_1^2 is not a divisor of n.

Proposition 2.20. If $n = 2^{p_i-1}M_1M_2...M_r$, where $M_i = 2^{p_i}-1$ are distinct Mersenne primes, $i = 1, 2, \ldots, r$, then n is not a near perfect number.

3. Conclusion

Near perfect numbers are natural extensions of classical perfect numbers. In this paper, we discuss near perfect numbers of certain form. There are very good scopes for studying other form of near perfect numbers.

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