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Applying Three Phase Lanchester Linear Model to the Ardennes Campaign

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Abstract: We constructed a new Lanchester linear model for the Ardennes Campaign with three phases and Bracken's tactical factor to obtain an improved goodness of fit with historical data. It indicates that the shift from defense to attack for the Blue force can be better described by adding a deadlock period. Our new model provides an improved explanation for the military operation research to calibrate a better fit between prediction and historical data.

Keywords: Ardennes Campaign, Bracken's Tactical Factor, Lanchester Linear Model

1. Introduction

The purpose of this study is not to show off the accuracy of the findings of this research team in terms of war casualty. Instead it is premised on respect for life because Karl Von Clausewitz [1] said in his book *On War* that war is act of collective violence. It is also an act of extreme used in absolute violence. The casualties of war calculated by scholars are just figures and signs; however, casualties of war on the battle field involve the loss of precious human life and also the attrition of national power. The number of war casualties on both sides, the winning and the losing, are either exaggerated or roughly estimated. In the evaluation of the history of war, cold weaponry in ancient times and hot weaponry in recent times, are both designed to eliminate human life, which is to destroy the enemy's ability to use weapons. However it takes about six months to train a man to be a solider and it costs a big fortune to attain the necessary materials for war. *Art of War* by Sun Tzi [2] stated, when in a battle, even if you are winning, lengthening the time spent on the battle field will dull your forces and blunt your edge; if you besiege a citadel, your strength will be exhausted. If an army is dispatched in the field for a long time, your supplies will be depleted. In ancient China, Sun Tzi proposed "The Cautious War" which stated that military action is important to a nation—it is the ground of death and life, the path of survival and destruction, so it is imperative to examine it. He then continues to say that the one who figures out a path to victory even before stepping foot on the battle field is the one who has the most favorable strategic factors on his side. The one with the inability to prevail at headquarters before a battle is the one who has the least favorable strategic factors on his side. The one with more beneficial strategic factors in his favor will win for the one with fewer beneficial strategic factors in his favor loses. Observing the matter in this way, I can see who will win and who will lose. These arguments mean that the gap between the rivaling parties must be calculated first before making decisions on whether to use war as a means to settle conflicts. This in terms of the modern concepts and technology means to realize and aggregate the power of one's own country and to use computer simulation and war gaming to calculate the various scenarios and results of war to be used as references for higher level decision makers. Many countries have the equipment to serve these purposes. These simulation systems combine the use of computers and research methodology. Lanchester, a scholar with a background in engineering and machinery, saw the launch of the First World War in 1914. In 1916, during the height of World War I, Lanchester [3] devised a series of differential equations to demonstrate the power of relationships between opposing forces. Among these are what is now known as Lanchester's Linear Law (for ancient combat) and Lanchester's Square Law (for modern war). For nearly a century, many scholars, such as Engel [4], Deitchman [5], Samz [6], Taylor [7], Helmbold [8], Chu and Chen [9], Hung et al. [10] continue to improve and apply Lanchester's Law to assess battle casualties. Battle casualty assessment involves the calculation of the sum of square estimation between the historical data and the predicted value. Based on historical record, the aim of this research is to determine the attack and defensive attrition coefficient and tactical factor under the assumption that there are three phases for the Ardennes Campaign: the Red force attack, the deadlock period, and the Blue force attack. There are two shift points in our proposed Lanchester Linear model. Some of these simulation systems are controversial but still wildly used to examine Lanchester's model.

Bracken [11] cited the record from the Data Memory Systems, Inc. [12] to provide historical data for the famous World War II Ardennes Campaign; from December 15, 1944 to January 16, 1945 daily information on the two-side battle casualty and equipment loses report. Bracken [11] used the sum of squared errors for a generalizing Lanchester model to fit the Ardennes Campaign data. He considered the exponents in the generalized Lanchester model as parameters. To evaluate the minimum sums of squared errors, he utilized a grid of parameters to derive a numerical estimation. Chen and Chu [9] used the Lanchester Linear Law to fit the Ardennes Campaign. They treated the shift time, accounting for the shift time between defense and attack, as a new variable. Hung *et al.* [10] adopted Lanchester Square Law to fit the Ardennes Campaign. They continued using the shift time variable to improve findings of Bracken [11] and Chen & Chu [9]. Time is always a critical factor for all levels of war. In this study, we consider the shift time between attack to deadlock as a new variable, the shift time between deadlock to defense as the other variable and use Lanchester's Linear Law with Bracken's tactical factor to fit the Ardennes Campaign. At the end of the Ardennes Campaign, the blue force had the superior air force. After December 25, 1944, the sky was clear so the Ardennes Campaign was dominated by the Blue force with their dominating air force advantage. The historical data on December 15, 1944 on the Red force is insufficient. Hence, we only consider the Ardennes Campaign for December 16, 1944 to December 25, 1944. Recently, there is a paper, Yang et al. [13] is partitioned the 10-day period of Ardennes Campaign also into three phases with Lanchester square law to study the historical data with their estimated casualty.

We divide this ten-day battle into three parts: (a) the first phase, Germans attacked and Allies defended; (b) the second phase, both sides attacked and defended, that is the deadlock period; and (c) the third phase, only Allies forces attacked and Germans defended. We consider the sum of squared errors between the actual and estimated attritions as our objective function. In the following section, we give the notation in this paper. Then, we describe Lanchester models, dealing with mathematical derivation, and prove some important theorems. In Section 4, we present the historical data from the Ardennes Campaign, compared with Chen and Chu [9] and Hung et al. [10], where we demonstrate that our optimal solution is closest to actual historical data. Finally, in the discussion we draw some conclusions.

2. Notation

We define the following notation:

B = The Blue (i.e., Allies) combat forces, including tanks, armored personnel carriers, artillery and personnel. The Allies include a British Corps as well as the U. S. forces.

$\dot{B}$ = The actual loss of Blue (Allies) combat forces.

R = The Red (i.e., German) combat forces, including tanks, armored personnel carriers, artillery and personnel.

$\dot{R}$ = The actual loss of Red (German) combat forces.

a = The Allies (Blue) attrition rate without Bracken's tactical factor.

b = the German (Red) attrition rate without Bracken's tactical factor.

d or $\frac{1}{d}$ = Bracken's tactical factor.

$j =$ The last day on which the Germans attacked.

$k =$ The first day on which the Allies attacked.

SSE = Sum of squared errors.

3. Mathematical Formulation

We consider the derivation of the traditional Lanchester's Linear Law.

$$\frac{dB}{dt} = -\alpha\, BR\,, \tag{1}$$

$$\frac{dR}{dt} = -\beta\, BR\,. \tag{2}$$

where α and β are defined as the attrition rate coefficients. It is believed that the attrition rate will be proportional to the reciprocal of the expected time to destroy enemy targets. The original Linear Law determines the full force of both sides operating at the same level of destruction. This rule governs the situation in general, but it is not applicable to some combat situations. Therefore, the tactical factor should be considered in these models for a more practical estimation. Owing to the previous discussion, Bracken proposed a modification of the familiar Lanchesterian formulation when his tactical factor, d or $\frac{1}{d}$, is introduced, giving a more accurate estimation of the historical data. Therefore, we assume that

(1) When the Blue force defends and the Red force attacks, we take

$$\alpha = ad,\ \beta = b\frac{1}{d}\,. \tag{3}$$

(2) When the Blue force and the Red force both seize the initiative of attack, we take

$$\alpha = a,\ \beta = b. \tag{4}$$

(3) When the Blue force attack and the Red force defend, we take

$$\alpha = a\frac{1}{d},\ \beta = bd. \tag{5}$$

From historical records, we have learned that at the beginning of the first ten days of the Ardennes Campaign the Germans attacked, and at the end of the first ten days the Allies attacked. Hence, in the first phase, the Red force attacked and in the third phase, the Blue force attacked. During the middle phase of the campaign, both sides tried to control the initiative of attack, therefore in this mid-field stand off phase, neither side has the advantage (or disadvantage) of the tactical factors. Therefore, we assume that some day j is the last day that the Germans had the attack initiative, during the period from day $j+1$ to day $k-1$, neither side had the attack initiative, and on day k, the attack initiative shifted to the Allies.

Our goal is to find the best fit a, b, d, j and k, to minimize the sum of the squared errors between the actual and theoretical attrition. For $2 \le j < k \le 11$, define the objective function $SSE_{jk}(a,b,d)$ as

$$SSE_{jk}(a,b,d) = \sum_{n=2}^{j}(\dot{B}_n - adB_nR_n)^2 + \sum_{n=2}^{j}(\dot{R}_n - b\frac{1}{d}B_nR_n)^2 + \sum_{n=j+1}^{k-1}(\dot{B}_n - aB_nR_n)^2 + \sum_{n=j+1}^{k-1}(\dot{R}_n - bB_nR_n)^2$$
$$+\sum_{n=k}^{11}(\dot{B}_n - a\frac{1}{d}B_nR_n)^2 + \sum_{n=k}^{11}(\dot{R}_n - bdB_nR_n)^2. \tag{6}$$

Given j and k with $2 \le j < k \le 11$, our procedure is to obtain the local critical points of $SSE_{jk}(a,b,d)$ under the restrictions 0< a, b and d.

We now consider the partial derivatives of $SSE_{jk}(a,b,d)$ to find the critical points of $SSE_{jk}(a,b,d)$.

First, we compute $\frac{\partial SSE_{jk}}{\partial a}$, $\frac{\partial SSE_{jk}}{\partial b}$ and $\frac{\partial SSE_{jk}}{\partial d}$.

$$\frac{\partial SSE_{jk}}{\partial a} = 2\sum_{n=2}^{j}(\dot{B}_n - adB_nR_n)(-dB_nR_n)$$
$$+2\sum_{n=j+1}^{k-1}(\dot{B}_n - aB_nR_n)(-B_nR_n) + 2\sum_{n=k}^{11}(\dot{B}_n - \frac{a}{d}B_nR_n)(\frac{-1}{d}B_nR_n). \tag{7}$$

$$\frac{\partial SSE_{jk}}{\partial b} = 2\sum_{n=2}^{j}(\dot{R}_n - \frac{b}{d}B_nR_n)(\frac{-1}{d}B_nR_n)$$
$$+2\sum_{n=j+1}^{k-1}(\dot{R}_n - bB_nR_n)(-B_nR_n) + 2\sum_{n=k}^{11}(\dot{R}_n - bdB_nR_n)(-dB_nR_n). \tag{8}$$

$$\frac{\partial SSE_{jk}}{\partial d} = 2\sum_{n=2}^{j}(\dot{B}_n - adB_nR_n)(-aB_nR_n) + 2\sum_{n=2}^{j}(\dot{R}_n - \frac{b}{d}B_nR_n)(\frac{b}{d^2}B_nR_n)$$
$$+2\sum_{n=k}^{11}(\dot{B}_n - \frac{a}{d}B_nR_n)(\frac{a}{d^2}B_nR_n) + 2\sum_{n=k}^{11}(\dot{R}_n - bdB_nR_n)(-bB_nR_n). \tag{9}$$

Secondly, we solve the critical points of $SSE_{jk}(a,b,d)$. To simplify the notation and mnemonics, put

$$g(1,j) = \sum_{n=2}^{j}\dot{B}_nB_nR_n,\ g(2,j,k) = \sum_{n=j+1}^{k-1}\dot{B}_nB_nR_n,\ g(3,k) = \sum_{n=k}^{11}\dot{B}_nB_nR_n,$$

$$g(4,j) = \sum_{n=2}^{j}B_n^2R_n^2,\ g(5,j,k) = \sum_{n=j+1}^{k-1}B_n^2R_n^2,\ g(6,k) = \sum_{n=k}^{11}B_n^2R_n^2,$$

$$g(7,k)=\sum_{n=k}^{11}\dot{R}_n B_n R_n,\ \ g(8,j,k)=\sum_{n=j+1}^{k-1}\dot{R}_n B_n R_n,\ \ g(9,j)=\sum_{n=2}^{j}\dot{R}_n B_n R_n\ . \tag{10}$$

We know the value of $g(n,j,k)$ depending on the value of j and (or) k. However, to simplify the expression, when we take a pair of j and k with 2 ≤ j< k ≤ 11, from the expression of $SSE_{jk}(a,b,d)$, without causing ambiguous confusion, we will use $g(n)$ to simplify $g(n,j)$, $g(n,j,k)$, and $g(n,k)$ when there is no confusion. We assume

$$(u,v,w,x,y)=g(u)g(v)g(w)g(x)g(y)+h(u)h(v)h(w)h(x)h(y)\ , \tag{11}$$

where $h(z)=g(z+6)$ for $1\le z\le 3$, $h(z)=g(z)$, for $4\le z\le 6$, and $h(z)=g(z-6)$ for $7\le z\le 9$. Moreover, we assume the following notation to simplify the expression:

$$\theta_{14}=(1,2,4,6,6)\ ,$$

$$\theta_{13}=2(1,3,4,6,6)+(2,2,4,6,6)-(1,1,5,6,6),$$

$$\theta_{12}=2(1,2,4,5,6)+3(2,3,4,6,6)-(1,2,5,6,6),$$

$$\theta_{11}=4(1,3,4,5,6)+2(2,2,4,5,6)+2(3,3,4,6,6)-2(1,1,5,5,6)-2(1,1,6,6,6),$$

$$\theta_{10}=2(1,2,4,4,6)+(1,2,4,5,5)+6(2,3,4,5,6)+(2,3,5,6,6)-2(1,2,5,5,6)-3(1,2,6,6,6),$$

$$\theta_9=4(1,3,4,4,6)+2(1,3,4,5,5)+2(2,2,4,4,6)+(2,2,4,5,5)+4(3,3,4,5,6)$$
$$+(3,3,5,6,6)-2(1,1,4,5,6)-(1,1,5,5,5)-4(1,1,5,6,6)-2(1,3,6,6,6)-(2,2,6,6,6),$$

$$\theta_8=2(1,2,4,4,5)+6(2,3,4,4,6)+3(2,3,4,5,5)+2(2,3,5,5,6)-2(1,2,4,5,6)$$
$$-(1,2,5,5,5)-6(1,2,5,6,6)-(2,3,6,6,6),$$

$$\theta_7=4(1,3,4,4,5)+2(2,2,4,4,5)+4(3,3,4,4,6)+2(3,3,4,5,5)+2(3,3,5,5,6)$$
$$-2(1,1,4,5,5)-2(1,1,5,5,6)-4(1,1,4,6,6)-4(1,3,5,6,6)-2(2,2,5,6,6),$$

$$\theta_6=(1,2,4,4,4)+6(2,3,4,4,5)+2(2,3,4,5,6)+(2,3,5,5,5)-2(1,2,4,5,5)$$
$$-6(1,2,4,6,6)-3(1,2,5,5,6)-2(2,3,5,6,6),$$

$$\theta_5=2(1,3,4,4,4)+(2,2,4,4,4)+4(3,3,4,4,5)+2(3,3,4,5,6)+(3,3,5,5,5)$$
$$-(1,1,4,4,5,)-4(1,1,4,5,6)-4(1,3,4,6,6)-2(1,3,5,5,6)-2(2,2,4,6,6)-(2,2,5,5,6),$$

$$\theta_4=3(2,3,4,4,4)+2(2,3,4,5,5)-(1,2,4,4,5)-6(1,2,4,5,6,)-2(2,3,4,6,6)-(2,3,5,5,6),$$

$$\theta_3=2(3,3,4,4,4)+2(3,3,4,5,5)-2(1,1,4,4,6)-4(1,3,4,5,6)-2(2,2,4,5,6),$$

$$\theta_2=(2,3,4,4,5)-3(1,2,4,4,6)-2(2,3,4,5,6),$$

$$\theta_1=(3,3,4,4,5)-2(1,3,4,4,6)-(2,2,4,4,6),$$

$$\theta_0=-(2,3,4,4,6). \tag{12}$$

By equations (7), (8) and (9), with notation of $g(n, j, k)$, we know that

$$\frac{\partial SSE_{jk}}{\partial a} = 0 \Leftrightarrow a = \frac{d^3 g(1,j) + d^2 g(2,j,k) + dg(3,k)}{d^4 g(4,j) + d^2 g(5,j,k) + g(6,k)}, \tag{13}$$

$$\frac{\partial SSE_{jk}}{\partial b} = 0 \Leftrightarrow b = \frac{d^3 g(7,k) + d^2 g(8,j,k) + dg(9,j)}{d^4 g(6,k) + d^2 g(5,j,k) + g(4,j)}, \tag{14}$$

$$\frac{\partial SSE_{jk}}{\partial d} = 0 \Leftrightarrow ag(1,j) + \frac{b^2}{d^3} g(4,j) + \frac{a^2}{d^3} g(6,k) + bg(7,k)$$

$$= a^2 dg(4,j) + \frac{b}{d^2} g(9,j) + \frac{a}{d^2} g(3,k) + b^2 dg(6,k). \tag{15}$$

If we substitute the expressions for a and b from equations (13) and (14) into equation (15), then we get an equation in d only. This equation can be derived as:

$$\sum_{i=0}^{14} \theta_i d^i = 0. \tag{16}$$

We summarize our findings in the next theorem.

Theorem 1. Given j and k, there exist local critical points for $SSE_{jk}(a,b,d)$ such that a, b and d satisfy the following equations:

$$a = \frac{d^3 g(1,j) + d^2 g(2,j,k) + dg(3,k)}{d^4 g(4,j) + d^2 g(5,j,k) + g(6,k)}, \quad b = \frac{d^3 g(7,k) + d^2 g(8,j,k) + dg(9,j)}{d^4 g(6,k) + d^2 g(5,j,k) + g(4,j)}, \text{ and } \sum_{i=0}^{14} \theta_i d^i = 0.$$

We now consider the positive roots of equation (16). Let $f(d) = \sum_{i=0}^{14} \theta_i d^i$. From $\theta_{14} > 0$, we have that $\lim_{d \to \infty} f(d) = \infty$. On the other hand, we know that $f(0)=\theta_0 < 0$. Since $f(d)$ is a continuous function, we deduce that $f(d)$ has positive roots.

4. Numerical Example

We will use the historical data of the Ardennes Campaign from December 16 to December 25, 1944 reported in Bracken's Table 5 in our following table 1.

Table 1. *Data on combat forces and losses.*

Dates	Blue forces	Blue losses	Red forces	Red losses
Dec. 15	558820	478	144	0
Dec. 16	555482	2594	577446	2656
Dec. 17	553625	3833	571923	4303
Dec. 18	562661	3615	567134	3415
Dec. 19	576795	4200	563255	3263
Dec. 20	644252	3424	570018	3275
Dec. 21	665764	1804	566877	3799
Dec. 22	681412	2350	578629	2866
Dec. 23	683076	2698	576223	4518
Dec. 24	698910	2858	580074	6985
Dec. 25	715159	2177	570005	5638

Owing to the fact that the historical data of the Red force is insufficient, researchers overlooked the first day's (December 15) data. According to possible mode shifts from Red force attack to the deadlock period, and then Red force defend, we partition the 10-day data (from December 16, 1944 to December 25, 1944) into three periods, $[2, j]$, $[j+1, k-1]$, and $[k, 11]$. For a given pair of j and k with $2 \le j < k \le 11$, using Theorem 1, with the help of Mathcad 14, only one positive solution is found from the equation $\sum_{i=0}^{14} \theta_i d^i = 0$. However, the uniqueness of the positive root is not guaranteed to be the case for other data sets. Hence, we examined each data set. The results are presented in Table 2.

Table 2. *Results of SSE, a, b and d.*

j	k	d	a($\times 10^{-9}$)	b($\times 10^{-9}$)	SSE($\times 10^{7}$)
2	3	1.097086	8.542788	10.468961	2.06164292
2	4	1.097296	8.511618	10.520138	2.05460342
2	5	1.129678	8.631787	10.359330	1.98989544
2	6	1.188005	8.834096	10.087462	1.83518260
2	7	1.236984	8.841282	10.021649	1.65657893
2	8	1.221370	8.552311	10.350499	1.71401651
2	9	1.317110	8.385806	10.369163	1.42286353
2	10	1.372509	8.137995	10.631470	1.35597322
2	11	1.239622	7.967669	11.161626	1.87125273
3	4	1.077435	8.366103	10.713062	2.06370649
3	5	1.104195	8.457441	10.584353	2.00871760
3	6	1.151872	8.605804	10.367735	1.87808736
3	7	1.195616	8.610122	10.304362	1.72037947
3	8	1.187710	8.390310	10.567650	1.76038750
3	9	1.270852	8.219011	10.605636	1.50071820
3	10	1.318382	7.990529	10.856561	1.44073694

j	k	d	a(×10 -9)	b(×10 -9)	SSE(×10 7)
3	11	1.204915	7.893680	11.288627	1.89298903
4	5	1.103110	8.396941	10.652465	1.97668952
4	6	1.142513	8.488651	10.503400	1.84834987
4	7	1.180815	8.467154	10.471868	1.69323261
4	8	1.178875	8.279121	10.700334	1.72082103
4	9	1.254370	8.080368	10.779860	1.45936035
4	10	1.300646	7.849971	11.040746	1.38671709
4	11	1.213318	7.794026	11.416426	1.81725606
5	6	1.151387	8.442557	10.531772	1.75326509
5	7	1.187443	8.388897	10.535877	1.58354840
5	8	1.190139	8.205717	10.759953	1.60009171
5	9	1.264307	7.965411	10.884982	1.30502869
5	10	1.314844	7.714423	11.176840	1.20050690
5	11	1.251249	7.665454	11.566268	1.63754230
6	7	1.175595	8.219687	10.743606	1.53257848
6	8	1.182849	8.050400	10.950850	1.53214418
6	9	1.251965	7.779983	11.131542	1.22234944
6	10	1.303337	7.514273	11.457341	1.08980522
6	11	1.261932	7.469769	11.831934	1.49437885
7	8	1.145783	7.886248	11.220887	1.66282436
7	9	1.204894	7.619137	11.442268	1.39966810
7	10	1.249957	7.361174	11.776617	1.28182845
7	11	1.216740	7.343181	12.055629	1.62191613
8	9	1.204257	7.399924	11.723900	1.30664065
8	10	1.254712	7.098237	12.135430	1.14492313
8	11	1.241698	7.039773	12.483575	1.45379849
9	10	1.235972	6.977475	12.394907	1.23192456
9	11	1.230568	6.903854	12.753378	1.50635136
10	11	1.143812	7,167910	12.448154	1.90011702

In Table 2, the smallest sum square estimation is marked by bold face.

For the linear law model, we compared our findings with previous results. In Bracken [11], the sum square estimation is 1.63×10^7 and in Chen and Chu [9], the sun square estimation is 1.23×10^7. On the other hand, our sum square estimation is 1.09×10^7 such that our improvement is 43.9% and 11.4% to that of Bracken [11] and Chen and Chu [9], respectively. It reveals that our approach provides better goodness of fit.

5. Conclusion

We examined the historical date with Lanchester linear model with Bracken tactical factor to discover that the best explanation for the first ten days of battle for the Ardennes Campaign of the World War II. The managerial meaning for our findings is that the tactical factor is roughly estimated 1.3 to demonstrate that attacking is the better military operation policy.

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