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An Analytic Approach to Weakly-Singular Integro-Dynamic Equation on Time Scales

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Abstract: In this paper, we present a new and simple approach to resolve linear and nonlinear weakly-singular Volterra integro-dynamic equations of first and second order on any time scales. In order to eliminate the singularity of the equation, nabla derivative is used and then transforming the given first-order integro-dynamic equations onto an first-order dynamic equations on time scales. The validity of the method is illustrated with some examples.

Keywords: Time Scales, Integro-Dynamic Equations, Volterra Integro-Differential Equation

1. Introduction

Linear and nonlinear Volterra integro-differential equations play an important role in mathematical modeling of many physical, chemical and biological phenomena in which it is necessary to take into account the effect of past history. Particularly in such field as heat transfer, nuclear reactor dynamics, dynamics of linear viscoelastic materials with long memory and thermoelectricity, optics, electromagnetics, electrodynamics, chemistry, electrochemistry, fluid flow, chemical reaction, population dynamics, statical physics, inverse scattering problems and many other practical applications.

During the last decades the researchers are considered the two of the most important types of mathematical equations that have been used to mathematically describe various dynamic procedure. One of them is differential and integral equations and the other is difference and summation equations, which model phenomena respectively: in continuous time; or discrete time. The researchers have used either differential and integral equations or difference and summation equations- but not a combination equations of the two areas to describe dynamic models.

Recently, it is now becaming apperent that certain phenomena do not involve only continuous aspect or only discrete aspects. Rather, they feature elements of both the continuous and discrete . These type of mixed processes can be seen, for example, in population dynamics where non-coincident generations [14] occurs. Additionaly, neither difference nor differential equations give a appropriate description of most population growth [9].

Some problems of mathematical physics are described in terms of nth-order linear and nonlinear Volterra integro-differential equation of the form

$$\sum_{i=0}^{n} u_i(t)y^{(i)}(t) = f(t) + \int_a^t K(t,\tau)y^m(\tau)d\tau, \quad a \le t \le b, \quad (1)$$

where $m \ge 1$, $y(t)$ is the unknown function and $K(t,s)$ is the kernel of integral equations in [1, 17].

In continuous case equations of this form with degenerate, difference and symmetric kernels have been approached by different methods including piecewise polynomials [6], the spline collocations method [7], the homotopy perturbation method [16], Hear wavelets [10], the wavelet-Galerkin method [12], the Tau method [8], Taylor polynomials [11] , the sine-collocations method [19], and the combined Laplace transforms-adomain decomposition method [18] to determine exact and approximate solutions. But if Equ. (1) is weakly-singular Volterra integro-differential equations there is still no viable analytic approach for solving Equ. (1). Recently in [5] the authors are considered the approximate solutions of a class of first and second order weakly-singular form of Equ. (1) with kernel $K(t,s) = \frac{1}{(s-t)^{\alpha}}$ is singular as $t \to s$, where $0 < \alpha < 1$ and in [15] D. B. Pachpatte gives an approximate procedure for first order dynamic integro-differential initial value problem.

In discrete case to our knowledge there isn't any analytic approaching method to the corresponding form of Equ. (1) with weakly singular kernel to discrete form and the time scale calculus is developed mainly to unify differential, difference and $q-$ calculus. Thus in this paper we are considered the first-order linear Volterra integro-dynamic equations in any time scales and we give an approaching method to the solution of the considered integro-dynamic equations with weakly singular kernel.

2. Some Preliminaries

The calculus of time scales was introduced by Aulbach and Hilger [2] in order to create a theory that can unify and extend discrete and continuos analysis.

Definition 1. A time scale $\mathbb{T}$, which inherits the standard topology on $\mathbb{R}$, is an arbitrary nonempty closed subset of the real numbers.

Example 1. The real numbers $\mathbb{R}$, the integers $\mathbb{Z}$, the natural numbers $\mathbb{N}$, the non-negative integers $\mathbb{N}_0$, the $h-$ numbers $h\mathbb{Z}=\{hk:k\in\mathbb{Z}\}$, where $k>0$ is a fixed real number, the $q-$ numbers $\mathbb{k}_q=q^{\mathbb{Z}}\cup\{0\}=\{q^k:k\in\mathbb{Z}\}\cup\{0\}$, where $q>1$ is a fixed real number, $[1,3]\cup[4,7]$, and $[-2,-1]\cup\mathbb{N}$ are examples of time scales.

In [2] Aulbach and Hilger introduced also dynamic equations on time scales in order to unify and extend the theory of ordinary differential equations, difference equations and quantum equations ($h-$ difference and $q-$ difference equations based on $h-$ calculus and $q-$ calculus). For a general introduction to the calculus on time scales we refer the reader to the textbooks by Bohner and Peterson [3, 4]. Here we give only those notions and facts concerned to time scales which we need for our purpose in this paper.

Any time scale $\mathbb{T}$ is a complete metric spaces with the metric (distance) $d(t,s)=|t-s|$ for $t,s\in\mathbb{T}$. Consequently, according to the well-known theory of general metric spaces, we have for $\mathbb{T}$ the fundamental concepts such as open balls (intervals), neighborhood of points, open set, closed sets, and so on. Also we have for function $f:\mathbb{T}\to\mathbb{R}$ the concept of the limit, continuity and properties of continuous functions on general complete metric spaces (note that, in particular, any function $f:\mathbb{Z}\to\mathbb{R}$ is continuous at each point of $\mathbb{Z}$). In order to introduce and investigate the derivative for a function $f:\mathbb{T}\to\mathbb{R}$, forward and backward operators play important roles.

Definition 2. *For $t\in\mathbb{T}$ the forward jump operator σ and backward operator ρ is defined by respectively as follows*

$$\sigma:\mathbb{T}\to\mathbb{T} \text{ by } \sigma(t)=\inf\{s\in\mathbb{T}:s>t\} \textit{ if } t\neq\sup\mathbb{T}, \quad \sigma(t)=t \quad \textit{if } t=\sup\mathbb{T} \quad , \tag{2}$$

and

$$\rho:\mathbb{T}\to\mathbb{T} \text{ by } \rho(t)=\sup\{s\in\mathbb{T}:s<t\} \textit{ if } t\neq\inf\mathbb{T}, \quad \sigma(t)=t \quad \textit{if } t=\inf\mathbb{T}. \tag{3}$$

These jump operators enable us to classify the points $\{t\}$ of a time scale as right-dense, right-scattered, left-dense, and left-scattered depending on whether $\sigma(t)=t$, $\sigma(t)>t$, $\rho(t)=t$, $\rho(t)<t$, respectively, for any $t\in\mathbb{T}$. If $\sup\mathbb{T}<\infty$ and $\sup\mathbb{T}$ is left-scattered we let $\mathbb{T}^{\kappa}=\mathbb{T}\setminus\{\sup\mathbb{T}\}$. Otherwise, we let $\mathbb{T}^{\kappa}=\mathbb{T}$. Similarly if $\mathbb{T}$ has a right-scattered minimum, we let $\mathbb{T}_{\kappa}=\mathbb{T}\setminus\{\min\mathbb{T}\}$, otherwise, we let $\mathbb{T}_{\kappa}=\mathbb{T}$. Finally, the graininess functions $\mu,\nu:\ \mathbb{T}\to 0,\infty)$ are defined by

$$\mu(t):=\sigma(t)-t \text{ and } \nu(t)=t-\rho(t) \text{ for all } t\in\mathbb{T}. \tag{4}$$

Example 2. If $\mathbb{T}=\mathbb{R}$, then $\sigma(t)=\rho(t)=t$ and $\mu(t)=\nu(t)=0$. If $\mathbb{T}=h\mathbb{Z}$, then $\sigma(t)=t+h$, $\rho(t)=t-h$ and $\mu(t)=\nu(t)=h$. If $\mathbb{T}=\mathbb{k}_q$, then $\sigma(t)=qt$, $\rho(t)=q^{-1}t$, $\mu(t)=(q-1)t$, and $\nu(t)=(1-q^{-1})t$.

Definition 3. For $f:\mathbb{T}\to\mathbb{R}$ and $t\in\mathbb{T}_{\kappa}$, we define the nabla derivative of f at t, denoted $f^{\nabla}(t)$, to be number (provided it exists) with the property that given any $\varepsilon>0$, there is a neighborhood U of t such that

$$\left|f(\rho(t))-f(s)-f^{\nabla}(t)[\rho(t)-s]\right|\leq\varepsilon|\rho(t)-s|$$

for all $s\in U$.

The following theorems delineate several properties of the nabla derivative; they are found in [3, 4].

Theorem 1. *Assume $f:\mathbb{T}\to\mathbb{R}$ is a function and $t\in\mathbb{T}_{\kappa}$. Then:*

(i)) If f is nabla differentiable at t, then f is continuous at t.

(ii) If f is continuous at t and t is left-scattered, then f is nabla differentiable at t with

$$f^{\nabla}(t)=\frac{f(t)-f(\rho(t))}{\nu(t)}.$$

(iii) If t is left-dense, then f is nabla differentiable at t iff the limit

$$\lim_{s\to t}\frac{f(t)-f(s)}{t-s}$$

exists as a finite number. In this case

$$f^{\nabla}(t)=\lim_{s\to t}\frac{f(t)-f(s)}{t-s}.$$

(iv) If f is nabla differential at t, then

$$f^{\rho}(t)=f(t)-\nu(t)f^{\nabla}(t).$$

Theorem 2. Assume $f,g:\mathbb{T}\to\mathbb{R}$ are nabla differential at $t\in\mathbb{T}_{\kappa}$. Then:

(i) The sum $f+g:\mathbb{T}\to\mathbb{R}$ is nabla differentiable at t with $(f+g)^{\nabla}(t)=f^{\nabla}(t)+g^{\nabla}(t)$.

(ii) The product $f.g:\mathbb{T}\to\mathbb{R}$ is nabla differentiable at t with

$$(fg)^{\nabla}(t)=f^{\nabla}(t)g(t)+f^{\rho}(t)g^{\nabla}(t)=f(t)g^{\nabla}(t)+f^{\nabla}(t)g^{\rho}(t).$$

(iii) If $g(t)g^{\rho}(t)\neq 0$, then g is nabla differentiable at t with

$$\left(\frac{f}{g}\right)^{\nabla}(t)=\frac{f^{\nabla}(t)g(t)-g^{\nabla}(t)f(t)}{g(t)g^{\rho}(t)}.$$

Example 3. If $\mathbb{T}=\mathbb{R}$ we have $f^{\nabla}=f^{'}$, the usual derivative, and if $\mathbb{T}=\mathbb{Z}$ we have the backward difference operator, $f^{\nabla}(t)=\nabla f(t):=f(t)-f(t-1)$.

Definition 4. A function $f:\mathbb{T}\to\mathbb{R}$ is left-dense continuous (or ld-continuous) provided it is continuous at left-dense points in $\mathbb{T}$ and its right-sided limits exists (finite) at right-dense points in $\mathbb{T}$.

Definition 5. Assume $f:\mathbb{T}\to\mathbb{R}$ is a regulated function. A function $F(t)$ is called an antiderivative of f provided $F^{\nabla}(t)=f(t)$ for all $t\in\mathbb{T}_{\kappa}$. In this case we define the nabla integral by

$$\int_a^b f(t)\nabla t=F(b)-F(a).$$

We now state some definitions and at goal we will define a function , called nabla exponential function, which solves the general first order linear nabla-dynamic IVP.

Definition 6. Let $\mathbb{T}$ be a time scale. We say that a function $p:\mathbb{T}\to\mathbb{R}$ is ν-regressive provided

$$1-\nu(t)p(t)\neq 0 \ \ for\, all\, t\in\mathbb{T}_{\kappa}.$$

Define the ν-regressive class of functions on $\mathbb{T}_{\kappa}$ to be

$\mathcal{R}_{\nu}=\{p:\mathbb{T}\to\mathbb{R}: p\,\text{is ld}-\text{continuous and}\,\nu-\text{regressive}\}.$

If $p\in\mathcal{R}_{\nu}$, then the first order linear dynamic equation

$$y^{\nabla} = p(t)y \tag{5}$$

called ν-regressive. In addition, if $f:\mathbb{T}\to\mathbb{R}$ is ld-continuous, then the first order inhomogenous linear dynamic equation

$$y^{\nabla} = p(t)y + f(t) \tag{6}$$

called ν-regressive. If $p,q\in\mathcal{R}_{\nu}$, then we define the circle plus and minus by

$$p\oplus_{\nu} q = p(t)+q(t)-p(t)q(t)\nu(t),$$

$$\ominus_{\nu} q(t) = -\frac{p(t)}{1-p(t)\nu(t)}.$$

Definition 7. For $h>0,$ let $\mathbb{Z}_h=\left\{z\in\mathbb{C}:\frac{-\pi}{h}\le Im(z)\le\frac{\pi}{h}\right\}$ and $\mathbb{C}_h=\left\{z\in\mathbb{C}:z\ne\frac{1}{h}\right\}.$ Define ν -cylinder transformation $\widehat{\xi}_h:\mathbb{C}_h\to\mathbb{Z}_h$ by $\widehat{\xi}_h(z)=-\frac{1}{h}Log(1-zh),$ where Log is the principal Logarithm function. For $h=0,$ we define $\widehat{\xi}_0(z)=z$ for all $z\in\mathbb{C}_0=\mathbb{C}$. If $p\in\mathcal{R}_{\nu}$, then we define the nabla exponential function by

$$\widehat{e}_p(t,s)=\exp\left(\int_s^t \widehat{\xi}_{\nu(\tau)}(p(\tau))\nabla\tau\right) \tag{7}$$

for $s,t\in\mathbb{T}.$

Theorem 3. Suppose (5) is ν-regressive and fix $t_0\in\mathbb{T}.$ Then $y_0\,\widehat{e}_p(t,t_0)$ is the unique solution of the IVP

$$y^{\nabla} = p(t)y,\ y(t_0)=y_0. \tag{8}$$

Next theorem gives some properties of the nabla exponential function, can be found in [3, 4].

Theorem 4. Let $p,q\in\mathcal{R}_{\nu}$, and $s,t,u\in\mathbb{T}.$ Then

(i) $\widehat{e}_0(t,s)\equiv 1$ and $\widehat{e}_p(t,t)\equiv 1,$

(ii) $\widehat{e}_p(\rho(t),s)=(1-\nu(t)p(t))\,\widehat{e}_p(t,s),$

(iii) $\dfrac{1}{\widehat{e}_p(t,s)}=\widehat{e}_{\ominus_{\nu}p}(t,s),$

(iv) $\widehat{e}_p(t,s)=\dfrac{1}{\widehat{e}_p(s,t)}=\widehat{e}_{\ominus_{\nu}p}(s,t),$

(v) $\widehat{e}_p(t,u)\,\widehat{e}_p(u,s)=\widehat{e}_p(t,s),$

(vi) $\widehat{e}_p(t,s)\,\widehat{e}_q(t,s)=\widehat{e}_{p\oplus_{\nu}q}(t,s),$

(vii) $\dfrac{\widehat{e}_p(t,s)}{\widehat{e}_q(t,s)} = \widehat{e}_{p\ominus_\nu q}(t,s),$

(viii) $\left(\dfrac{1}{\widehat{e}_p(t,s)}\right)^{\nabla} = \dfrac{-p(t)}{\widehat{e}_p^{\rho}(t,s)}.$

Example 4. It is clear that $\widehat{e}_\alpha(t,t_0) = e^{(t-t_0)}$, where α is constant, for $\mathbb{T} = \mathbb{R}$. Now let $\mathbb{T} = h\mathbb{Z}$ for $h > 0$. Let $\alpha \in \mathcal{R}_\nu$ be a constant, i.e., $\alpha \in \mathbb{C} - \left\{\dfrac{1}{h}\right\}$. Then

$$\widehat{e}_\alpha(t,t_0) = \left(\frac{1}{1-\alpha h}\right)^{\frac{t-t_0}{h}} \text{ for all } t \in \mathbb{T}.$$

Theorem 5. [3] Suppose (6) is ν-regressive. Let $t_0 \in \mathbb{T}$, and $y_0 \in \mathbb{R}$. The unique solution of the IVP

$$y^{\nabla} = p(t)y + f(t),\ y(t_0) = y_0 \tag{9}$$

is given by

$$y(t) = y_0\,\widehat{e}_p(t,t_0) + \int_{t_0}^{t} \widehat{e}_p(\tau,\rho(\tau))f(\tau)\nabla\tau.$$

3. Solutions by Approximation Method

We start this section with the recalling the concept of an approximate method for solving linear and nonlinear weakly-singular Volterra integro-dynamic equations as in [14]. This concept will help us to constract the approximation solution of first-order nonlinear weakly-singular Volterra integro-dynamic equations and second-order linear weakly-singular Volterra integro-dynamic equations on time scales, which will be given in subsection 3.1 and 3.2 respectively.

Consider the following first-order linear weakly-singular Volterra integro-dynamic equation

$$y^{\nabla}(t) + p(t)y^{\rho}(t) = f(t) + \int_a^t \frac{y(s)\nabla s}{(s-\rho(t))^{\alpha}} \quad , \text{for } a \le t \le b \text{ and } 0 < \alpha < 1, \tag{10}$$

where $p(t)$ and $f(t)$ are given functions that at least ld-continuous on $[a,b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \le t \le b\}$. Rewriting the integral part of Equ. (3.1) as

$$\int_a^t \frac{y(s)\nabla s}{(s-\rho(t))^{\alpha}} = \int_a^t \frac{y(s) + y(\rho(t)) - y(\rho(t))}{(s-\rho(t))^{\alpha}}\nabla s \tag{11}$$

$$= y(\rho(t))\int_a^t \frac{\nabla s}{(s-\rho(t))^{\alpha}} + \int_a^t \frac{y(s) - y(\rho(t))}{(s-\rho(t))}(s-\rho(t))^{1-\alpha}\nabla s.$$

Thus Equ. (10) can be written as

$$y^{\nabla}(t) + p(t)y^{\rho}(t) = f(t) + y(\rho(t))\int_a^t \frac{\nabla s}{(s-\rho(t))^{\alpha}} + \int_a^t \frac{y(s) - y(\rho(t))}{(s-\rho(t))}(s-\rho(t))^{1-\alpha}\nabla s. \tag{12}$$

If we use the fact $y^{\nabla}(t) = \lim_{s\to t} \dfrac{y(s) - y(\rho(t))}{s-\rho(t)}$, we can take the fraction $\dfrac{y(s) - y(\rho(t))}{s-\rho(t)}$ in the second integral of Equ.

(12) as approximately $y^{\nabla}(t)$. Substituting the approximate relation into the right side of Equ. (10) we can get

$$y^{\nabla}(t)+p(t)y^{\rho}(t)=f(t)+y(\rho(t))g(t)+y^{\nabla}(t)h(t). \tag{13}$$

Therefore, Equ. (10) can be approximated by the following first-order linear dynamic equation

$$y^{\nabla}(t)+P(t)y^{\rho}(t)=F(t). \tag{14}$$

Note that if $\mathbb{T}=\mathbb{R}$ than Equ. (14) becomes first-order linear differential equation $y'(t)+p(t)y(t)=F(t)$ and the general solution may be readily written as $y(t)=e^{-\int P(t)dt}\left[\int e^{\int P(t)dt}F(t)dt+c\right]$. Moreover for $\mathbb{T}=\mathbb{R}$ we can calculate $g(t)$ and $h(t)$ as

$$g(t)=\int_a^t\frac{\nabla s}{(s-\rho(t))^{\alpha}}=\int_a^t\frac{ds}{(s-t)^{\alpha}}=\frac{(t-a)^{1-\alpha}}{\alpha-1} \text{ and } h(t)=\int_a^t(s-\rho(t))^{1-\alpha}\nabla s=\int_a^t(s-t)^{1-\alpha}ds=\frac{(a-t)^{2-\alpha}}{\alpha-2},$$

which is coincide with the section 2.1 of [5].

For the points $\rho(t)=t$ we can calculate $g(t)$ and $h(t)$ as

$$g(t)=\int_a^t\frac{\nabla s}{(s-\rho(t))^{\alpha}}=\lim_{\delta\to 0}\int_a^{t-\delta}\frac{\nabla s}{(s-t)^{\alpha}} \text{ and } h(t)=\int_a^t(s-\rho(t))^{1-\alpha}\nabla s=\lim_{\delta\to 0}\int_a^{t-\delta}(s-t)^{1-\alpha}\nabla s.$$

By the help of Theorem 5 we can write the solution of the Equ. (14) of the form

$$y(t)=\widehat{e_{\ominus\nu P}}(t,a)c+\int_a^t e_{\ominus\nu P}(t,\tau)F(\tau)\nabla\tau \tag{15}$$

under the initial condition $y(a)=c$ for $\rho(t)<t$ [3].

Theorem 6. [14]Let $p(t)$ and $f(t)$ are given functions as in Equ. (10) and $x(t)$ be the solution of Equ. (14) under the condition $x(a)=c$. Then $x(t)$ can be taken the approximate solution of Equ. (10)with the error

$$E(t)=c\left[\left(\widehat{e_{\ominus\nu P}}(t,a)\right)^{\nabla}+p(t)\left(\widehat{e_{\ominus\nu P}}(t,a)\right)^{\rho}-\int_a^t\frac{\widehat{e_{\ominus\nu P}}(s,a)\nabla s}{(s-\rho(t))^{\alpha}}\right]-f(t)$$

$$+\left(\int_a^t\widehat{e_{\ominus\nu P}}(t,\tau)F(\tau)\nabla\tau\right)^{\nabla}+p(t)\left(\int_a^t\widehat{e_{\ominus\nu P}}(t,\tau)F(\tau)\nabla\tau\right)^{\rho}$$

$$-\int_a^t\frac{\int_a^s\widehat{e_{\ominus\nu P}}(s,\tau)F(\tau)\nabla\tau}{(s-\rho(t))^{\alpha}}\nabla s.$$

Remark 1. If we take $x(a)=0$ and $f(t)=0$ then the solution of Equ. (14) under the condition $x(a)=0$ will be exact solution of Equ. (10).

Example 5. Let $f(t)=\frac{1}{t^2}$, $p(t)=\sum_{s=a}^{t-2}\frac{1}{\sqrt{t-1-s}}-\frac{1}{9}\left(1+\sum_{s=a}^{t-1}\sqrt{t-1-s}\right)$ and $\alpha=\frac{1}{2}$. Then for $a=5$ we get

$$g(t)=\sum_{s=5}^{t-2}\frac{1}{\sqrt{t-1-s}},\ h(t)=\sum_{s=5}^{t-1}\sqrt{t-1-s},\ P(t)=-\frac{1}{9},$$

$$F(t) = \frac{1}{t^2\left(1 + \sum_{s=5}^{t-1}\sqrt{t-1-s}\right)},\ \ominus_{\nu P}(t) = \frac{1}{10} \text{ and } \widehat{e}_{\ominus \nu P}(t,5) = \left(\frac{10}{9}\right)^{t-5}$$

respectively. Thus from Theorem 6 we find that for $t = 10$

$$E(t) = E(10) = 0.001722530658 - 0.183136796 \cdot c,$$

as an error. For $a = 10$ we find that

$$g(t) = \sum_{s=10}^{t-2}\frac{1}{\sqrt{t-1-s}},\ h(t) = \sum_{s=10}^{t-1}\sqrt{t-1-s},\ P(t) = -\frac{1}{9},$$

$$F(t) = \frac{1}{t^2\left(1 + \sum_{s=10}^{t-1}\sqrt{t-1-s}\right)},\ \ominus_{\nu P}(t) = \frac{1}{10} \text{ and } \widehat{e}_{\ominus \nu P}(t,10) = \left(\frac{10}{9}\right)^{t-10}$$

respectively and the error will be approximately

$$E(t) = E(10) = 0.0006734070565 - 0.183136796 \cdot c,$$

for $t = 10$. Finally if we choice $a = 10$ and $f(t) = \frac{1}{t}$ we get

$$g(t) = \sum_{s=10}^{t-2}\frac{1}{\sqrt{t-1-s}},\ h(t) = \sum_{s=10}^{t-1}\sqrt{t-1-s},\ P(t) = -\frac{1}{9},$$

$$F(t) = \frac{1}{t\left(1 + \sum_{s=10}^{t-1}\sqrt{t-1-s}\right)},\ \ominus_{\nu P}(t) = \frac{1}{10} \text{ and } \widehat{e}_{\ominus \nu P}(t,10) = \left(\frac{10}{9}\right)^{t-10}$$

respectively and the error will be approximately

$$E(t) = E(15) = 0.002336477015 - 0.183136796 \cdot c,$$

for $t = 15$.

3.1. First-Order Nonlinear Weakly-Singular Volterra Integro-Dynamic Equations

In this subsection we consider the first-order nonlinear Volterra integro-dynamic equation of the form

$$y^{\nabla}(t) + p(t)y^{\rho}(t) = f(t) + \int_a^t \frac{G(y(s))}{(\rho(t)-s)^{\alpha}}\nabla s;\ a \le t \le b,\ 0 < \alpha < 1, \tag{16}$$

where $G(y)$ is analytic in the solution y.

We have by setting

$$u(t) = G(y(t)), \tag{17}$$

$$y^{\nabla}(t) + p(t)y^{\rho}(t) = f(t) + \int_a^t \frac{u(s)}{(\rho(t)-s)^{\alpha}}\nabla s; a \le t \le b,\ 0 < \alpha < 1,$$

as before, we write this equation in the following form

$$y^{\nabla}(t)+p(t)y^{\rho}(t)=f(t)+\int_a^t \frac{u(s)-u(t)+u(t)}{(\rho(t)-s)^{\alpha}}\nabla s;\ a\le t\le b,\ 0<\alpha<1,$$

or

$$y^{\nabla}(t)+p(t)y^{\rho}(t)=f(t)+u(t)\int_a^t \frac{1}{(\rho(t)-s)^{\alpha}}\nabla s+\int_a^t \frac{u(s)-u(t)}{(\rho(t)-s)}(\rho(t)-s)^{1-\alpha}\nabla s\,.$$

An approximate solution can be found by considering $\dfrac{u(t)-u(s)}{\rho(t)-s}\simeq G^{\nabla}(t)$

$$y^{\nabla}(t)+p(t)y^{\rho}(t)=f(t)+G(y(\rho(t)))g(t)-G^{\nabla}(y(t))h(t), \tag{18}$$

where $g(t)$ and $h(t)$ are the same as before.

Therefore, Equ. (16) can be approximated by the first-order nonlinear dynamic equation Equ. (18).

Naturally Equ. (16) and Equ. (18)coincide with Equ. (2) and Equ. (8) of [5] when $\mathbb{T}=\mathbb{R}$.

3.2. Second-Order Linear Weakly-Singular Volterra Integro-Dynamic Equations

The same procedure can be adopted to transform a second-order linear weakly-singular Volterra integro-dynamic equation into an first-order dynamic equation, which permits convenient resolution of these equations.

Consider the following second-order linear weakly-linear Volterra integro-dynamic equation

$$y^{\nabla\nabla}(t)+p(t)y^{\nabla}(t)+g(t)y^{\rho}(t)=f(t)+\int_a^t \frac{y(s)}{(\rho(t)-s)^{\alpha}}\nabla s\,,\ a\le t\le b,\ 0<\alpha<1, \tag{19}$$

where $f(t)$, $g(t)$ and $p(t)$ are given functions as in previous section. If we use the same procedure as in previous section we can write

$$\int_a^t \frac{y(s)}{(\rho(t)-s)^{\alpha}}\nabla s=y(\rho(t))\int_a^t \frac{\nabla s}{(s-\rho(t))^{\alpha}}+y^{\nabla}(t)\int_a^t (s-\rho(t))^{1-\alpha}\nabla s,$$

or

$$\int_a^t \frac{y(s)}{(\rho(t)-s)^{\alpha}}\nabla s=y^{\rho}(t)g(t)+y^{\nabla}(t)h(t),$$

where $h(t)=\int_a^t (s-\rho(t))^{1-\alpha}\nabla s$ as in section 3.

Thus we can rewrite Equ.(3.10) as

$$y^{\nabla\nabla}(t)+p(t)y^{\nabla}(t)=f(t)+y^{\nabla}(t)h(t). \tag{20}$$

By setting $y^{\nabla}(t)=z(t)$ in Equ. (20) we get first-order linear dynamic equation as

$$z^{\nabla}(t)+Q(t)z(t)=f(t), \tag{21}$$

where $Q(t)=p(t)-h(t)$.

Therefore, Equ.(3.10) can be approximated by the first-order nonlinear dynamic equation Equ. (21).

If we use the same procedure of the Theorem 6 we get the error function as

$$E(t) = z^{\nabla}(t) + p(t)z(t) + g(t)\left(\int_a^t z(s)\nabla s\right)^{\rho} - f(t) - \int_a^t \frac{\int_a^s z(\tau)\nabla\tau}{(\rho(t)-s)^{\alpha}}\nabla s. \tag{22}$$

Example 6 Let $p(t) = h(t)$ and $f(t) = \lambda$, where λ is an abritrary constant. Then we find that

$$z(t) = y^{\nabla}(t) = c\lambda(t-a) = c_1(t-a), \text{where } c_1 = c\lambda,$$

$$z^{\nabla}(t) = y^{\nabla\nabla}(t) = c_1,$$

$$\int_a^t z(s)\nabla s = y^{\nabla}(t) = c_1\int_a^t s\nabla s - c_1 a(t-a).$$

If we use these facts in Equ. (22) we find that

$$E(t) = c_1\left(1 - p(t)(t-a) - g(t)\left[\left(\int_a^t s\nabla s\right)^{\rho} + a^2 - a\rho(t)\right] - \int_a^t \left(\frac{\int_a^s \tau\nabla\tau + a^2 - as}{(\rho(t)-s)^{\alpha}}\right)\nabla s\right) - k$$

$$= c_1\left(1 - h(t)(t-a) - g(t)\left[\left(\int_a^t s\nabla s\right)^{\rho} + a^2 - a\rho(t)\right] - \int_a^t \left(\frac{\int_a^s \tau\nabla\tau + a^2 - as}{(\rho(t)-s)^{\alpha}}\right)\nabla s\right) - k.$$

It is easy to sea that $E(t) = 0$ for $\lambda = 0$, $E(t) = \lambda(c-1)$ for $a = 1$ and $t = 2$, $E(t) = \lambda(-3c-1)$ for $a = 1$ and $t = 3$ and finally $E(t) = \lambda(-10.65685425\cdot c - 1)$ for $a = 1$ and $t = 4$. Naturally as $t - a$ increases $E(t)$ increases.

Note: In order to calculate $E(t)$ in the above examples Maple 13 software has been used.

4. Some Remarks

We have reduced the solution of a class of linear and nonlinear weakly-singular Volterra integro-dynamic equations to the solution of ordinary dynamic equations by removing the singularity using an approximate nabla derivative. Then we have demonstrated the solution of these ordinary dynamic equations, which approximate the solution for the original weakly-singular Volterra integro-dynamic equations.

5. Conclusions

We have considered several distinct examples to illustrate our new approach and have verified our solution, beginning with first-order and second order linear weakly-singular Volterra integro-dynamic equations. Of course, it would be better to obtain a similar procedure if $g(t)$ is an arbitrary ld-contunious function in Equ.(3.10). It seems that ones can get over the problem by using the Taylor expansions of a function on time scales [3, 4].

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