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On Sparre Andersen Model with Partial Premium Payment Strategy to Shareholders with Dependence via Sperman Copula

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Abstract: This paper is based on the Poisson composite risk model, popularised for its flexibility in modelling loss occurrences. However, it innovates by incorporating a strategy of distributing dividends to shareholders, adding a realistic dimension to the financial implications. A key element is the introduction of a constant threshold 'b', representing a critical amount beyond which claims become significant. This threshold makes it possible to distinguish between small, routine claims and major events with a significant impact on reserves. In addition, the model introduces a dependency between the amount of claims and the time between claims via the Spearman copula. This copula captures the non-independence often observed in insurance data, where large claims tend to be followed by claim-free periods or vice versa. The analysis then focuses on the integro-differential equation associated with the model, which describes the evolution of Gerber's Shiu function, a fundamental element in assessing the reserve required to cover future obligations. The Laplace transform of this function is also studied, providing valuable information on the distribution of the long-term reserve.

Keywords: Gerber-Shiu Functions, Dependence, Spearman Copula, Dividends, Integro-Differential Equation

1. Introduction

Mathematical models are constantly being developed in response to the need for better knowledge of risks with the simplifying assumption of independence between the random variables the random variables involved in risk modelling (See, for example, references [11, 17]). However, in certain practical contexts, this assumption needs to be relaxed, as it is inappropriate and too restrictive (see [15, 16]). In flood insurance, for example, the occurrence of several floods in a short space of time in a short space of time can lead to major damage, and therefore large claims, as a result of accumulation of water. In earthquake insurance, it's the other way around, because in a high-risk zone, the longer the time between two earthquakes, the greater the impact of the second earthquake, due to the accumulation of energy.

To make up for this shortcoming (see [1, 9, 10, 13],), many works include in the risk model the dependence between certain dependence between certain random variables, in particular the variables claim amount and inter-claim time, thanks to the Farlie Gumbel Morgenstern copula (see, for example, references [5-8, 12, 18]). Although this copula is commonly used in the literature, encounters certain limitations. It fails to model tail dependencies (see references [2-4]).

To remedy the inadequacy of the Farlie Gumbel Morgenstern copula, while taking into account the reality of insurance companies, we consider in this article, the Compound Poisson risk model in which we integrate not only the dependence between the variables claim amounts and interclaim times via the Spearman copula, with also a strategy of partial payment of dividends to shareholders of constant threshold b.

In this model, when the surplus process reaches the constant threshold barrier b set, bonuses are partially granted to shareholders at a contant rate θ such that $0 < \theta < 1$. Noting by $U_b(t)$ the surplus process in the presence of the threshold dividend barrier b (with $U_b(0) = u$), the model follows the following dynamics:

$$dU_b(t) = \begin{cases} cdt - dS(t) \; si \; U_b(t) < b \\ (1-\theta)cdt - dS(t) \; si \; U_b(t) = b \end{cases} \quad (1)$$

where:

$U_b(t)$ is the surplus process in the presence of a b threshold dividend barrier b (with $U_b(0) = u$ the initial surplus and $0 < u \le b$);

c is the constant rate of premium received by the insurer per unit of time;

t_b is the first instant when the surplus reaches the horizontal barrier b so $t_b = \frac{b-u}{c}$.

$S(t) = \sum_{i=1}^{N(t)} X_i$ s the aggregate Poisson loss process composed of:

$\{N(t), t \ge 0\}$ the total number of claims recorded up to time t, which follows a Poisson process of intensity $\lambda > 0$; (Note that$S(t) = 0 \; if \; N(t) = 0$);

$\{X_i, i \ge 1\}$ a sequence of random representing the individual amounts of claims with common density function f and distribution function F and assumed to have an exponential distribution Erlang (2) de paramètre β.

The interclaim times $\{V_i, i \ge 1\}$ form a sequence of random variables with Erlang (2) law of parameter λ, probability density function $k(t) = \lambda^2 t e^{-\lambda t}$ and distribution function

$$K(t) = 1 - e^{-\lambda t} - \lambda t e^{-\lambda t};$$

The aim of this work is to determine the integro-differential equation and the Laplace transform of the Gerber Shiu function in the risk model defined by the relation (1). The rest of the article is structured as follows: In section 2, we discuss the preliminaries of the risk model defined by the relation (1). In Section 3, we study the integro-differential equation satisfied by the Gerber Shiu function in the risk model defined by relation (1).

2. Preliminaries

2.1. Ruin Probability

The insurer's probability of ruin is the probability of ruin occurring either over a finite horizon or over an infinite horizon. In the latter case, we speak of the ultimate probability of risk.

Let τ be the insurance company's instant of ruin. τ is defined by:

$$\tau = inf\{t \ge 0, U(t) < 0 | U(0) = u\} \quad (2)$$

When the probability of ruin is always zero, by convention we note $\tau = \infty$ in this case

$$U(t) \ge 0 \; \forall \; t \ge 0.$$

The probability of ultimate failure is defined by:

$$\psi(u) = \psi(u, \infty) = Pr[\tau < \infty, U(t) < 0 | U(0) = u\,] \quad (3)$$

2.2. Gerber-Shiu Discounted Penalty Function

The Gerber-Shiu expected penalty function or Gerber-Shiu function appeared in 1998 in the work of Gerber and Shiu. Nowadays, this function is of great interest for research. Its analysis remains a central issue in both insurance and finance, as it is a valuable tool not only in the study of the probability of ruin, but also in the calculation of pension and reinsurance premiums, the pricing of options and so on. It is defined by:

$$\phi(u) = E[e^{-\delta\tau} w\big(U_{(\tau^-)}, |U_\tau|\big) I(\tau < \infty) | U(0) = u] \quad (4)$$

where:

τ is the instant of failure defined by the relation (2);

τ^- is the moment just before ruin;

δ is a force of interest;

The penalty function $w(x, y)$ is a positive function of the surplus just before ruin $U_{(\tau^-)}$ and of the ruin deficit $|U_\tau|, \forall \; x, y \ge 0$;

I is the indicator function which is worth 1 if event A occurs and 0 otherwise.

2.3. Dependency Model Based on Spearman's Copula

In this work, the dependency structure is provided by the Spearman copula defined by: $\forall \; (u, v) \in [0,1]^2$ and $\alpha \in [0,1]$ par:

$$C_\alpha(u, v) = (1-\alpha) C_I(u, v) + \alpha C_M(u, v) \quad (5)$$

Where: $C_I(u, v) = uv$; $C_M(u, v) = min(u, v)$; α is dependency parameter.

Spearman's copula can be used to express positive dependencies and also tail dependencies in many situations in many situations (see [18, 19]). Using formula (5), the random vector claims amount and inter-claim times $(X, \; V\,)$ has the joint distribution function given by:

$$\begin{aligned} F_{X,V}(x, t) &= C_\alpha(F_X(x), F_V(t)) \\ &= (1-\alpha) C_I(F_X(x), F_V(t)) + \alpha C_M(F_X(x), F_V(t)) \\ &= (1-\alpha) F_I(x, t) + \alpha F_M(x, t) \end{aligned} \quad (6)$$

Where: $F_X \; F_V$ are are the respective marginals of the random variables X and V.

The copula support C_M est $D = \{\, (u, v) \in [0,1]^2 : u = v\}$

On the domain $[0,1]^2 \setminus D, \frac{\partial^2 C_M}{\partial u \partial v} = 0$; and on D, C_M is uniformly distributed.

Since the dependency structure is described by the copula C_M then they are monotonic and there is almost certainly an increasing function l, such that $X = l(V)$ (See [8], page 27).

The distribution function of X is:

$$F_X(x) = F_V(l^{-1}(x))$$

$$\Leftrightarrow 1 - e^{-\beta x} = 1 - e^{-\lambda l^{-1}(x)}$$

$$\Leftrightarrow -\lambda l^{-1}(x) = -\beta x$$

$$\Leftrightarrow l^{-1}(x) = \frac{\beta x}{\lambda} \quad (7)$$

From relation (7) we deduce that: $l(t) = \frac{\lambda}{\beta} t$.

The joint distribution $F_{X,V}(x,t)$ of the random vector (X, V) is singular, whose support is the domain $D' = \{(x,t): F_X(x) = F_V(t)\} = \{(x,t): \mathrm{x} = l(t)\}$.

Its distribution is $G(t) = F_M(l(t), t) = 1 - e^{-\lambda t}$ on the domain $D' = \left\{(x,t): \mathrm{x} = \frac{\lambda}{\beta} t\right\}$.

3. Integro-Differential Equation Satisfied by the Gerber Shiu Function

The aim of this section is to determine the differential equation satisfied by the function $\phi_b(u)$ in a risk model with constant threshold dividend payment b and dependence between the random variables claim amount and inter-claim time via Spearman's copula. In this risk model (see reference [2, 3, 4]), the Gerber Shiu function $\phi_b(u)$ is given by:

$$\phi_b(u) = (1-\alpha)\left(I_{b,1}(u) + I_{b,2}(u)\right) + \alpha\left(I_{b,3}(u) + I_{b,4}(u)\right) \quad (8)$$

Where:

$$I_{b,1}(u) = \int_0^\infty \int_0^{u+ct} e^{-\delta t}\phi_b(u+ct-x)dF_I(x,t);$$

$$I_{b,2}(u) = \int_0^\infty \int_{u+ct}^\infty e^{-\delta t} W(u+ct, x-u-ct)dF_I(x,t);$$

$$I_{b,3}(u) = \int_0^\infty \int_0^{u+ct} e^{-\delta t}\phi_b(u+ct-x)dF_M(x,t);$$

$$I_{b,4}(u) = \int_0^\infty \int_{u+ct}^\infty e^{-\delta t} W(u+ct, x-u-ct)dF_M(x,t)$$

To determine the integro-differential equation satisfied by the Gerber Shiu function in the risk model defined by relation (1), we adopt the following approach:

1) The first loss occurs at time t before the surplus process reaches the barrier $b\left(t < \frac{b-u}{c}\right)$. The amount x is such that $x < u + ct$.
2) The first loss occurs at time t before the surplus process reaches the barrier $b\left(t < \frac{b-u}{c}\right)$. The amount x is such that $x > u + ct$.
3) The first loss occurs at time t after the surplus process has crossed the barrier. $b\left(t > \frac{b-u}{c}\right)$. The amount x is such that $x < b + (1-\theta)c(t - t_b)$.
4) The first loss occurs at time t after the surplus process has crossed the barrier $b\left(t > \frac{b-u}{c}\right)$. The amount x is such that $x > b + (1-\theta)c(t - t_b)$.

By conditioning on the time and amount of the first claim, and taking into account the different scenarios above, we have:

$$I_{b,1}(u) = \int_0^{t_b} \int_0^{u+ct} e^{-\delta t}\phi_b(u+ct-x)\,dF_I(x,t) + \int_{t_b}^\infty \int_0^{b+\varepsilon_1 c(t-t_b)} e^{-\delta t}\phi_b(b+\varepsilon_1 c(t-t_b)-x)\,dF_I(x,t)$$

$$I_{b,2}(u) = \int_0^{t_b} \int_{u+ct}^\infty e^{-\delta t} W(u+ct, x-u-ct)dF_I(x,t) + \int_{t_b}^\infty \int_{b+\varepsilon_1 c(t-t_b)}^\infty e^{-\delta t} W\big(b+\varepsilon_1 c(t-t_b), x-b-\varepsilon_1 c(t-t_b)\big)\,dF_I(x,t) \quad (9)$$

Where $t_b = (b-u)/c$ and : $\varepsilon_1 = 1 - \theta$.

The copula C_I being the independent part of the Spearman copula, we have:

$$dF_I(x,t) = \lambda e^{-\lambda t} f_X(x)dxdt \quad (10)$$

BY posing $I_b(u) = I_{b,1}(u) + I_{b,2}(u)$, and with using the relations (7); (9), (10), we have:

$$I_b(u) = \lambda^2 \int_0^{t_b}\int_0^{u+ct} te^{-\delta t}\phi_b(u+ct-x)\,e^{-\lambda t} f_X(x)dxdt + \lambda^2 \int_{t_b}^\infty \int_0^{b+\varepsilon_1 c(t-t_b)} te^{-\delta t}\phi_b(b+\varepsilon_1 c(t-t_b) - x)\,e^{-\lambda t} f_X(x)dxdt + \lambda^2 \int_0^{t_b}\int_{u+ct}^\infty te^{-\delta t} W(u+ct, x-u-ct)e^{-\lambda t} f_X(x)dxdt + \lambda^2 \int_{t_b}^\infty \int_{b+\varepsilon_1 c(t-t_b)}^\infty te^{-\delta t} W\big(b+\varepsilon_1 c(t-t_b), x-b-\varepsilon_1 c(t-t_b)\big)\,e^{-\lambda t} f_X(x)dxdt \quad (11)$$

To simplify the notation of relation (11), we pose:

$$\omega(u) = \int_u^\infty w(u, x-u)f(x)dx;\ \sigma_b(u) = \int_0^u \phi_b(u-x)f(x)dx + \omega(u) \quad (12)$$

The relation (11) becomes:

$$I_b(u) = \lambda^2 \int_0^{\frac{b-u}{c}} te^{-(\delta+\lambda)t}\sigma_b(u+ct)dt + \lambda^2 \int_{\frac{b-u}{c}}^\infty te^{-(\delta+\lambda)t}\sigma_b\left(b + c\varepsilon_1\left(t - \frac{b-u}{c}\right)\right)dt \quad (13)$$

Let's move on to calculating integrals $I_{b,3}(u)$ and $I_{b,4}(u)$ in the relation (8).

$$I_{b,3}(u) = \int_0^{t_b}\int_0^{u+ct} e^{-\delta t}\phi_b(u+ct-x)\,dF_M(x;t) + \int_0^{t_b}\int_{u+ct}^\infty e^{-\delta t} W(u+ct, x-u-ct)\,F_M(x;t)$$

$$= \int_K e^{-\delta t}\phi_b(u+ct-x)\,G(t) + \int_J e^{-\delta t} W(u+ct, x-u-ct)\,dG(t) \quad (14)$$

where:

$$K = \left\{t \in \mathbb{R}^+: 0 \le t \le \frac{b-u}{c}\ and\ 0 \le x = \frac{\lambda}{\beta}t \le u + ct\right\}$$

$$= \left\{t \in \mathbb{R}^+: 0 \le t \le \frac{b-u}{c}\ and\ \left(c - \frac{\lambda}{\beta}\right)t \ge -u\right\}$$

$$= \left\{t \in \mathbb{R}^+: 0 \le t \le \frac{b-u}{c}\ and\ t \in \mathbb{R}^+\right\} \text{ because } c > \frac{\lambda}{\beta}; t \ge 0\ and\ u > 0$$

Hence:

$$K = \left[0;\ \frac{b-u}{c}\right] \tag{15}$$

$$J = \left\{t \in \mathbb{R}^+ : 0 \le t \le \frac{b-u}{c}\ and\ u + ct \le x = \frac{\lambda}{\beta}t\right\}$$

$$= \left\{t \colon \in \mathbb{R}^+ : 0 \le t \le \frac{b-u}{c}\ and \left(\frac{\lambda}{\beta} - c\right)t \ge u\right\}$$

$\frac{\lambda}{\beta} - c < 0;\ t \ge 0\ and\ u > 0\ \Longrightarrow \left\{t \colon \in \mathbb{R}^+\ and\ \left(\frac{\lambda}{\beta} - c\right)t \ge u\right\} = \emptyset$

Hence:

$$J = \emptyset \tag{16}$$

By injecting relations (15) and (16) into relation (14), we obtain:

$$I_{b,3}(u) = \int_0^{t_b} e^{-\delta t}\phi_b(u + ct - x)\, G(t)$$

$$= \lambda^2 \int_0^{t_b} te^{-(\delta+\lambda)t}\phi_b\left(u + ct - \frac{\lambda}{\beta}t\right)dt$$

$$I_{b,3}(u)\ = \lambda^2 \int_0^{\frac{b-u}{c}} te^{-(\delta+\lambda)t}\phi_b\left(u + \left(c - \frac{\lambda}{\beta}\right)t\right)dt \tag{17}$$

$$I_{b,4}(u) = \int_{t_b}^{\infty}\int_0^{b+c\varepsilon_1(t-t_b)} e^{-\delta t}\phi_b(b + c\varepsilon_1(t - t_b) - x)\, dF_M(x;t) + \int_{t_b}^{\infty}\int_{b+c\varepsilon_1(t-t_b)}^{\infty} e^{-\delta t}W(b + c\varepsilon_1(t - t_b), b + c\varepsilon_1(t - t_b) - b)\, F_M(x;t)$$

$$I_{b,4}(u)\ = \int_{K'} e^{-\delta t}\phi_b(b + c\varepsilon_1(t - t_b) - x)\, G(t) + \int_{J'} e^{-\delta t}W(b + c\varepsilon_1(t - t_b), x - b)\, dG(t) \tag{18}$$

where:

$$K' = \left\{t \in \mathbb{R}^+ : t \ge \frac{b-u}{c}\ and\ 0 \le x = \frac{\lambda}{\beta}t \le b + c\varepsilon_1(t - t_b)\right\}$$

$$= \left\{t \in \mathbb{R}^+ : t \ge \frac{b-u}{c}\ and\ \left(\frac{\lambda}{\beta} - c\varepsilon_1\right)t \le b - \varepsilon_1(b - u)\right\}$$

To guarantee solvency, it is assumed that: $c\varepsilon_1 \ge \frac{\lambda}{\beta} \Longrightarrow \frac{\lambda}{\beta} - c\varepsilon_1 \le 0$;
We also have: $t \ge 0;\ b - \varepsilon_1(b - u) \ge 0$ because $0 < \varepsilon_1 \le 1$ and $b - u < b$
Hence:

$$K' = \left\{t \in \mathbb{R}^+ : t \ge \frac{b-u}{c}\ and\ t \in \mathbb{R}^+\right\}$$

Subsequently:

$$K' = \left[\frac{b-u}{c};\ +\infty\right[\tag{19}$$

$$J' = \left\{t \in \mathbb{R}^+ : t \ge \frac{b-u}{c}\ and\ x = \frac{\lambda}{\beta}t \ge b + c\varepsilon_1(t - t_b)\right\}$$

$$= \left\{t \colon \in \mathbb{R}^+ : t \ge \frac{b-u}{c}\ and\ \left(\frac{\lambda}{\beta} - c\varepsilon_1\right)t \ge b - \varepsilon_1(b - u)\right\}$$

We have: $\frac{\lambda}{\beta} - c\varepsilon_1 < 0; t \ge 0;\ b - \varepsilon_1(b - u)$ Hence: $\left\{t \in \mathbb{R}^+ : \left(\frac{\lambda}{\beta} - c\varepsilon_1\right)t \ge b - \varepsilon_1(b - u)\right\} = \emptyset$.
We also have:

$$J' = \emptyset \tag{20}$$

By injecting relations (19) and (20) into relation (18), we have:

$$I_{b,4}(u) = \lambda^2 \int_{\frac{b-u}{c}}^{\infty} te^{-(\delta+\lambda)t}\phi_b\left(b + c\varepsilon_1\left(t - \frac{b-u}{c}\right) - \frac{\lambda}{\beta}t\right)dt \tag{21}$$

By posing: $I_b^{\star}(u) = I_{b,3}(u) + I_{b,4}(u)$.

$$I_b^{\star}(u) = \lambda^2 \int_0^{\frac{b-u}{c}} te^{-(\delta+\lambda)t}\phi_b\left(u + \left(c - \frac{\lambda}{\beta}\right)t\right)dt + \lambda^2 \int_{\frac{b-u}{c}}^{\infty} te^{-(\delta+\lambda)t}\phi_b\left(b + c\varepsilon_1\left(t - \frac{b-u}{c}\right) - \frac{\lambda}{\beta}t\right)dt \quad (22)$$

By relations (13) and (22) relation (8) becomes:

$$\phi_b(u) = (1-\alpha)\left(\lambda^2 \int_0^{\frac{b-u}{c}} te^{-(\delta+\lambda)t}\sigma_b(u+ct)dt + \lambda^2 \int_{\frac{b-u}{c}}^{\infty} te^{-(\delta+\lambda)t}\sigma_b\left(b + c\varepsilon_1\left(t - \frac{b-u}{c}\right)\right)dt\right) + \alpha\left(\lambda^2 \int_0^{\frac{b-u}{c}} te^{-(\delta+\lambda)t}\phi_b\left(u + \left(c - \frac{\lambda}{\beta}\right)t\right)dt + \lambda^2 \int_{\frac{b-u}{c}}^{\infty} te^{-(\delta+\lambda)t}\phi_b\left(b + c\varepsilon_1\left(t - \frac{b-u}{c}\right) - \frac{\lambda}{\beta}t\right)dt\right) \quad (23)$$

The relation (23) can be expressed as:

$$\phi_b(u) = \lambda^2(1-\alpha)\int_0^{\infty} te^{-(\delta+\lambda)t}\sigma_b\left((u+ct) \wedge \left(b + c\varepsilon_1\left(t - \frac{b-u}{c}\right)\right)\right)dt + \alpha\lambda^2 \int_0^{\infty} te^{-(\delta+\lambda)t}\phi_b\left(\left(u + \left(c - \frac{\lambda}{\beta}\right)t\right) \wedge \left(b + c\varepsilon_1\left(t - \frac{b-u}{c}\right) - \frac{\lambda}{\beta}t\right)\right)dt$$

By a change of variable by posing $s = b + c\varepsilon_1\left(t - \frac{b-u}{c}\right)$ *and* $s = u + \left(c - \frac{\lambda}{\beta}\right)t$, we obtain:

$$\phi_b(u) = \frac{\lambda^2(1-\alpha)}{c^2\varepsilon_1} \times \int_{b-\varepsilon_1(b-u)}^{\infty} \left(\frac{s-b}{\varepsilon_1} + b - u\right)e^{-\left(\frac{\delta+\lambda}{c}\right)\left(\frac{s-b}{\varepsilon_1}+b-u\right)}\sigma_b\left(\left(b + \frac{s-b}{\varepsilon_1}\right) \wedge s\right)ds + \frac{\alpha(\beta\lambda)^2}{\beta c-\lambda} \times \int_u^{\infty}\left(\frac{s-u}{\beta c-\lambda}\right)e^{-\beta\left(\frac{\delta+\lambda}{\beta c-\lambda}\right)(s-u)}\phi_b\left(s \wedge \left(b + (\beta c\varepsilon_1 - \lambda)\left(\frac{s-u}{\beta c-\lambda}\right) - \varepsilon_1(b-u)\right)\right) \quad (24)$$

Theorem 3.1: *The Gerber Shiu function $\phi_b(u)$ in a risk model defined by relation (1) satisfies the following integro-differential equation*:

$$\left[\left(\mathcal{D} - \beta\left(\frac{\delta+\lambda}{\beta c-\lambda}\right)\ell\right)\right]^2\left[\left(\mathcal{D} - \left(\frac{\delta+\lambda}{c}\right)\ell\right)\right]^2\phi_b(u) = \left[\frac{(1-\alpha)\left(\beta\lambda(\delta+\lambda)\right)^2}{\left(c(\beta c-\lambda)\right)^2}\ell - \frac{2\beta\lambda^2(1-\alpha)(\delta+\lambda)}{c^2(\beta c-\lambda)}\mathcal{D} + \frac{\lambda^2(1-\alpha)}{c^2}\mathcal{D}^2\right]\sigma_b(u) + \left[\alpha(\delta+\lambda)^2\left(\frac{\beta\lambda}{\beta c-\lambda}\right)^2\left[\left(\frac{\beta}{\beta c-\lambda} - \frac{1}{c}\right)^2 - \frac{\beta}{\beta c-\lambda}\left(\frac{2\beta}{\beta c-\lambda} - \frac{\beta c+2}{c}\right)\right]\ell + 2\alpha(\delta+\lambda)\left(\frac{\beta\lambda}{\beta c-\lambda}\right)^2\left(\frac{\beta}{\beta c-\lambda} - \frac{\beta c+1}{c}\right)\mathcal{D} + \alpha\frac{(\beta\lambda)^2}{\beta c-\lambda}\mathcal{D}^2\right]\phi_b(u) \quad (25)$$

Proof:

We derive $\phi_b(u)$ in relation (24) with respect to u.

$$\phi'_b(u) = \frac{\lambda^2(1-\alpha)}{c^2\varepsilon_1}\left(\frac{\delta+\lambda}{c}\right)\int_{b-\varepsilon_1(b-u)}^{\infty}\left(\frac{s-b}{\varepsilon_1} + b - u\right)e^{-\left(\frac{\delta+\lambda}{c}\right)\left(\frac{s-b}{\varepsilon_1}+b-u\right)}\sigma_b\left(\left(b + \frac{s-b}{\varepsilon_1}\right) \wedge s\right)ds - \frac{\lambda^2(1-\alpha)}{c^2\varepsilon_1}\int_{b-\varepsilon_1(b-u)}^{\infty} e^{-\left(\frac{\delta+\lambda}{c}\right)\left(\frac{s-b}{\varepsilon_1}+b-u\right)}\sigma_b\left(\left(b + \frac{s-b}{\varepsilon_1}\right) \wedge s\right)ds + \alpha\beta(\delta+\lambda)\left(\frac{\beta\lambda}{\beta c-\lambda}\right)^2\int_u^{\infty}\left(\frac{s-u}{\beta c-\lambda}\right)e^{-\beta\left(\frac{\delta+\lambda}{\beta c-\lambda}\right)(s-u)}\phi_b\left(s \wedge \left(b + (\beta c\varepsilon_1 - \lambda)\left(\frac{s-u}{\beta c-\lambda}\right) - \varepsilon_1(b-u)\right)\right) - \alpha\left(\frac{\beta\lambda}{\beta c-\lambda}\right)^2\int_u^{\infty} e^{-\beta\left(\frac{\delta+\lambda}{\beta c-\lambda}\right)(s-u)}\phi_b\left(s \wedge \left(b + (\beta c\varepsilon_1 - \lambda)\left(\frac{s-u}{\beta c-\lambda}\right) - \varepsilon_1(b-u)\right)\right) \quad (26)$$

Noting by $\mathcal{D}$ *and* ℓ the respective differentiation and identity operators, let's calculate

$$g(u) = \left(\mathcal{D} - \frac{\delta+\lambda}{c}\ell\right)\phi_b(u).$$

$$g(u) = -\frac{\lambda^2(1-\alpha)}{c^2\varepsilon_1}\int_{b-\varepsilon_1(b-u)}^{\infty} e^{-\left(\frac{\delta+\lambda}{c}\right)\left(\frac{s-b}{\varepsilon_1}+b-u\right)}\sigma_b\left(\left(b + \frac{s-b}{\varepsilon_1}\right) \wedge s\right)ds + \alpha(\delta+\lambda)\frac{(\beta\lambda)^2}{\beta c-\lambda}\left(\frac{\beta}{\beta c-\lambda} - \frac{1}{c}\right)\int_u^{\infty}\left(\frac{s-u}{\beta c-\lambda}\right)e^{-\beta\left(\frac{\delta+\lambda}{\beta c-\lambda}\right)(s-u)}\phi_b\left(s \wedge \left(b + (\beta c\varepsilon_1 - \lambda)\left(\frac{s-u}{\beta c-\lambda}\right) - \varepsilon_1(b-u)\right)\right) - \alpha\left(\frac{\beta\lambda}{\beta c-\lambda}\right)^2\int_u^{\infty} e^{-\beta\left(\frac{\delta+\lambda}{\beta c-\lambda}\right)(s-u)}\phi_b\left(s \wedge \left(b + (\beta c\varepsilon_1 - \lambda)\left(\frac{s-u}{\beta c-\lambda}\right) - \varepsilon_1(b-u)\right)\right) \quad (27)$$

We derive $g(u)$ in the relation (27) with respect to u.

$$g'(u) = -\frac{\lambda^2(1-\alpha)}{c^2\varepsilon_1}\left(\frac{\delta+\lambda}{c}\right)\int_{b-\varepsilon_1(b-u)}^{\infty} e^{-\left(\frac{\delta+\lambda}{c}\right)\left(\frac{s-b}{\varepsilon_1}+b-u\right)}\sigma_b\left(\left(b+\frac{s-b}{\varepsilon_1}\right)\wedge s\right)ds + \frac{\lambda^2(1-\alpha)}{c^2}\sigma_b(u) + \alpha\beta(\delta+\lambda)^2\left(\frac{\beta\lambda}{\beta c-\lambda}\right)^2\left(\frac{\beta}{\beta c-\lambda}-\frac{1}{c}\right)\int_u^{\infty}\left(\frac{s-u}{\beta c-\lambda}\right)e^{-\beta\left(\frac{\delta+\lambda}{\beta c-\lambda}\right)(s-u)}\phi_b\left(s\wedge\left(b+(\beta c\varepsilon_1-\lambda)\left(\frac{s-u}{\beta c-\lambda}\right)-\varepsilon_1(b-u)\right)\right) - \alpha(\delta+\lambda)\left(\frac{\beta\lambda}{\beta c-\lambda}\right)^2\left(\frac{2\beta}{\beta c-\lambda}-\frac{1}{c}\right)\int_u^{\infty} e^{-\beta\left(\frac{\delta+\lambda}{\beta c-\lambda}\right)(s-u)}\phi_b\left(s\wedge\left(b+(\beta c\varepsilon_1-\lambda)\left(\frac{s-u}{\beta c-\lambda}\right)-\varepsilon_1(b-u)\right)\right) + \alpha\frac{(\beta\lambda)^2}{\beta c-\lambda}\phi_b(u) \quad (28)$$

We calcule $h(u) = \left(\mathcal{D}-\left(\frac{\delta+\lambda}{c}\right)\ell\right)g(u)$.

$$h(u) = \frac{\lambda^2(1-\alpha)}{c^2}\sigma_b(u) + \alpha\frac{(\beta\lambda)^2}{\beta c-\lambda}\phi_b(u) + \alpha(\delta+\lambda)^2\left(\frac{(\beta\lambda)^2}{\beta c-\lambda}\right)\left(\frac{\beta}{\beta c-\lambda}-\frac{1}{c}\right)^2\int_u^{\infty}\left(\frac{s-u}{\beta c-\lambda}\right)e^{-\beta\left(\frac{\delta+\lambda}{\beta c-\lambda}\right)(s-u)}\phi_b\left(s\wedge\left(b+(\beta c\varepsilon_1-\lambda)\left(\frac{s-u}{\beta c-\lambda}\right)-\varepsilon_1(b-u)\right)\right) - 2\alpha(\delta+\lambda)\left(\frac{\beta\lambda}{\beta c-\lambda}\right)^2\left(\frac{\beta}{\beta c-\lambda}-\frac{1}{c}\right)\int_u^{\infty} e^{-\beta\left(\frac{\delta+\lambda}{\beta c-\lambda}\right)(s-u)}\phi_b\left(s\wedge\left(b+(\beta c\varepsilon_1-\lambda)\left(\frac{s-u}{\beta c-\lambda}\right)-\varepsilon_1(b-u)\right)\right) \quad (29)$$

We derive the function $h(u)$ in the relation (29) with respect to u.

$$h'(u) = \frac{\lambda^2(1-\alpha)}{c^2}\sigma'_b(u) + \alpha\frac{(\beta\lambda)^2}{\beta c-\lambda}\phi'_b(u) + \alpha\beta(\delta+\lambda)^2\left(\frac{(\beta\lambda)^2}{\beta c-\lambda}\right)\left(\frac{\beta}{\beta c-\lambda}-\frac{1}{c}\right)^2\left(\frac{\delta+\lambda}{\beta c-\lambda}\right)\int_u^{\infty}\left(\frac{s-u}{\beta c-\lambda}\right)e^{-\beta\left(\frac{\delta+\lambda}{\beta c-\lambda}\right)(s-u)}\phi_b\left(s\wedge\left(b+(\beta c\varepsilon_1-\lambda)\left(\frac{s-u}{\beta c-\lambda}\right)-\varepsilon_1(b-u)\right)\right) - \alpha(\delta+\lambda)^2\left(\frac{\beta\lambda}{\beta c-\lambda}\right)^2\left(\frac{\beta}{\beta c-\lambda}-\frac{1}{c}\right)\left(\frac{3\beta}{\beta c-\lambda}-\frac{1}{c}\right)\int_u^{\infty} e^{-\beta\left(\frac{\delta+\lambda}{\beta c-\lambda}\right)(s-u)}\phi_b\left(s\wedge\left(b+(\beta c\varepsilon_1-\lambda)\left(\frac{s-u}{\beta c-\lambda}\right)-\varepsilon_1(b-u)\right)\right) + 2\alpha(\delta+\lambda)\left(\frac{\beta\lambda}{\beta c-\lambda}\right)^2\left(\frac{\beta}{\beta c-\lambda}-\frac{1}{c}\right)\phi_b(u) \quad (30)$$

We determine $k(u) = \left(\mathcal{D}-\beta\left(\frac{\delta+\lambda}{\beta c-\lambda}\right)\ell\right)h(u)$

$$k(u) = -\frac{\beta\lambda^2(1-\alpha)(\delta+\lambda)}{c^2(\beta c-\lambda)}\sigma_b(u) + \alpha(\delta+\lambda)\left(\frac{\beta\lambda}{\beta c-\lambda}\right)^2\left(\frac{2\beta}{\beta c-\lambda}-\frac{\beta c+2}{c}\right)\phi_b(u) + \frac{\lambda^2(1-\alpha)}{c^2}\sigma'_b(u) + \alpha\frac{(\beta\lambda)^2}{\beta c-\lambda}\phi'_b(u) - \alpha(\delta+\lambda)^2\left(\frac{\beta\lambda}{\beta c-\lambda}\right)^2\left(\frac{\beta}{\beta c-\lambda}-\frac{1}{c}\right)^2\int_u^{\infty} e^{-\beta\left(\frac{\delta+\lambda}{\beta c-\lambda}\right)(s-u)}\phi_b\left(s\wedge\left(b+(\beta c\varepsilon_1-\lambda)\left(\frac{s-u}{\beta c-\lambda}\right)-\varepsilon_1(b-u)\right)\right) \quad (31)$$

We derive still $k(u)$ in the relation (31) with respect to u.

$$k'(u) = -\frac{\beta\lambda^2(1-\alpha)(\delta+\lambda)}{c^2(\beta c-\lambda)}\sigma'_b(u) + \alpha(\delta+\lambda)\left(\frac{\beta\lambda}{\beta c-\lambda}\right)^2\left(\frac{2\beta}{\beta c-\lambda}-\frac{\beta c+2}{c}\right)\phi'_b(u) + \frac{\lambda^2(1-\alpha)}{c^2}\sigma''_b(u) + \alpha\frac{(\beta\lambda)^2}{\beta c-\lambda}\phi''_b(u) - \alpha\beta(\delta+\lambda)^2\left(\frac{\beta\lambda}{\beta c-\lambda}\right)^2\left(\frac{\beta}{\beta c-\lambda}+\frac{1}{c}\right)^2\left(\frac{\delta+\lambda}{\beta c-\lambda}\right)\int_u^{\infty} e^{-\beta\left(\frac{\delta+\lambda}{\beta c-\lambda}\right)(s-u)}\phi_b\left(s\wedge\left(b+(\beta c\varepsilon_1-\lambda)\left(\frac{s-u}{\beta c-\lambda}\right)-\varepsilon_1(b-u)\right)\right) + \alpha(\delta+\lambda)^2\left(\frac{\beta\lambda}{\beta c-\lambda}\right)^2\left(\frac{\beta}{\beta c-\lambda}-\frac{1}{c}\right)^2\phi_b(u) \quad (32)$$

Let's calculate $t(u) = \left(\mathcal{D}-\beta\left(\frac{\delta+\lambda}{\beta c-\lambda}\right)\ell\right)k(u)$

$$t(u) = \frac{(1-\alpha)\left(\beta\lambda(\delta+\lambda)\right)^2}{\left(c(\beta c-\lambda)\right)^2}\sigma_b(u) - \frac{2\beta\lambda^2(1-\alpha)(\delta+\lambda)}{c^2(\beta c-\lambda)}\sigma'_b(u) + \alpha(\delta+\lambda)^2\left(\frac{\beta\lambda}{\beta c-\lambda}\right)^2\left[\left(\frac{\beta}{\beta c-\lambda}-\frac{1}{c}\right)^2-\frac{\beta}{\beta c-\lambda}\left(\frac{2\beta}{\beta c-\lambda}-\frac{\beta c+2}{c}\right)\right]\phi_b(u) + 2\alpha(\delta+\lambda)\left(\frac{\beta\lambda}{\beta c-\lambda}\right)^2\left(\frac{\beta}{\beta c-\lambda}-\frac{\beta c+1}{c}\right)\phi'_b(u) + \frac{\lambda^2(1-\alpha)}{c^2}\sigma''_b(u) + \alpha\frac{(\beta\lambda)^2}{\beta c-\lambda}\phi''_b(u) \quad (33)$$

From relations (26) to (33), we deduce relation (25).

4. Conclusion

In this paper, we have determined the integro-differential equation satisfied by the Gerber Shiu function in a Sparre Andersen risk model with a strategy of partial dividend payment to shareholders and a dependence between claim amounts and inter-claim times via the Spearman copula. Determining the Laplace transforms of the Gerber Shiu function and the probability of ultimate ruin is our next objective.

Conflicts of Interest

The authors declare no conflicts of interest.

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