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Performance Rating of the Exponentiated Generalized Gompertz Makeham Distribution: An Analytical Approach

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Abstract: We developed a five parameter distribution known as the Generalized Exponentiated Gompertz Makeham distribution which is quite flexible and can have a decreasing, increasing and bathtub-shaped failure rate function depending on its parameters making it more effective in modeling survival data and reliability problems. Some comprehensive properties of the new distribution, such as closed-form expressions for the density function, cumulative distribution function, hazard rate function, moment generating function and order Statistics were provided as well as maximum likelihood estimation of the Generalized Exponentiated Gompertz Makeham distribution parameters and at the end, in order to show the distribution flexibility, an application using a real data set was presented.

Keywords: Generalized Exponentiated Gompertz Makeham Distribution, Maximum Likelihood Estimation, Bathtub-Shape Failure Rate, Distribution Flexibility

1. Introduction

Generalized Exponentiated Class of distribution

Cordeiro G. M. et al. Proposed a new method of adding two shape parameters to a continuous distribution that extends an idea which was first introduced by Lehmann and studied by Nadarajah and Kotz. The idea produces a new class of exponentiated generalized distributions that can be interpreted as a double construction of Lehmann alternatives. Given a continuous cumulative density function, G. M Cordeiro et al define the exponentiated generalized class of distribution by

$$F(x) = [1 - [1 - G(x)]^{a}]^{b} \tag{1}$$

And the probability density function given by

$$f(x) = ab[G(x)]^{a-1}[1 - [G(x)]^{a}]^{b-1}g(x) \tag{2}$$

Where are two additional shape parameters in equations can control the both the tail weight and possibly adding entropies to the center of the exponentiated generalized density function.

2. Gompertz Makeham Distribution

The Gompertz distribution was first introduced by Benjamin Gompertz a British actuary. The distribution has been used frequently to describe human mortality, growth model and actuarial tables.

A different version of Gompertz distribution which is called Gompertz Makeham (GM) distribution was introduced by another British actuary, Makeham. He introduced a constant (Makeham terms) that describe the age independent mortality and has received considerable attention in the literature. The GM family has been studied by Baily et al. and an expression using the Lambert W function for the quantile function was given by Jodra, P. Suppose now is a GM random variable with the cumulative density function given by

$$G(x) = 1 - e^{-\lambda x - \frac{\alpha}{\beta}(e^{\beta x} - 1)} \tag{3}$$

And the probability density function given by

$$g(x) = \left(\lambda + \alpha e^{\beta x}\right) e^{-\lambda x - \frac{\alpha}{\beta}\left(e^{\beta x} - 1\right)} \qquad \alpha, \beta, \lambda > 0 \quad (4)$$

According to Finch, the Gompertz Makeham distribution produces a better fit between the age windows 30 to 85 years. An extension of the distribution will induce flexibility and enable it to cope with early failure or infant mortality.

3. The Proposed Generalized Exponentiated Gompertz Makeham Distributions

Putting (3) in (1), the cumulative density function of generalized exponentiated Gompertz Makeham (EGGM) distribution can be obtained as follows

$$F(x) = \left[1 - e^{a\left[-\lambda x - \frac{\alpha}{\beta}\left(e^{\beta x} - 1\right)\right]}\right]^{b} \qquad (5)$$

The graph below depicts the behaviour of the Cumulative density function of the EGGM distribution.

Also putting (4) in (2), we obtain an expression for the probability density function of the Generalized Exponentiated Gompertz Makeham (EGGM) distribution as follows

$$f(x) = ab\left(\lambda + \alpha e^{\beta x}\right) e^{a\left[-\lambda x - \frac{\alpha}{\beta}\left(e^{\beta x} - 1\right)\right]} \left[1 - e^{a\left[-\lambda x - \frac{\alpha}{\beta}\left(e^{\beta x} - 1\right)\right]}\right]^{b-1} \qquad (6)$$

The graph below depicts the behaviour of EGGM at different values of the shape parameters.

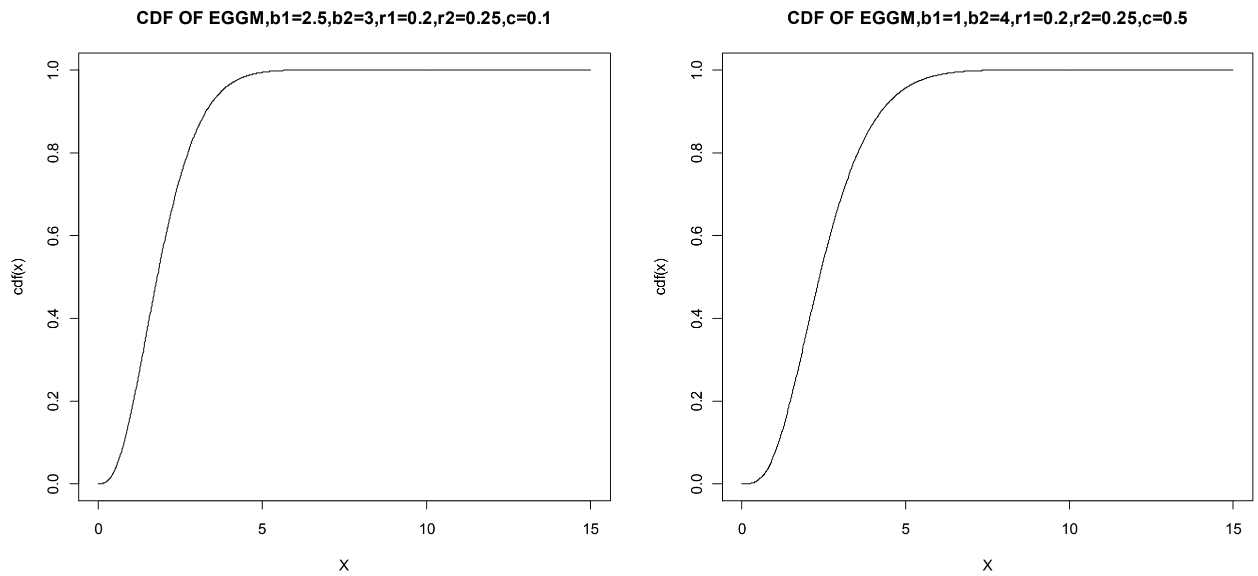

Figure 1. *The cdf of EGGM for various parameters.*

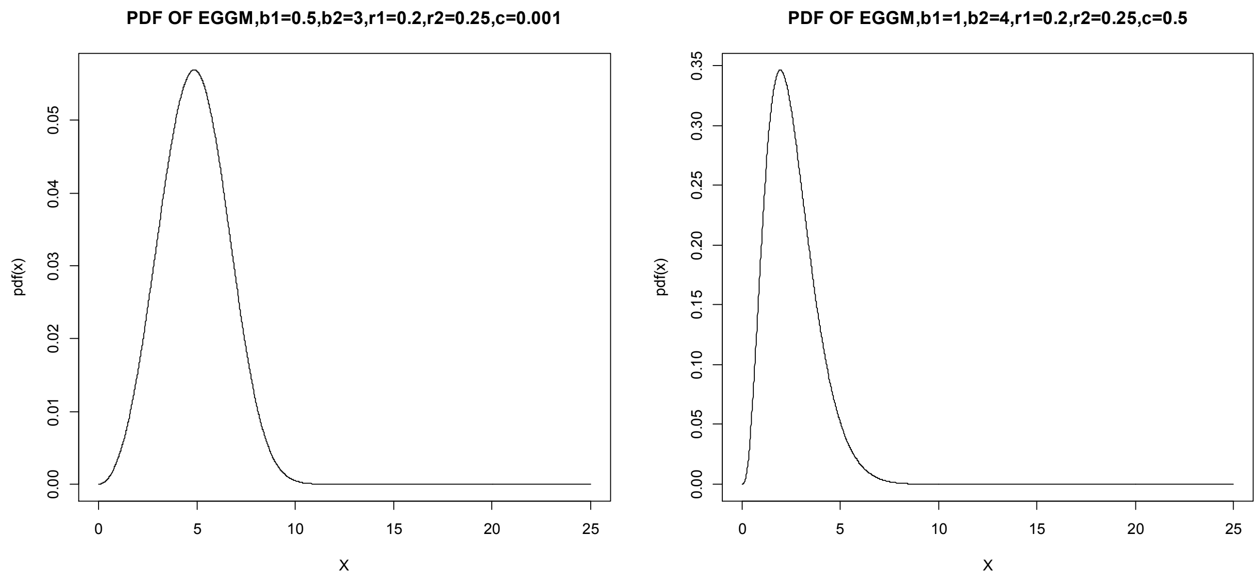

Figure 2. *The pdf of EGGM for various parameters.*

The graph drawn above indicates that the pdf of EGGM is positively skewed

3.1. Expansion for the Density Function

For any real non integer b, we consider the binomial series,

$$(1-z)^b = \sum_{k=0}^{\infty}(-1)^k \binom{b}{k} z^k \tag{7}$$

Which is valid for $|z| < 1$
Applying equation (7) in (5), we have

$$F(x) = \sum_{j=0}^{\infty}(-1)^j \binom{b}{j} e^{a\left[-\lambda x-\frac{\alpha}{\beta}(e^{\beta x}-1)\right]j} \tag{8}$$

Also for the probability density function we have

$$f(x) = ab(\lambda+\alpha e^{\beta x})e^{a\left[-\lambda x-\frac{\alpha}{\beta}(e^{\beta x}-1)\right]}\sum_{j=0}^{\infty}(-1)^j \binom{b-1}{j} e^{a\left[-\lambda x-\frac{\alpha}{\beta}(e^{\beta x}-1)\right]j} \tag{9}$$

Finally we have

$$f(x) = abg(x)\sum_{j=0}^{\infty}(-1)^j \binom{b-1}{j} e^{\left[-\lambda x-\frac{\alpha}{\beta}(e^{\beta x}-1)\right](a+aj-1)} \tag{10}$$

3.2. Verification of Exponentiated Generalized Distribution to Be a Proper Pdf

Here, we want to show that the integral of the EGGM distribution equal to 1; that is

$$\int_{-\infty}^{\infty} f(x) = 1 \tag{11}$$

$$\int_{-\infty}^{\infty} f(x) = \int_{-\infty}^{\infty} ab(\lambda+\alpha e^{\beta x})e^{a\left[-\lambda x-\frac{\alpha}{\beta}(e^{\beta x}-1)\right]}\left[1- e^{a\left[-\lambda x-\frac{\alpha}{\beta}(e^{\beta x}-1)\right]}\right]^{b-1} dx \tag{12}$$

Let $M = e^{-\lambda x-\frac{\alpha}{\beta}(e^{\beta x}-1)}$, then
$dM = -(\lambda+\alpha e^{\beta x})e^{-\lambda x-\frac{\alpha}{\beta}(e^{\beta x}-1)}$,
this implies that $dx = \frac{-dM}{(\lambda+\alpha e^{\beta x})M}$ substitute this in equation (12), we have

$$\int_{-\infty}^{\infty} f(x) = \int_{-\infty}^{\infty} ab(\lambda+\alpha e^{\beta x})M^a(1-M^a)^{b-1}\frac{dM}{(\lambda+\alpha e^{\beta x})M}$$

On simplification this gives

$$\int_{-\infty}^{\infty} f(x) = ab\int_{-\infty}^{\infty} M^{a-1}(1-M^a)^{b-1}dM \tag{13}$$

Further if we let $P = M^a, dP = aM^{a-1}dM$, $dM = \frac{dP}{aM^{a-1}}$, substitute this in equation (13), we have

$$\int_{-\infty}^{\infty} f(x) = ab\int_{-\infty}^{\infty}\left(P^{\frac{1}{a}}\right)^{a-1}(1-P)^{b-1}\frac{dP}{a\left(P^{\frac{1}{a}}\right)^{a-1}},$$

Finally we have

$$\int_{-\infty}^{\infty} f(x) = \int_{-\infty}^{\infty} b(1-P)^{b-1}dP = 1$$

This verified that the pdf of EGGM distribution function is a proper pdf.

3.3. Investigation of the Asymptotic Properties of EEGM Distribution

We seek to investigate the behaviour of the model in Equation (6) as $x \longrightarrow 0$
We have

$$\lim_{x\to 0} b(\lambda+\alpha e^{\beta x})e^{\left[-\lambda x-\frac{\alpha}{\beta}(e^{\beta x}-1)\right]}\left[1-e^{\left[-\lambda x-\frac{\alpha}{\beta}(e^{\beta x}-1)\right]}\right]^{b-1}$$

$$= b(\boldsymbol{\lambda}+\boldsymbol{\alpha})$$

Also as $x \longrightarrow 0$ and $b = 1$, we have

$$\lim_{x\to 0} a(\lambda+\alpha e^{\beta x})e^{a\left[-\lambda x-\frac{\alpha}{\beta}(e^{\beta x}-1)\right]}$$

$$= a(\boldsymbol{\lambda}+\boldsymbol{\alpha})$$

It has been shown that as $x \longrightarrow 0, a = 1, b = 1$ the EGGM distribution depends mainly on the shape parameters namely, a, b, α, β.

4. Well-known Distributions That Are Special Cases of the EGGM

(i) If then we get the EGM distribution
(ii) If, then we get, GM distribution
(iii) If $a = 1$, $b = 1$, $\beta \to 0$, then we get the E distribution.
(iv) If $b= 1$, $\beta \to 0$,, then we get the GE distribution which is introduced by Gupta & Kundu (1999)
(v) If $b = 1$, then we get the GG distribution which is introduced by El-Gohary & Al-Otaibi (2013).
(vi) If $a = 1$, $b = 1$, then we get the G distribution. (ii)

5. Hazard Function

The hazard function is define as

$$h(x) = \frac{f(x)}{1-F(x)} \tag{14}$$

Putting equation (5) and (6) in (14) we obtain the hazard function of the EGGM distribution as

$$h(x) = \frac{ab(\lambda+\alpha e^{\beta x})e^{a\left[-\lambda x-\frac{\alpha}{\beta}(e^{\beta x}-1)\right]}\left[1-e^{a\left[-\lambda x-\frac{\alpha}{\beta}(e^{\beta x}-1)\right]}\right]^{b-1}}{1-\left[1-e^{a\left[-\lambda x-\frac{\alpha}{\beta}(e^{\beta x}-1)\right]}\right]^b} \tag{15}$$

Equation (15) above can also be called the Exponentiated Generalized Gompertz Makeham model.

Putting $a = b = 1$ in equation (15); it will reduce to

$$h(x) = \frac{(\lambda + \alpha e^{\beta x})e^{\left[-\lambda x - \frac{\alpha}{\beta}(e^{\beta x} - 1)\right]}}{e^{a\left[-\lambda x - \frac{\alpha}{\beta}(e^{\beta x} - 1)\right]}}$$

Finally,

$$h(x) = (\lambda + \alpha e^{\beta x}) \tag{16}$$

Equation (16) represents the Gompertz Makeham model.

The reliability function can be obtained as

$$\mathcal{R}(x) = 1 - F(x) \tag{17}$$

Putting equation (5) in (17) we obtain the reliability function of EGGM distribution as

$$\mathcal{R}(x) = 1 - \sum_{j=0}^{\infty}(-1)^j \binom{b}{j} e^{a\left[-\lambda x - \frac{\alpha}{\beta}(e^{\beta x} - 1)\right]j} \tag{18}$$

6. Generating Functions

Here, we derive the moment generating function for a random variable X having the Exponentiated Generalized Gompertz Makeham distribution given in equation (9) as follows:

The moment generating function of a random variable X is defined as

$$M_x(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)dx\ , \qquad where\ |t| < 1$$

$$M_x(t) = \int_0^{\infty} e^{tx} ab(\lambda + \alpha e^{\beta x}) \sum_{j=0}^{\infty}(-1)^j \binom{b-1}{j} e^{a\left[-\lambda x - \frac{\alpha}{\beta}(e^{\beta x} - 1)\right](j+1)} dx$$

This can be simplified as

$$M_x(t) = ab \sum_{j=0}^{\infty}(-1)^j \binom{b-1}{j} \int_0^{\infty}(\lambda + \alpha e^{\beta x}) e^{tx + a\left[-\lambda x - \frac{\alpha}{\beta}(e^{\beta x} - 1)\right](j+1)} dx \tag{19}$$

Where

$$\int_{-\infty}^{\infty}(\lambda + \alpha e^{\beta x}) e^{tx + a\left[-\lambda x - \frac{\alpha}{\beta}(e^{\beta x} - 1)\right](j+1)} dx = \int_0^{\infty} \lambda e^{tx + a\left[-\lambda x - \frac{\alpha}{\beta}(e^{\beta x} - 1)\right](j+1)} dx + \int_0^{\infty} \alpha e^{tx + \beta x + a\left[-\lambda x - \frac{\alpha}{\beta}(e^{\beta x} - 1)\right](j+1)} dx$$

We let,

$$I_1 = \int_0^{\infty} \lambda e^{tx + a\left[-\lambda x - \frac{\alpha}{\beta}(e^{\beta x} - 1)\right](j+1)} dx$$

$$and \qquad I_2 = \int_0^{\infty} \alpha e^{tx + \beta x + a\left[-\lambda x - \frac{\alpha}{\beta}(e^{\beta x} - 1)\right](j+1)} dx$$

Solving for I_1 $we\ have$

$$I_1 = \lambda \int_0^{\infty} e^{tx + a\left[-\lambda x - \frac{\alpha}{\beta}(e^{\beta x} - 1)\right](j+1)} dx$$

$$I_1 = \lambda \left[\frac{e^{tx + a\left[-\lambda x - \frac{\alpha}{\beta}(e^{\beta x} - 1)\right](j+1)}}{t - a(\lambda + \alpha e^{\beta x})(i + j)}\right]_0^{\infty}$$

Then we have,

$$I_1 = -\left[\frac{\lambda}{t - a(\lambda + \alpha)(i + j)}\right] \tag{20}$$

Also for I_2, we have

$$I_2 = \alpha \int_0^{\infty} e^{tx + \beta x + a\left[-\lambda x - \frac{\alpha}{\beta}(e^{\beta x} - 1)\right](j+1)} dx$$

$$I_1 = \alpha\left[\frac{e^{tx+\beta x+a\left[-\lambda x-\frac{\alpha}{\beta}(e^{\beta x}-1)\right](j+1)}}{t+\beta-a(\lambda+\alpha e^{\beta x})(i+j)}\right]_0^\infty$$

Therefore,

$$I_2 = -\left[\frac{\alpha}{t+\beta-a(\lambda+\alpha)(i+j)}\right] \quad (21)$$

Finally the moment generating function of EGGM distribution is given as

$$M_x(t) = -ab\sum_{j=0}^{\infty}(-1)^j\binom{b-1}{j}\left[\frac{\lambda}{t-a(\lambda+\alpha)(i+j)}+\frac{\alpha}{t+\beta-a(\lambda+\alpha)(i+j)}\right] \quad (22)$$

Order Statistics

The density $f_{i:n}(x)$ of the ith order statistics for $i = 1,2,\dots,n$ from the independent identically distributed random variable $Y_{1,..}Y_n$ is given by

$$f_{i:n}(x) = \frac{f(x)}{B(i,n-i-1)}F(x)^{i-1}[1-F(x)]^{n-i} \quad (23)$$

Substituting equation (5) and (6) in equation (23), we obtain the ith order statistics of EGGM which is given as

$$f_{i:n}(x) = \frac{ab(\lambda+\alpha e^{\beta x})}{B(i,n-i-1)}e^{a\left[-\lambda x-\frac{\alpha}{\beta}(e^{\beta x}-1)\right]}\left[1-e^{a\left[-\lambda x-\frac{\alpha}{\beta}(e^{\beta x}-1)\right]}\right]^{b-1}\left[\left\{1-e^{a\left[-\lambda x-\frac{\alpha}{\beta}(e^{\beta x}-1)\right]}\right\}^b\right]^{i-1} \quad (24)$$

For b real non-integer by applying equation (7) and let $\xi = -\lambda x - \frac{\alpha}{\beta}\left(e^{\beta x}-1\right)$, we have

$$f_{i:n}(x) = \frac{ab\left(\lambda+\alpha e^{\beta x}\right)}{B(i,n-i-1)}e^{a\xi}\sum_{k=0}^{\infty}(-1)^k\binom{b-1}{k}e^{ak\xi}\sum_{l=0}^{\infty}(-1)^l\binom{b(i-1)}{l}e^{al\xi}\sum_{m=0}^{\infty}(-1)^m\binom{n-i}{m}\left[1-e^{a\xi}\right]^{bm}$$

Further simplification we have,

$$f_{i:n}(x) = \frac{ab\left(\lambda+\alpha e^{\beta x}\right)}{B(i,n-i-1)}e^{a\xi}\sum_{k=0}^{\infty}(-1)^k\binom{b-1}{k}e^{ak\xi}\sum_{l=0}^{\infty}(-1)^l\binom{b(i-1)}{l}e^{al\xi}\sum_{m=0}^{\infty}(-1)^m\binom{n-i}{m}\sum_{p=0}^{\infty}(-1)^p\binom{bm}{p}e^{a\xi p}$$

Finally, we have

$$f_{i:n}(x) = \frac{ab(\lambda+\alpha e^{\beta x})}{B(i,n-i-1)}e^{a\xi}\sum_{k=0}^{\infty}(-1)^{k+l+m+p}\binom{b-1}{k}\binom{b(i-1)}{l}\binom{n-i}{m}\binom{bm}{p}e^{a\xi(1+k+l+p)} \quad (25)$$

7. Estimation of Statistical Inference

Let $x_1, x_2, \dots, x_n$ be random variable distributed according to (8) the likelihood function of a vector of parameters given as $\Omega(a,b,\alpha,\beta,\lambda)$.

$$l(\Omega) = nlog(a) + nlog(b) + \sum_{i=1}^{n} log\left[\left(\lambda+\alpha e^{\beta x_i}\right)e^{-\lambda x_i-\frac{\alpha}{\beta}\left(e^{\beta x_i}-1\right)}\right] + (a-1)\sum_{i=1}^{n} log\left[e^{-\lambda x_i-\frac{\alpha}{\beta}\left(e^{\beta x_i}-1\right)}\right] + (b-1)\sum_{i=1}^{n} log\left[1-\left[e^{-\lambda x_i-\frac{\alpha}{\beta}\left(e^{\beta x_i}-1\right)}\right]^a\right] \quad (26)$$

Then the score vector $\nabla l = \frac{\delta l}{\delta a}, \frac{\delta l}{\delta b}, \frac{\delta l}{\delta \alpha}, \frac{\delta l}{\delta \lambda}, \frac{\delta l}{\delta \beta}$ has components,

let $\phi = -\lambda x_i - \frac{\alpha}{\beta}\left(e^{\beta x_i}-1\right)$

$$\frac{\delta l}{\delta a} = \frac{n}{a} + \sum_{i=1}^{n}\log(\phi)\left[\frac{(b-1)e^{\phi a}}{1-e^{\phi a}}\right] \quad (27)$$

$$\frac{\delta l}{\delta b} = \frac{n}{b} + \sum_{i=1}^{n} log\left[1-e^{\phi a}\right] \quad (28)$$

$$\frac{\delta l}{\delta \lambda} = \sum_{i=1}^{n}\frac{e^{\phi}-x_i e^{\phi}(\alpha e^{\beta x_i}+\lambda)}{(\alpha e^{\beta x_i}+\lambda)e^{\phi}} + (a-1)x + \frac{a(b-1)-\left[1-e^{\phi}\right]^{a-1}x_i e^{\phi}}{1-e^{\phi a}} \quad (29)$$

$$\frac{\delta l}{\delta \alpha} = \sum_{i=n}^{n} \frac{e^{\beta x_i+\phi} - \lambda\left\{\frac{1}{\beta}\left(e^{\beta x_i}-1\right)\right\}e^{\phi} - \alpha\left\{\frac{1}{\beta}\left(e^{\beta x_i}-1\right)\right\}e^{\beta x_i+\phi}}{(\alpha e^{\beta x_i}+\lambda)e^{\phi}} - \frac{(a-1)\left\{\frac{1}{\beta}\left(e^{\beta x_i}-1\right)\right\}e^{\phi}}{e^{\phi}} + \frac{a(b-1)-\left[1-e^{\phi}\right]^{a-1}\left\{\frac{1}{\beta}\left(e^{\beta x_i}-1\right)\right\}e^{\phi}}{1-e^{\phi a}} \tag{30}$$

$$\frac{\delta l}{\delta \beta} = \sum_{i=1}^{n} \frac{\frac{\alpha\lambda}{\beta^2}\left[e^{\beta x_i}-\beta x e^{\beta x_i}-1\right]e^{\phi} + \left[\frac{\alpha^2}{\beta^2}e^{\beta x_i}(1-x_i\beta)+\frac{\alpha(\beta^3-1)}{\beta^2}\right]e^{\beta x_i+\phi}}{(\alpha e^{\beta x_i}+\lambda)e^{\phi}} - \frac{(a-1)\left[\frac{\alpha x_i}{\beta}e^{\beta x_i}-\frac{\alpha}{\beta^2}\left(e^{\beta x_i}-1\right)\right]e^{\phi}}{e^{\phi}} + \frac{a(b-1)-\left[1-e^{\phi}\right]^{a-1}\left[\frac{\alpha x_i}{\beta}e^{\beta x_i}-\frac{\alpha}{\beta^2}\left(e^{\beta x_i}-1\right)\right]e^{\phi}}{1-e^{\phi a}} \tag{31}$$

8. Application

To illustrate the new results presented in this paper, we fit the EGGM distribution to an uncensored data set from Nichols and Padgett, (2006) considering 100 observations on breaking stress of carbon fibres (in Gba). The data are as follows: 3.7, 2.74, 2.73, 2.5, 3.6, 3.11, 3.27, 2.87, 1.47, 3.11,4.42, 2.41, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.9, 3.75, 2.43, 2.95, 2.97, 3.39, 2.96, 2.53,2.67, 2.93, 3.22, 3.39, 2.81, 4.2, 3.33, 2.55, 3.31, 3.31, 2.85, 2.56, 3.56, 3.15, 2.35, 2.55, 2.59,2.38, 2.81, 2.77, 2.17, 2.83, 1.92, 1.41, 3.68, 2.97, 1.36, 0.98, 2.76, 4.91, 3.68, 1.84, 1.59,3.19,1.57, 0.81, 5.56, 1.73, 1.59, 2, 1.22, 1.12, 1.71, 2.17, 1.17, 5.08, 2.48, 1.18, 3.51, 2.17, 1.69,1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.7, 2.03, 1.8, 1.57, 1.08, 2.03, 1.61, 2.12,1.89, 2.88, 2.82, 2.05, 3.65. These data were previously studied by Souza *et al.* for beta Frechet (BF), exponentiated Frechet (EF) and Frechet distributions. In the following, we shall compare the proposed KGM and its sub-model (GM) with several other three- and four-parameter lifetime distributions, namely: the Zografos-Balakrishnan log-logistic (ZBLL), Kumaraswamy Pareto (KP) and recently the Kumaraswamy Gompertz Makeham (KGM) distribution with corresponding densities:

Where

$$f_{KGM}(x,a,b,\beta,\alpha,\lambda) = ab\left(\alpha e^{\beta x}+\lambda\right)\left(e^{-\lambda x-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)\left(1-e^{-\lambda x-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)^{a-1}$$

$$\left(1-\left(1-e^{-\lambda x-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}\right)^{a}\right)^{b-1}$$

$$f_{ZBLL}(x,a,\beta,\theta) = \frac{\beta}{\theta^{\beta}\tau(a)}x^{\beta-1}(1+(\frac{x}{\theta})^{\beta})^{-2}\left[\ln(1+(\frac{x}{\theta})^{\beta})\right]^{a-1} \quad x>0$$

$$f_{BF}(x,a,b,\theta,\beta) = \frac{\beta\theta^{\beta}}{B(a,b)}x^{-(\beta+1)}e^{-a(\frac{\theta}{x})^{\beta}}(1-e^{-(\frac{\theta}{x})^{\beta}})^{b-1} \quad x>0$$

$$f_{KP}(x,a,b,\theta,\beta) = ab\beta\theta^{\beta}x^{-(\beta+1)}\left[1-(\frac{\theta}{x})^{\beta}\right]^{a-1}\left[1-(1-(\frac{\theta}{x})^{\beta})^{a}\right]^{b-1}$$

Where $a,b,\beta,\theta,\alpha,\lambda>0$

Table1 gives the descriptive statistics of the data and Table 2 gives the likelihood ratio estimates of the parameters and table 3 gives the values of AIC, BIC, CAIC and HQIC for EGGM, KGM, GM, BF, KP, ZBLL, BF and EF distributions, the corresponding errors(given in parenthesis) and the statistics $l(\hat{\theta})$ (where $l(\hat{\theta})$ denotes the log-likelihood function evaluated at the maximum likelihood estimates), Akaike information criterion (AIC), the Bayesian information criterion (BIC), Consistent Akaike information criterion (CAIC) and Hannan-Quinn information criterion (HQIC). We also construct the Total Time on Test (TTT) plot for the data as well as its empirical density and cumulative density function.

Table 1. *Descriptive Statistics on Breaking stress of Carbon fibres.*

Min	Q_1	Med.	$mean$	Q_3	Max	$kurtosis$	Skewness
0.390	1.840	2.700	2.6214	3.220	5.560	0.10494	0.36815

Table 2. *Likelihood Estimates of Parameters.*

$Model$	$Estimates$				
$EGGM$	4.1581	4.1757	0.5705	10^{-11}	0.55113
(a,b,θ,β)	(1.6264)	(1.6094)	(0.4949)	(0.0037)	$(-)$
KGM	3.2590	6.7422	$10e^{-11}$	0.22148	0.130941
$(a,b,\lambda,\alpha,\beta)$	(1.8545)	(1.18545)	(17.4572)	(0.74510)	(0.71868)
KP	4.69523	236.2335	0.39	0.19204	-
(a,b,θ,β)	(0.502)	(149.552)	-	(0.045)	
$ZBLL$	1.5501	1.90903	3.61259	-	-
(a,θ,β)	(0.104)	(0.0093)	0.288		
BF	0.42934	138.0664	34.38484	0.72474	-
(a,b,θ,β)	(0.236)	(113.552)	(21.52)	(0.19)	
GM	10^{-11}	0.076941	0.790997	-	-
(λ,α,β)	(0.0829)	(0.03399)	(0.10837)		

Table 3. *Criteria for Comparison.*

$\boldsymbol{Model}$	$l(\hat{\theta})$	AIC	BIC	HQIC	CAIC
$EGGM$ (a, b, θ, β)	-29.548	69.096	82.122	74.368	69.734
KGM $(a, b, \lambda, \alpha, \beta)$	-141.332	292.664	305.690	297.936	293.599
KP (a, b, θ, β)	-166.751	339.502	347.318	338.084	339.923
$ZBLL$ (α, θ, β)	-162.913	331.826	339.642	330.408	332.076
BF (a, b, θ, β)	-142.866	293.733	304.154	291.842	294.154
GM (λ, α, β)	-149.125	304.25	312.066	307.413	304.50

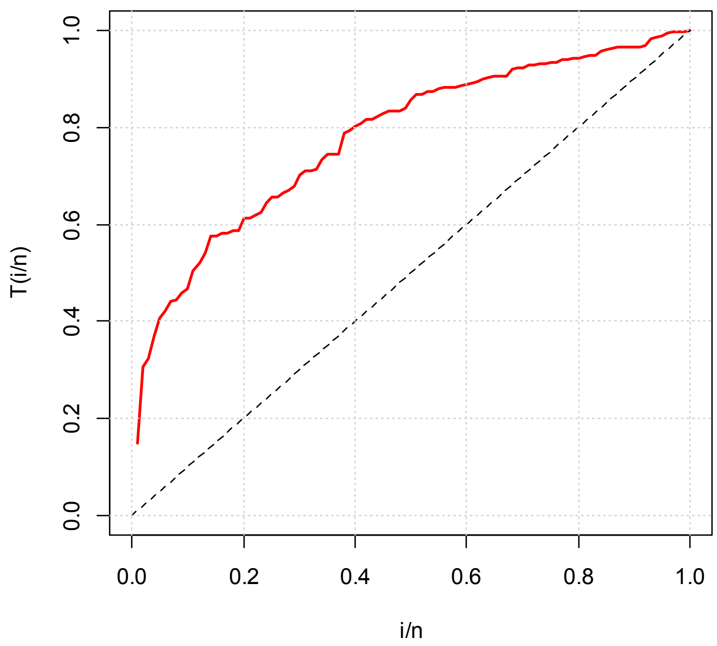

Figure 3. *The graph of Total Time on Test Plot for the breaking stress of carbon data.*

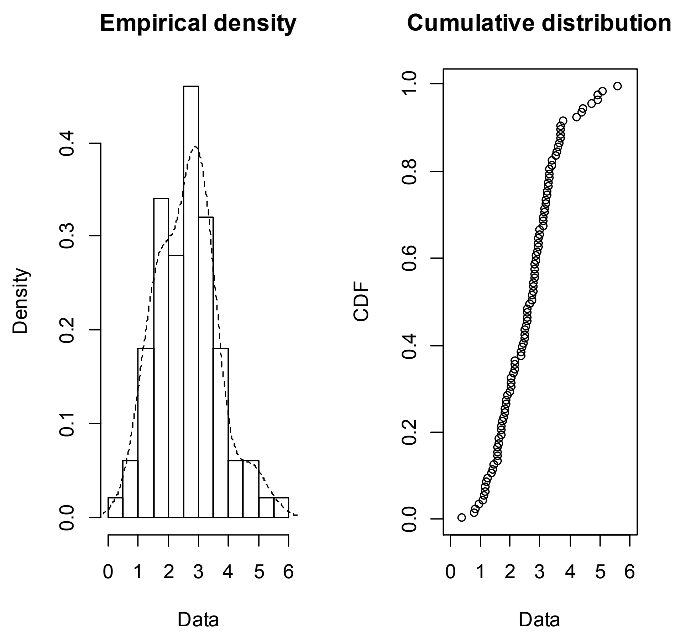

Figure 4. *The graph of the Emprical density and the cummulative density of the carbon data.*

9. Conclusion

Since the EGGM distribution has the lowest, AIC, BIC, CAIC and HQIC values among all other models and its sub-model so it could be chosen as the best model.

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