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Estimation of the Parameters of Poisson-Exponential Distribution Based on Progressively Type II Censoring Using the Expectation Maximization (Em) Algorithm

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Abstract: This paper considers the parameter estimation problem of test units from Poisson-Exponential distribution based on progressively type II right censoring scheme. The maximum likelihood estimators (MLEs) for Poisson-Exponential parameters are derived using Expectation Maximization (EM) algorithm. EM-algorithm is also used to obtain the estimates as well as the asymptotic variance-covariance matrix. By using the obtained variance-covariance matrix of the MLEs, the asymptotic 95% confidence interval for the parameters are constructed. Through simulation, the behavior of these estimates are studied and compared under different censoring schemes and parameter values. It is concluded that for an increasing sample size; the estimated value of the parameters converges to the true value, the variances decrease and the width of the confidence interval become narrower.

Keywords: Poisson-Exponential Distribution, Progressive Type II Censoring, Maximum Likelihood Estimation, EM Algorithm

1. Introduction

In the statistical literature one can find numerous distributions for modelling life time data. In life time study, exponential distribution is one of the most discussed distributions due to its simplicity and easy mathematical manipulations. However, its use is inappropriate in those situations where associated hazard rate is not constant. A number of life time distributions having non-constant hazard rate are available in the literature e.g., Gamma, Weibull, Exponentiated Exponential etc. These distributions are generalization of Exponential distribution and possess increasing, decreasing or constant hazard rate depending on the value of the shape parameters and reduce to exponential distribution for their specific choices of the shape parameter. A modification in exponential distribution was proposed by Kus [1] to get a decreasing failure rate distribution by finding the distribution of the minimum of n independently, identically and exponentially distributed random variables where n is random following zero truncated poisson distributions. Since the distribution is obtained through the compounding of poisson and exponential. Further Barreto and Cribari [2] generalized the distribution proposed by Kus by including a power parameter. Cancho *et al*. [3] proposed a new family of distribution, called Poisson-Exponential (PE) distribution having increasing failure rate. The distribution has been obtained by finding the distributionof the maimum of n independently,identically and exponentially distributed random variables where n is randomfollowing,zero truncated Poisson distribution.

In this study, we assume that the lifetimes have Poisson-Exponential distribution. The motivation for this family of distribution can also be traced in the study of complementary risk(CR) problems in presence of latent risks i.e, for those situations when only life time values are observed but no information is available about the factors responsible for component failures. For other datails regarding CR and related models,the readers may refer Basu and Klein [4] and

Adamidis and Loukas [5].

In real life, sometimes it is hard to get a complete data set; often the data are censored. Scientific experiments might have to stop before all items fail because of the limit of time or lack of money. This results to availability of censored data. Type-I and Type-II censoring are the most basic among the different censoring schemes. Type-I censoring happens when the experimental time T is fixed, but the number of failures is random. Type-II censoring occurs when the number of failures r is fixed, the experimental time is random. Vast literature is available on these two censoring schemes and one may refer to Bain and Engelhardt [6] for detailed discussion on various aspects of these schemes. Unfortunately,these methods do not allow the removal of units before the completion of the experiment. However, in medical and engeneering survival analysis,removal of items may occur at intermediate steps also due to various reasons which are beyond the control of the experimenter. For such a situation,progressive censoring is an appropriate censoring scheme as it allow the removal of surving items before the termination point of the test. Therefore, in this study, we will focus on progressive censoring due to its flexibility that allows the experimenter to remove active units during the experiment.

Many authors have discussed inference under progressive censoring using different lifetime distributions, including Cohen, [7], Aggarwala [8] and Amal et al [9]. For a comprehensive recent review of progressive censoring, readers may refer to Balakrishinan [10].

Let X be a non-negative random variable denoting the life time of a component/system. The random variable X is said to have a PE distribution with parameters θ *and* λ, if its probability density function (pdf) is given by,

$$f\left(x,\theta,\lambda\right)=\frac{\theta\lambda e^{-\lambda x-\theta e^{-\lambda x}}}{1-e^{-\theta}},\quad x>0\quad \theta>0,\lambda>0 \tag{1}$$

The corresponding cumulative distribution function (cdf) is given by,

$$F(x)=1-\frac{1-e^{-\theta e^{-\lambda x}}}{1-e^{-\theta}},\quad x>0\quad \theta>0,\lambda>0 \tag{2}$$

Where λ is the scale parameter, while θ is shape parameter of the distribution. Louzada-Neto *et al.* [11], pointed out that the parameters θ and λ of the distribution have direct interpretation in terms of complementary risk. In fact θ represents the mean of the number of complementary risk whereas λ denotes the lifetime failure rate.

Inferential issues for the Poisson-Exponential distribution based on complete data have been addressed by Louzada-Neto *et al* who studied the statistical properties of PE distribution and discussed about the Bayes estimators under squared error loss function (SELF). Singh *et al.* [12] obtained the maximum likelihood estimators and Bayes estimators of the parameters under symmetric and asymmetric loss function for Poisson-exponential distribution and compared the proposed estimators with maximum likelihood estimators in terms of their risks. Raqab and Madi [13] discussed the classical and Bayesian inferential procedure for progressively type II censored data from the generalized Rayleigh distribution. The results showed that the maximum likelihood estimators of the scale and shape parameters can be obtained via EM algorithm based on progressive censoring. Krishna and Kumar [14] discussed the inference problems in Lindley distribution and the results shows that Lindley distribution provide good parametric fit under progressive censoring scheme for some real life situations. Also, some of the recent work on progressive censoring include but not limited to Kumar *et al*. [15], Pak *et al.* [16] and Rastogi and Tripathi [17]. As far as we know, no one has described the EM algorithm for determining the MLEs of the parameters of the Poisson-Exponential distribution based on progressive type-II censoring scheme.

In this study, we propose to use EM algorithm for computing MLEs. This is because the EM algorithm is relatively robust against the initial values compared to the traditional Newton-Raphson (NR) method as shown by Watanabe and Yamaguchi [18] and Ng *et al.* [19]. It guarantees a single uniform non-decreasing likelihood trial from the initial value to the convergence value. Moreover, with the EM algorithm, there is no need to evaluate the first and second derivatives of the log-likelihood function, which helps save the central processing unit (CPU) time of each iteration. The Expectation maximization algorithm is computational stable, easy to implement and asymptotic variances and covariance are also obtained. For more recent relevant references on EM algorithm and censoring include [20-22].

The purpose of this study is to estimate the shape and scale parameters of the Poisson-Exponential distribution under progressive type-II censoring using the EM algorithm and to compare the results under different censoring schemes.

The rest of this paper is organized as follows:Section 2,provides a brief description of Progressive type II censoring scheme. Furthermore, the asymptotic variance and covariance of the maximum likelihood estimates which are generated through EM algorithm are given. Simulation study is conducted in section 3. Finally, conclusion and recommendation are presented in section 4.

2. Parameter Estimation

2.1. Progressive Type-II Censoring Scheme

Suppose that n units are placed on a life test at time 0. Prior to the experiment, a number m (< n) is fixed and the censoring scheme $R=R_1, R_2, \ldots, R_m$ are predetermined with $R_j \geq 0$ and $\sum_{j=1}^{m} R_j + m = n$ is specified. At the first failure time $X_{1:m:n}$, R_1 units, chosen at random, are removed from the n−1 surviving units. At the second failure time $X_{2:m:n}$,

R_2 randomly chosen units from the remaining $n-2-R_1$ units are removed. The test continues until the m^{th} failure time $X_{m:m:n}$. At this time, all remaining units are removed; there are $R_m = n-m-\sum_{j=1}^{m-1} R_j$ of these. The set of observed lifetime $X = X_{1:m:n}, X_{2:m:n}, ..., X_{m:m:n}$ is a progressively Type II right censored sample as reffered by Balakrishnan and Aggarwala [23].

2.2. Maximum Likelihood Estimation Based on Progressive Type-II Censoring

Suppose n identical units are placed on a lifetime test. At the time of the i^{th} failure, R_i surviving units are randomly withdrawn from the experiment, $1 \le i \le m$. Thus, if m failures are observed then $R_1 + R_2 + ... + R_m$ units are progressively censored; hence $n = m + R_1 + R_2 + ... + R_m$, $X_{1:m:n}^R \le X_{2:m:n}^R \le ....X_{m:m:n}^R$ describe the progressively censored failure times, where $R = (R_1, R_2 ..., R_m)$ denotes the censoring scheme. If the failure times of the n items originally on test are from a continuous population with $p.d.f\ f(x)$ and cdf F(x) given by equation (1) and (2) respectively, then the joint probability density function for $X_{1:m:n}^R \le X_{2:m:n}^R \le ....X_{m:m:n}^R$ is given by,

$$f_{1,\,2,\,...,\,m}\left(x_{1:m:n}^R \le x_{2:m:n}^R \le ....x_{m:m:n}^R\right) = A\prod_{j=1}^{m} f\left(x_{j:m:n}^R\right)\left(1-F\left(x_{j:m:n}^R\right)\right)^{R_j} \tag{3}$$

Where $-\infty < x_{1:m:n}^R \le x_{2:m:n}^R \le ....x_{m:m:n}^R < \infty$ and $A = n(n-R_1-1)(n-R_1-R_2-2)...(n-R_1-R_2-...-R_{m-1}-m+1)$

From equation (1) and (2), the likelihood function based on progressively Type II censored sample is given by;

$$L(\theta,\lambda \mid x) = A\prod_{j=1}^{m} \frac{\theta\lambda e^{-\lambda x_{(j)}-\theta e^{-\lambda x_{(j)}}}}{1-e^{-\theta}}\left\{\frac{1-e^{-\theta e^{-\lambda x_{(j)}}}}{1-e^{-\theta}}\right\}^{R_j} \tag{4}$$

The log-likelihood function of equation (4) can be written as follows

$$\begin{aligned} lnL(\theta,\lambda \mid x) &= const + mln(\theta\lambda) - 2mln(1-e^{-\theta}) \\ &- \lambda\sum_{j=1}^{m} x_{(j)} - \theta\sum_{j=1}^{m} e^{-\lambda x_j} + \sum_{j=1}^{m} R_j ln\left(1-e^{-\theta e^{-\lambda x_j}}\right) \end{aligned} \tag{5}$$

Differentiating (5) w. r. t. (with respect to) to θ and λ and equating the derivatives to zero, we get the following normal equations:

$$\frac{m}{\theta} - \sum_{j=1}^{m} e^{-\lambda x_j} - \frac{2me^{-\theta}}{\left(1-e^{-\theta}\right)} + \sum_{j=1}^{m} R_j \frac{e^{-\lambda x_j} e^{-\theta e^{-\lambda x_j}}}{1-e^{-\theta e^{-\lambda x_j}}} = 0 \tag{6}$$

$$\frac{m}{\lambda} - \sum_{j=1}^{m} x_j + \theta\sum_{j=1}^{m} x_j e^{-\lambda x_j} - \theta\sum_{j=1}^{m} R_j \frac{x_{(j)} e^{-\lambda x_j} e^{-\theta e^{-\lambda x_j}}}{1-e^{-\theta e^{-\lambda x_j}}} = 0 \tag{7}$$

The normal equations (6) and (7) are implicit system of equations in θ and λ. They cannot be solved analytically. Therefore, we propose to use EM algorithm for solving these equations numerically, for maximum likelihood estimate of θ and λ.

2.3. Expectation-Maximization (EM) Algorithm

The E M algorithm was introduced by Dempster *et al.* [24] to handle any missing or incomplete data situation. McLachlan and Krishnan[25] discussed EM algorithm and its applications. The progressive type-II censoring can be viewed as an incomplete data set, and therefore, the EM algorithm is a good alternative to the NR method for numerically finding the MLEs.

Let $X = X_{1:m:n}, X_{2:m:n}, ..., X_{m:m:n}$ with $X_{1:m:n} < X_{2:m:n} < ... < X_{m:m:n}$ denotes the progressive type-II right-censored data from a population with pdf and cdf given in Equations (1) and (2), respectively. For notation simplicity, we will write X_j for $X_{j:m:n}$.

Let $Z = (Z_1, Z_2, ..., Z_m)$ with $Z_j = \left(Z_{j1}, Z_{j2}, ..., Z_{jR_j}\right)$, $j = 1, 2, ...,m$ be the censored data. We consider the censored data as missing data. The combination of $(X, Z) = Y$ forms the complete data set. The Likelihood function based on Y is

$$L(Y,\theta,\lambda) = \prod_{j=1}^{m}\left[\frac{\theta\lambda e^{-\lambda x_j - \theta e^{-\lambda x_j}}}{1-e^{-\theta}}\prod_{k=1}^{R_j}\frac{\theta\lambda e^{-\lambda z_{jk}-\theta e^{-\lambda z_{jk}}}}{1-e^{-\theta}}\right] \tag{8}$$

The log-likelihood function based on Y is

$$lnL(Y,\theta,\lambda) = nln(\theta\lambda) - nln\left(1-e^{-\theta}\right) - \sum_{j=1}^{m}\left(\lambda x_j + \theta e^{-\lambda x_j}\right) - \sum_{j=1}^{m}\sum_{k=1}^{R_j}\left(\lambda z_{jk} + \theta e^{-\lambda z_{jk}}\right) \tag{9}$$

The MLEs of the parameters θ and λ for complete sample Y can be obtained by deriving the log-likelihood function in Equation (9) with respect to θ and λ and equating the normal equations to 0 as follows:

$$\frac{\partial lnL(Y,\theta,\lambda)}{\partial\theta} = \frac{n}{\theta} - \frac{ne^{-\theta}}{1-e^{-\theta}} - \sum_{j=1}^{m} e^{-\lambda x_j} - \sum_{j=1}^{m}\sum_{k=1}^{R_j} e^{-\lambda z_{jk}} = 0 \tag{10}$$

$$\frac{\partial lnL(Y,\theta,\lambda)}{\partial\lambda} = \frac{n}{\lambda} - \sum_{j=1}^{m} x_j + \sum_{j=1}^{m}\theta x_j e^{-\lambda x_j} - \sum_{j=1}^{m}\sum_{k=1}^{R_j} z_{jk} + \sum_{j=1}^{m}\sum_{k=1}^{R_j}\theta z_{jk} e^{-\lambda z_{jk}} = 0 \tag{11}$$

To start the algorithm, the joint distribution of x and z is given by,

$$f(x,z) = p(z)(x \mid z) = z\lambda\left(1-e^{-\lambda x}\right)^{z-1} e^{-\lambda x}\frac{e^{-\theta}\theta^z}{z!\left(1-e^{-\theta}\right)} \quad x > 0, z = 1,\ 2,\ 3... \tag{12}$$

Where $\lambda > 0$ and $\theta > 0$ are parameters. It is straightforward to verify that the computation of the conditional expectation of (Z|X) using the pdf is given by

$$p(z \mid x)=\frac{f(x,z)}{f(x)}=\frac{z\lambda\left(1-e^{-\lambda x}\right)^{z-1} e^{-\lambda x} \dfrac{e^{-\theta}\theta^{z}}{z!\left(1-e^{-\theta}\right)}}{\dfrac{\theta\lambda e^{-\lambda x-\theta e^{-\lambda x}}}{1-e^{-\theta}}} \quad (13)$$

Simplifying (13), we get

$$p(z \mid x)=\frac{f(x,z)}{f(x)}=\frac{\theta^{z-1}\left(1-e^{-\lambda x}\right)^{z-1} e^{-\theta+\theta e^{-\lambda x}}}{(z-1)!} \quad (14)$$

Thus it is straightforward to verify that the E-step of an EM cycle requires the computation of the conditional expectation $\left(Z \mid X,\lambda^{h},\theta^{h}\right)$ where $\left(\lambda^{h},\theta^{h}\right)$ is the current estimates of (λ,θ).

$$E(z \mid x,\lambda^{h},\theta^{h})=\sum_{z=1}^{\infty} z p(z \mid x,\lambda,\theta) \quad (15)$$

Using equation (15), we get,

$$E(z \mid x,\lambda^{h},\theta^{h})=\sum_{z=1}^{\infty} z\frac{\theta^{z-1}\left(1-e^{-\lambda x}\right)^{z-1} e^{-\theta+\theta e^{-\lambda x}}}{(z-1)!} \quad (16)$$

Simplifying (16), we get

$E(z \mid x,\lambda^{h},\theta^{h})=1+\theta^{h}\left(1-e^{-\lambda^{h}x}\right)$ see Sadegh and Rasool [26] (17)

The EM cycle is completed with M-step, which is complete data maximum likelihood over (λ,θ), with missing Z's replaced by their conditional expectations $\left(Z \mid X,\lambda^{h},\theta^{h}\right)$. Thus an EM iteration, taking $\left(\lambda^{h},\theta^{h}\right)$ into $\left(\lambda^{h+1},\theta^{h+1}\right)$ is given by

$$\frac{\partial lnL(Y,\theta,\lambda)}{\partial\theta}=\frac{n}{\theta}-\frac{ne^{-\theta}}{1-e^{-\theta}}-\sum_{j=1}^{m} e^{-\lambda x_j}-\sum_{j=1}^{m}\sum_{k=1}^{R_j} e^{-\lambda\left(1+\theta^{h}\left(1-e^{-\lambda^{h}x}\right)\right)}=0 \quad (18)$$

$$\frac{\partial lnL(Y,\theta,\lambda)}{\partial\lambda}=\frac{n}{\lambda}-\sum_{j=1}^{m} x_j+\sum_{j=1}^{m}\theta x_j e^{-\lambda x_j}-\sum_{j=1}^{m}\sum_{k=1}^{R_j}\left(1+\theta^{h}\left(1-e^{-\lambda^{h}x}\right)\right)+ \sum_{j=1}^{m}\sum_{k=1}^{R_j}\theta\left(1+\theta^{h}\left(1-e^{-\lambda^{h}x}\right)\right)e^{-\lambda\left(1+\theta^{h}\left(1-e^{-\lambda^{h}x}\right)\right)}=0 \quad (19)$$

We obtain the iterative procedure of the EM-algorithm as

$$\theta^{h+1}=\frac{n}{\dfrac{ne^{-\theta^{h}}}{1-e^{-\theta^{h}}}+\sum_{j=1}^{m} e^{-\lambda^{h}x_j}+\sum_{j=1}^{m} R_j e^{-\lambda\left(1+\theta^{h}\left(1-e^{-\lambda^{h}x_j}\right)\right)}} \quad (20)$$

and

$$\lambda^{h+1}=\frac{n}{\sum_{j=1}^{m} x_j-\sum_{j=1}^{m}\theta^{h}x_j e^{-\lambda^{h}x_j}+\sum_{j=1}^{m} R_j\left(1+\theta^{h}\left(1-e^{-\lambda^{h}x_j}\right)\right)-\sum_{j=1}^{m} R_j\theta\left(1+\theta^{h}\left(1-e^{-\lambda^{h}x_j}\right)\right)e^{-\lambda\left(1+\theta^{h}\left(1-e^{-\lambda^{h}x_j}\right)\right)}} \quad (21)$$

The $\left(\theta^{(h+1)},\lambda^{(h+1)}\right)$ is then used as a new value of (θ,λ) in the subsequent iteration. The MLEs of (θ,λ) can be obtained by repeating the E-step and M-step until convergence. Each iteration is guaranteed to increase the log-likelihood and the algorithm is guaranteed to converge to a local maximum of the likelihood function, i.e. starting from an arbitrary point in the parameter space, the EM algorithm will always converge to a local maximum. In this work, the MLEs of θ and λ based on complete sample are used as initial values for θ and λ in the EM algorithm.

2.4. Asymptotic Variances and Covariance

The variance–covariance matrix is used to provide a measure of precision for parameter estimators by utilizing the log-likelihood function. Applying the usual large sample approximation, the MLE of $\beta=(\theta,\lambda)$ can be treated as being approximately bivariate normal with mean β and variance-covariance matrix, which is the inverse of the expected information matrix $J(\beta)=E(I,\beta)$, where $I=I(\beta:x_{obs})$ is the observed information matrix with elements $I_{ij}=\dfrac{-\partial^{2}l}{\partial\theta_i\partial\lambda_j}$ with i, j = 1,2 and the expectation is to be taken with respect to the distribution of X.

For a complete data set from the Poisson-exponential distribution, the variance–covariance matrix of parameters θ and λ is given by the likelihood function of $\beta=(\theta,\lambda)$ based on the observed sample of size n, $x=(x_1, x_2, ..., x_n)$, from the PE distribution is given by,

$$L(\beta)=e^{n\log(\theta\lambda)-\lambda\sum_{j=1}^{n}x_j-\theta\sum_{j=1}^{n}e^{-\lambda x_j}-n\log\left(1-e^{-\theta}\right)} \quad (22)$$

Theorem

Some Cramer-Rao regularity conditions hold and $\beta=(\theta,\lambda)$ belongs to an open interval of the real line. If the variance of an unbiased estimator attains the Cramer's-Rao Lower Bound, the likelihood equation has a unique solution $\hat{\beta}$ that maximizes the likelihood function.

It is known that under such regularity conditions, as the sample size increases, the distribution of the MLE tends to the bivariate normal distribution with mean (θ,λ) and covariance matrix equal to the inverse of the Fisher information matrix, see Cox & Hinkley [27]. The Fisher information matrix is given by,

$$\begin{bmatrix} \mathrm{var}(\hat{\theta}) & \mathrm{cov}(\hat{\theta},\hat{\lambda}) \\ \mathrm{cov}(\hat{\theta},\hat{\lambda}) & \mathrm{var}(\hat{\lambda}) \end{bmatrix} = \begin{bmatrix} -E\left[\frac{\partial^2 lnL(Y,\theta,\lambda)}{\partial\theta^2}\right] & -E\left[\frac{\partial^2 lnL(Y,\theta,\lambda)}{\partial\theta\partial\lambda}\right] \\ -E\left[\frac{\partial^2 lnL(Y,\theta,\lambda)}{\partial\theta\partial\lambda}\right] & -E\left[\frac{\partial^2 lnL(Y,\theta,\lambda)}{\partial\lambda^2}\right] \end{bmatrix}^{-1} \quad (23)$$

Where

$$\frac{\partial lnL(Y,\theta,\lambda)}{\partial\theta} = \frac{n}{\theta} - \frac{ne^{-\theta}}{1-e^{-\theta}} - \sum_{j=1}^{n} e^{-\lambda x_j}$$

$$\frac{\partial^2 lnL(Y,\theta,\lambda)}{\partial\theta^2} = -\frac{n}{\theta^2} + \frac{ne^{-\theta}}{\left(1-e^{-\theta}\right)^2}$$

$$\frac{\partial lnL(Y,\theta,\lambda)}{\partial\lambda} = \frac{n}{\lambda} - \sum_{j=1}^{n} x_j + \theta\sum_{j=1}^{n} x_j e^{-\lambda x_j}$$

$$\frac{\partial^2 lnL(Y,\theta,\lambda)}{\partial\lambda^2} = -\frac{n}{\lambda^2} - \theta\sum_{j=1}^{n} x_j^2 e^{-\lambda x_j}$$

$$\frac{\partial^2 lnL(Y,\theta,\lambda)}{\partial\lambda\partial\theta} = \sum_{j=1}^{n} x_j e^{-\lambda x_j}$$

$$\frac{\partial^2 lnL(Y,\theta,\lambda)}{\partial\theta\partial\lambda} = \sum_{j=1}^{n} x_j e^{-\lambda x_j}$$

The expectations are given by,

$$E\left(xe^{-\lambda x}\right) = \frac{n\theta}{4\lambda\left(1-e^{-\theta}\right)} F_{2,2}\left([2,2],[3,3],-\theta\right) \quad (24)$$

$$E\left(x^2e^{-\lambda x}\right) = \frac{n\theta}{4\lambda^2\left(1-e^{-\theta}\right)} F_{3,3}\left([2,2,2],[3,3,3],-\theta\right) \quad (25)$$

In matrix form, we get,

$$\begin{bmatrix} \mathrm{var}(\hat{\theta}) & \mathrm{cov}(\hat{\theta},\hat{\lambda}) \\ \mathrm{cov}(\hat{\theta},\hat{\lambda}) & \mathrm{var}(\hat{\lambda}) \end{bmatrix} = \begin{bmatrix} \frac{n}{\theta^2} - \frac{ne^{-\theta}}{\left(1-e^{-\theta}\right)^2} & -\frac{n\theta}{4\lambda\left(1-e^{-\theta}\right)} F_{2,2}\left([2,2],[3,3],-\theta\right) \\ -\frac{n\theta}{4\lambda\left(1-e^{-\theta}\right)} F_{2,2}\left([2,2],[3,3],-\theta\right) & \frac{n}{\lambda^2} + \frac{n\theta^2}{4\lambda^2\left(1-e^{-\theta}\right)} F_{3,3}\left([2,2,2],[3,3,3],-\theta\right) \end{bmatrix}^{-1} \quad (26)$$

The inverse of $J(\beta)$, evaluated at $\hat{\beta}$ provides the asymptotic variance-covariance matrix of the MLEs.

In this study, the procedure developed by Louis and Tanner [28] is used to derive the asymptotic variance–covariance matrix for the MLEs based on the EM algorithm. The idea of this procedure is given by

$$I_{obs}(\eta) = I_c(\eta) - I_{miss}(\eta) \quad (27)$$

Where $I_{obs}(\eta)$, $I_c(\eta)$ and $I_{miss}(\eta)$ denote the complete, observed, and missing (expected) information, respectively, and $\eta = (\theta, \lambda)$. The Fisher information matrix for a single observation which is censored at the time of the j^{th} failure is given by

$$I_{miss}^{(j)}(\eta) = -E\left[\left(\frac{\partial^2 lnf_{z|y}\left(z_{jk} \mid z_{jk} > x_j;\eta\right)}{\partial\eta^2}\right)\right] \quad (28)$$

Given $X_j = x_j$, the conditional distribution of Z_{jk} follows a truncated Poisson-Exponential distribution with left truncation at x_j. That is,

$$f_{z|x}\left(z_{jk} \mid z_{jk} > x_j;\eta\right) = \frac{f_X\left(z_{jk}\right)}{1-F_X\left(x_j\right)}, \quad z_{jk} > x_j \quad (29)$$

Hence,

$$f\left(z_{jk} \mid z_{jk} > x_j;\eta\right) = \frac{\frac{\theta\lambda e^{-\lambda z_j - \theta e^{-\lambda z_j}}}{1-e^{-\theta}}}{\frac{1-e^{-\theta e^{-\lambda x_j}}}{1-e^{-\theta}}} = \frac{\theta\lambda e^{-\lambda z_j - \theta e^{-\lambda z_j}}}{1-e^{-\theta e^{-\lambda x_j}}}, \quad z_j > x_j \quad (30)$$

Taking the logarithm of both sides, we get,

$$lnf_{z/x}\left(z_{jk} \mid z_{jk} > x_j;\eta\right) = ln(\theta) + ln(\lambda) - \lambda z_j - \theta e^{-\lambda z_j} - ln\left(1-e^{-\theta e^{-\lambda x_j}}\right) \quad (31)$$

Differentiating (30) with respect to $\beta = (\theta, \lambda)$, we get

$$\frac{\partial lnf_{z/x}\left(z_{jk} \mid z_{jk} > x_j;\eta\right)}{\partial\theta} = \frac{1}{\theta} - e^{-\lambda z_j} - \frac{e^{-\lambda x_j - \theta e^{-\lambda x_j}}}{1-e^{-\theta e^{-\lambda x_j}}}$$

$$\frac{\partial^2 lnf_{z|x}\left(z_{jk} \mid z_{jk} > x_j;\eta\right)}{\partial\theta^2} = -\frac{1}{\theta^2} + \frac{e^{-2\lambda x_j - \theta e^{-\lambda x_j}}}{\left(1-e^{-\theta e^{-\lambda x_j}}\right)^2}$$

$$\frac{\partial lnf_{z|x}\left(z_{jk} \mid z_{jk} > x_j;\eta\right)}{\partial\lambda} = \frac{1}{\lambda} - z_j + z_j\theta e^{-\lambda z_j} + \frac{x_j\theta e^{-\lambda x_j - \theta e^{-\lambda x_j}}}{1-e^{-\theta e^{-\lambda x_j}}}$$

$$\frac{\partial^2 lnf_{z/x}\left(z_{jk} \mid z_{jk} > x_j;\eta\right)}{\partial\lambda^2} = -\frac{1}{\lambda^2} - z_j{}^2\theta e^{-\lambda z_j} + x_j{}^2\theta\frac{\left(-e^{-\lambda x_j - \theta e^{-\lambda x_j}} + \theta e^{-2\lambda x_j - \theta e^{-\lambda x_j}} + e^{-\lambda x_j - 2\theta e^{-\lambda x_j}}\right)}{\left(1-e^{-\theta e^{-\lambda x_j}}\right)^2}$$

$$\frac{\partial lnf_{z/x}\left(z_{jk} \mid z_{jk} > x_j;\eta\right)}{\partial\theta\partial\lambda} = z_j e^{-\lambda z_j} - \frac{x_j\theta\left(e^{-2\lambda x_j - \theta e^{-\lambda x_j}}\right)}{\left(1-e^{-\theta e^{-\lambda x_j}}\right)^2}$$

The expectations are given by,

$$E\left(ze^{-\lambda z}\right) = \frac{\theta}{4\lambda\left(1-e^{-\theta}\right)} F_{2,2}\left([2,2],[3,3],-\theta\right) \tag{32}$$

$$E\left(z^2 e^{-\lambda z}\right) = \frac{\theta}{4\lambda^2\left(1-e^{-\theta}\right)} F_{3,3}\left([2,2,2],[3,3,3],-\theta\right) \tag{33}$$

The expected values of the second partial of the log-likelihood function of Z given X are calculated as,

$$-E\left(\frac{\partial^2 lnf_{z|x}\left(z_{jk} \mid z_{jk} > x_j;\eta\right)}{\partial\theta^2}\right) = -\left(-\frac{1}{\theta^2} + \frac{e^{-2\lambda x_j - \theta e^{-\lambda x_j}}}{\left(1-e^{-\theta e^{-\lambda x}{}_j}\right)^2}\right) = I_{11}^j$$

$$-E\left(\frac{\partial Inf_{z/x}\left(z_{jk} \mid z_{jk} > x_j;\eta\right)}{\partial\theta\partial\lambda}\right) = -\left(\frac{\theta}{4\lambda\left(1-e^{-\theta}\right)} F_{2,2}\left([2,2],[3,3],-\theta\right) - \frac{x\theta\left(e^{-2\lambda x - \theta e^{-\lambda x}}\right)}{\left(1-e^{-\theta e^{-\lambda x}}\right)^2}\right) = I_{12}^j$$

$$-E\left(\frac{\partial^2 lnf_{z/x}\left(z_{jk} \mid z_{jk} > x_j;\eta\right)}{\partial\lambda^2}\right) = -\left(\begin{array}{l} -\frac{1}{\lambda^2} - \frac{\theta^2}{4\lambda^2\left(1-e^{-\theta}\right)} F_{3,3}\left([2,2,2],[3,3,3],-\theta\right) + \\ x_j{}^2\theta\frac{\left(-e^{-\lambda x_j - \theta e^{-\lambda x_j}} + \theta e^{-2\lambda x_j - \theta e^{-\lambda x}{}_j} + e^{-\lambda x_j - 2\theta e^{-\lambda x_j}}\right)}{\left(1-e^{-\theta e^{-\lambda x_j}}\right)^2} \end{array}\right) = I_{22}^j$$

Where

$$I_{miss}^{(j)}\left(\eta\right) = \begin{pmatrix} I^j{}_{11} & I^j{}_{12} \\ I^j{}_{21} & I^j{}_{22} \end{pmatrix} \tag{34}$$

Note that $I_{miss}^{(j)}\left(\eta\right)$ miss is a function of x_j and η, since the expectation is taken with respect z_j ; therefore, the expected information matrix is simply

$$I_{miss}\left(\eta\right) = \sum_{j=1}^{m} R_j I_{miss}^{(j)}\left(\eta\right) \tag{35}$$

Therefore, the variance–covariance matrix of parameter η can be obtained by

$$I_{obs}^{-1}\left(\eta\right) = [I_{comp}\left(\eta\right) - I_{miss}\left(\eta\right)]^{-1} \tag{36}$$

An approximate $(1-\alpha)$ 100% confidence interval for θ and λ is obtained as

$\hat{\theta} \pm z_{\alpha/2}\sqrt{\text{var}(\hat{\theta})}$ and $\hat{\lambda} \pm z_{\alpha/2}\sqrt{\text{var}(\hat{\lambda})}$ where $z\alpha/2$ is the $(\alpha/2)100^{\text{th}}$ percentile of standard normal distribution.

3. Results and Discussions

In this section, a simulation study is conducted to investigate how the proposed estimators perform in estimating the parameters of Poisson-Exponential distribution based on progressive type II censored data. The samples were generated based on the algorithms of Balakrishnan and Sandhu [26].

In this study, samples of sizes 20, 50, and 100 were used and the censoring schemes considered are given in Table 1, 2 and 3 below.

Table 1. *Censoring scheme* $R = (r_1, r_2, ..., r_m)$ *for* $\beta = (\theta = 1.5, \lambda = 1.5)$.

n	m	θ	λ	Censoring scheme
20	10	1.5	1.5	1, 0, 1, 1, 0, 2, 0, 2, 0, 3
	15	1.5	1.5	1, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 0, 0, 1
50	20	1.5	1.5	1, 2, 3, 3, 0, 3, 2, 1, 2, 0, 3, 0, 1, 2, 0, 2, 1, 0, 2, 2
	40	1.5	1.5	1,0,0,0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 2, 0, 0, 0, 0, 0, 1, 0, 0,1
100	50	1.5	1.5	1, 0, 1, 0, 2, 1, 1, 2, 0, 0, 1, 2, 0, 1, 0, 4, 0, 2, 0, 1, 3, 0, 2, 2, 0, 2, 0, 0, 0, 0, 3, 1, 2, 0, 1, 0, 1, 1, 0, 1, 1, 2, 0, 2, 0, 1, 0, 1, 0, 5
	80	1.5	1.5	0, 0, 1, 0, 2, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 2, 0, 0, 0, 0,1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 1

Table 1 represents the progressive censoring scheme for different samples size and different numbers of failures for parameters $\beta = (\theta = 1.5,\ \lambda = 1.5)$.

Table 2. *Censoring scheme* $R = (r_1,\ r_2,\ ....,\ r_m)$ *for* $\beta = (\theta = 1.5,\ \lambda = 2)$.

n	m	θ	λ	Censoring scheme
20	10	1.5	2	1, 0, 1, 1, 0, 2, 0, 2, 0, 3
	15	1.5	2	1, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 0, 0, 1
50	20	1.5	2	1, 2, 3, 3, 0, 3, 2, 1, 2, 0, 3, 0, 1, 2, 0, 2, 1, 0, 2, 2
	40	1.5	2	1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 2, 0, 0, 0, 0, 0, 1, 0, 0,1
100	50	1.5	2	1, 0, 1, 0, 2, 1, 1, 2, 0, 0, 1, 2, 0, 1, 0, 4, 0, 2, 0, 1, 3, 0, 2, 2, 0, 2, 0, 0, 0, 0, 3, 1, 2, 0, 1, 0, 1, 1, 0, 1, 1, 2, 0, 2, 0, 1, 0, 1, 0, 5
	80	1.5	2	0, 0, 1, 0, 2, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 2, 0, 0, 0, 0,1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 1

Table 2 represents the progressive censoring scheme for different samples size and different numbers of failures for parameters $\beta = (\theta = 1.5,\ \lambda = 2)$.

Table 3. *Censoring scheme* $R = (r_1,\ r_2,\ ....,\ r_m)$ *for* $\beta = (\theta = 2.3,\ \lambda = 2)$.

n	m	θ	λ	Censoring scheme
20	15	2.3	2	1, 0, 0, 1, 1, 1, 0, 0, 1, 0, 0, 0, 0, 0, 0
30	15	2.3	2	1, 0, 1, 1, 0, 1, 2, 0, 1, 2, 2, 0, 2, 0, 2
40	15	2.3	2	1, 0, 1, 1, 0, 1, 2, 5, 0, 2, 2, 3, 2, 0, 5

Table 3 represents the progressive censoring scheme for increasing samples size but fixed numbers of failures for parameters $\beta(\theta = 2.3,\ \lambda = 2)$.

No restriction has been imposed on the maximum number of iterations and convergence is assumed when the absolute differences between successive estimates are less than 10^{-5}. All computational results were computed using R software.

Table 4. *The MLEs, Variances, covariance and 95% confidence limits of the MLEs for the parameters of Poisson-exponential distribution under progressive type II censored sample when* $\beta = (\theta = 1.5,\ \lambda = 1.5)$

n	m	$\hat{\theta}$	$\hat{\lambda}$	var($\hat{\theta}$)	var($\hat{\lambda}$)	cov($\hat{\lambda},\hat{\theta}$)	CL(θ)		CL(λ)	
							LCL	UCL	LCL	UCL
20	10	1.0349	1.4462	4.3642	0.8841	1.6737	-2.6101	5.5791	-0.3429	3.3429
	15	1.6676	1.6068	2.4254	0.4323	0.8316	-1.5680	4.5370	0.2112	2.7887
50	20	1.3241	1.4792	1.6817	0.3864	0.6517	-1.0572	4.0262	0.2817	2.1783
	40	1.5392	1.2672	1.1076	0.1958	0.3950	-0.5783	3.5472	0.6325	2.3675
100	50	1.8236	1.3161	0.5689	0.1184	0.1979	0.0049	2.9641	0.8257	2.1742
	80	1.3900	1.5574	0.4647	0.0799	0.1572	0.1484	2.8406	0.9460	2.0540

From the above table, it is observed that irrespective of the censoring rate and at which point the censored units are removed from the sample, for increasing sample size;

a the estimated value of the parameter converge to the true value,

b the variances and covariance of the MLEs decrease.

It is also observed that, for the case when *n* is fixed (i.e. *n* = 20), we note that as *m* increases (i.e. from 10 to 15) the variances and the covariance values decrease (also see table 5).

Table 5. *The MLEs, Variances, covariance and 95% confidence limits of the MLEs for the parameters of Poisson-exponential distribution under progressive type II censored sample when* $\beta = (\theta = 1.5,\ \lambda = 2)$.

n	m	$\hat{\theta}$	$\hat{\lambda}$	var($\hat{\theta}$)	var($\hat{\lambda}$)	cov($\hat{\lambda},\hat{\theta}$)	CL(θ)		CL(λ)	
							LCL	UCL	LCL	UCL
20	10	1.2113	1.4259	3.0293	0.3955	0.8517	-1.9267	4.8959	0.7673	3.2327
	15	1.2024	1.7038	2.3895	0.3516	0.7415	-1.5453	4.5153	0.8378	3.1624
50	20	1.7095	1.3944	1.8924	0.2146	0.5302	-1.2128	4.1809	1.0920	2.9080
	40	1.5662	1.7159	0.9372	0.1422	0.2984	-0.4130	3.3820	1.2610	2.7390
100	50	1.3479	1.4699	0.6956	0.0876	0.2002	-0.1502	3.1191	1.4192	2.5810
	80	1.4799	1.7093	0.4574	0.0692	0.1445	0.1589	2.8101	1.4844	2.5156

From the above table, it is observed that irrespective of the censoring rate and at which point the censored units are removed from the sample, for increasing sample size;

a the estimated value of the parameter converge to the true value.

b the variances and covariance of the MLEs decrease.

Table 6. *Confidence intervals of* $\beta = (\theta = 1.5,\ \lambda{=}1.5)$.

n	m	Width of C. I (θ)	Width of C. I (λ)
20	10	8.1891	3.6858
	15	6.1049	2.5774
50	20	5.0835	2.4366
	40	4.1256	1.7350
100	50	2.9591	1.3486
	80	2.6722	1.1080

From the above table, it is observed that the widths of 95% confidence intervals tend to be narrower for an increasing sample size.

Table 7. *Confidence intervals of* $\beta = (\theta = 1.5,\ \lambda{=}2)$.

n	m	Width of C. I (θ)	Width of C. I(λ)
20	10	6.8227	2.4354
	15	6.0596	2.3245
50	20	5.3926	1.8160
	40	3.7950	1.4780
100	50	3.2693	1.1617
	80	2.6512	1.0313

From the above table, it is also observed that the widths of 95% confidence intervals tend to be narrower for an increasing sample size.

Table 8. *The MLEs, Variances, covariance and 95% confidence limits of the MLEs for the parameters of Poisson-exponential distribution under progressive type II censored sample when* $\beta(\theta = 2.3,\ \lambda{=}2)$ *with different sample size but fixed number of failures completely observed.*

n	m	$\hat{\theta}$	$\hat{\lambda}$	var($\hat{\theta}$)	var($\hat{\lambda}$)	cov($\hat{\lambda}$,$\hat{\theta}$)	CL(θ)			CL(λ)		
							LCL	UCL	Width (θ)	LCL	UCL	Width (λ)
20		1.7231	1.8731	2.0768	0.3193	0.6386	-0.5074	5.1334	5.6409	0.8925	3.1075	2.2151
30	15	1.3858	1.4527	1.5537	0.2397	0.4489	-0.1300	4.4761	4.8861	1.0405	2.9595	1.9190
40		1.5681	1.6230	1.0741	0.2070	0.2973	-0.2817	4.3443	4.0625	1.0626	2.8918	1.7836

From the above table, it is observed that irrespective of the censoring rate and at which point the censored units are removed from the sample with fixed number of failures completely observed for increasing sample size;

a the estimated value of the parameter converge to the true value.

b the variances and covariance of the MLEs decreases.

4. Conclusions

This study has addressed the problem of estimation of parameters of the Poisson-exponential distribution based on progressive Type-II censored data. The maximum likelihood estimators of the scale and shape parameters were obtained by using EM algorithm.

A comparison of the MLEs and their variances as well as their confidence intervals was made by simulation for different censoring schemes. It was observed that:

(i) for an increasing sample size, the estimated value of the parameter becomes closer to the true value, the variances and covariance of the MLEs decrease and the widths of the confidence intervals become narrower.

(ii) reducing the number of units to be removed in the censoring scheme, leads to better estimates for a fixed sample size.

The results provide the EM algorithm that is relatively robust against the initial values. It guarantees a single uniform non-decreasing likelihood trial from the initial value to the convergence value. Moreover, with the EM algorithm, there is no need to evaluate the first and second derivatives of the log-likelihood function, which helps save the central processing unit (CPU) time of each iteration. The Expectation Maximization algorithm is computational stable, easy to implement and asymptotic variances and covariance of estimates are also obtained.

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