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Characterization of the Linear Failure Rate Distribution by General Progressively Type-II Right Censored Order Statistics

M. M. Mohie El-Din[1], A. Sadek[1], Marwa M. Mohie El-Din[2], A. M. Sharawy[2, *]

[1]Department of Mathematics, Faculty of Science (Men), Al-Azhar University, Cairo, Egypt

[2]Department of Mathematics, Faculty of Engineering, Egyptian Russian University, Cairo, Egypt

Email address:

ali.elsharawy82@yahoo.com (A. M. Sharawy)

[*]Corresponding author

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Abstract: In this article, we establish recurrence relations among single moments and among product moments of general progressively Type-II right censored order statistics and characterization for linear failure rate distribution using recurrence relations of single moments and product moments of general progressively Type-II right censored order statistics. Further, the results are specialized to the progressively Type-II right censored order statistics.

Keywords: Characterization, General Progressively Type-II Right Censored, Linear Failure Rate, Order Statistics, Recurrence Relations, Single and Product Moments

1. Introduction

In failure data analysis, it is common that some individuals cannot be observed for the full time to failure. the progressive Type-II right censored is a useful and more general scheme in which a specific fraction of individuals at risk may be removed from the study at each of several ordered failure times. Progressively censored samples have been considered, among others, by Balakrishnan et al. [6] and Davis and Feldstein [8]. Bain [5] derived analysis for the Linear Failure-Rate life-testing distribution. Aggarwala and Balakrishnan [3] derived recurrence relations for single and product moments of progressive Type-II right censored order statistics from exponential, Pareto and power function distributions and their truncated forms. Abd El-Aty and Marwa Mohie El-Din [1] derived recurrence relations for single and double moments of GOS from the inverted linear exponential distribution and any continuous function. Mokhlis et al. [13] derived recurrence relations for moments of GOS from Marshall-Olkin-Extended burr XII distribution. Mohie El-Din, and Kotb [12] derived recurrence relations for single and product moments of generalized order statistics for modified burr XII-Geometric distribution and characterization. Mohie El-Din et al. [11] derived Statistical Inference and Characterizations from Independent and Identical Exponential-Bernoulli Mixture Distribution. Athar et al. [4] discussed some new moments of progressively Type-II right censored order statistics from Lindley distribution. Saran and Pushkarna [9] derived moments of progressive Type-II right censored order statistics from a general class of doubly truncated continuous distributions. Abd El-Hamid et al. [2] derived inference and optimal design based on step-partially accelerated life tests for the generalized Pareto distribution under progressive Type-I censoring.

This scheme of censoring was generalized by Balakrishnan and Sandhu [7] as follows: at time $X_0 \equiv 0, n$ units are placed on test; the first r failure times, $X_1, \dots, X_r$, are not observed; at time $X_i + 0$, where X_i is the i^{th} ordered failure time $(i = r+1, \dots, m-1), R_i$ units are removed from the test randomly, so prior to the $(i+1)^{th}$ failure there are $n_i = n - i - \sum_{j=r+1}^{i} R_j$ units on test; finally, at the time of the m^{th} failure, X_m, the experiment is terminated, i.e., the remaining R_m units are removed from the test. The R_i's, m

and r are prespecified integers which must satisfy the conditions: $0 \leq r < m \leq n, 0 \leq R_i \leq n_{i-1}\ for\ i = r + 1, \dots, m-1$ with $n_r = n - r$ and $R_m = n_{m-1} - 1$.

The joint probability density function of the general progressively Type-II right censored order statistics failure times $X_{r+1:m:,n}, X_{r+2:m:,n}, \dots, X_{m:m:,n}$, is given by

$$f_{X_{r+1:m:,n},\dots,X_{m:m:,n}}(x_{r+1}, \dots, x_m) =$$

$$A_{(n,m-1)}[F(x_{r+1}, \theta)]^r \prod_{i=r+1}^{m} f(x_i, \theta)[1 - F(x_i, \theta)]^{R_i}$$

$$x_{r+1} < x_{r+2} < \dots < x_m \quad (1)$$

where,

$$A_{(n,m-1)} = \frac{n!}{r!\,(n-r)!}\left(\prod_{j=r}^{m-1} n_j\right), n_i = n - i - \sum_{j=r+1}^{i} R_j,$$

$$i = r+1, \dots, m-1.$$

In this paper, we shall introduce recurrence relations among single and product moments of general progressively Type-II right censored order statistics. Characterization for linear failure rate distribution using recurrence relations of single and product moments of general progressively Type-II right censored order statistics.

Let $X_{r+1:m:n}^{(R_{r+1},R_{r+2},\dots,R_m)} < X_{r+2:m:n}^{(R_{r+1},R_{r+2},\dots,R_m)} < \dots < X_{m:m:n}^{(R_{r+1},R_{r+2},\dots,R_m)}$ be the m ordered observed failure times in a sample of size $(n-r)$ under general progressively Type-II right censored order statistics from the linear failure rate distribution with probability density function (pdf) is given by

$$f(x, \theta) = [\alpha + \theta x]e^{-\alpha x - \frac{\theta x^2}{2}}, \theta, \alpha > 0, x \geq 0 \quad (2)$$

The corresponding cumulative distribution function (cdf) is given by

$$F(x, \theta) = 1 - e^{-\alpha x - \frac{\theta x^2}{2}} \quad (3)$$

It may be noted that from (2) and (3) the relation between pdf and cdf is given by,

$$f(x) = [\alpha + \theta x][1 - F(x)] \quad (4)$$

For any continuous distribution, we shall denote the i^{th} single moment of the general progressively Type-II right censored order statistics in view of Eq. (1) as

$$\mu_{q:m:n}^{(R_{r+1},R_{r+2},\dots,R_m)^{(i)}} = E\left[X_{q:m:n}^{(R_{r+1},R_{r+2},\dots,R_m)}\right]^i$$

$$= A_{(n,m-1)} \iint \dots \int_{0<x_{r+1}<\dots<x_m<\infty} x_q^i [F(x_{r+1})]^r f(x_{r+1})[1 - F(x_{r+1})]^{R_{r+1}} \times$$

$$f(x_{r+2})[1 - F(x_{r+2})]^{R_{r+2}} \dots f(x_m)[1 - F(x_m)]^{R_m} dx_{r+1} \dots dx_m \quad (5)$$

and the i^{th} and j^{th} product moments as

$$\mu_{q,s:m:n}^{(R_{r+1},R_{r+2},\dots,R_m)^{(i,j)}} = E\left[X_{q:m:n}^{(R_{r+1},R_{r+2},\dots,R_m)^i} X_{s:m:n}^{(R_{r+1},R_{r+2},\dots,R_m)^j}\right] =$$

$$A_{(n,m-1)} \iint \dots \int_{0<x_{r+1}<\dots<x_m<\infty} x_q^i x_s^j [F(x_{r+1})]^r f(x_{r+1})[1 - F(x_{r+1})]^{R_{r+1}} \times$$

$$f(x_{r+2})[1 - F(x_{r+2})]^{R_{r+2}} \dots f(x_m)[1 - F(x_m)]^{R_m} dx_{r+1} \dots dx_m \quad (6)$$

2. Recurrence Relations for Single and Product Moments

In this section we introduce the recurrence relation for single and product moments of general progressively Type-II right censored order statistics from linear failure rate distribution.

In the next theorem we introduce the recurrence relation for single moment of general progressively Type-II right censored order statistics.

Theorem 2.1

For $r + 2 \leq q \leq m - 1, m \leq n\ and\ i \geq 0$, then

$$\mu_{q:m:n}^{(R_{r+1},\dots,R_m)^{(i+2)}} = \frac{(i+2)}{\theta(R_q+1)}\mu_{q:m:n}^{(R_{r+1},\dots,R_m)^{(i)}} - \frac{\alpha(i+2)}{\theta(i+1)}\mu_{q:m:n}^{(R_{r+1},\dots,R_m)^{(i+1)}}$$

$$+\frac{(n - R_{r+1} - \dots - R_{q-1} - q + 1)}{(R_q+1)}\left[\mu_{q-1:m-1:n}^{(R_{r+1},\dots,R_{q-2},(R_{q-1}+R_q+1),R_{q+1},\dots,R_m)^{(i+2)}} + \frac{\alpha(i+2)}{\theta(i+1)}\mu_{q-1:m-1:n}^{(R_{r+1},\dots,R_{q-2},(R_{q-1}+R_q+1),R_{q+1},\dots,R_m)^{(i+1)}}\right]$$

$$-\frac{(n-R_{r+1}-\dots-R_q-q)}{(R_q+1)}\left[\mu_{q:m-1:n}^{(R_{r+1},\dots,R_{q-1},(R_q+R_{q+1}+1),R_{q+2},\dots,R_m)^{(i+2)}} + \frac{\alpha(i+2)}{\theta(i+1)}\mu_{q:m-1:n}^{(R_{r+1},\dots,R_{q-1},(R_q+R_{q+1}+1),R_{q+2},\dots,R_m)^{(i+1)}}\right] \quad (7)$$

Proof

From Eq. (4) and Eq. (5), we get

$$\mu_{q:m:n}^{(R_{r+1},\dots,R_m)^{(i)}} = A_{(n,m-1)} \iint \dots \int_{0<x_{r+1}<\dots<x_{q-1}<x_{q+1}<\dots<x_m<\infty} I_1(x_{q-1},x_{q+1})\,[F(x_{r+1})]^r \times$$
$$f(x_{r+1})[1-F(x_{r+1})]^{R_{r+1}} \dots f(x_{q-1})[1-F(x_{q-1})]^{R_{q-1}} f(x_{q+1})[1-F(x_{q+1})]^{R_{q+1}} \dots \times$$
$$f(x_m)[1-F(x_m)]^{R_m} dx_{r+1}dx_{r+2}\dots dx_{q-1}dx_{q+1}\dots dx_m \tag{8}$$

where

$$I_1(x_{q-1},x_{q+1}) = \int_{x_{q-1}}^{x_{q+1}} x_q^i\,(\alpha+\theta x_q)\,[1-F(x_q)]^{R_q+1}\,dx_q \tag{9}$$

Now, integrating by parts gives

$$I_1(x_{q-1},x_{q+1}) = \frac{\alpha x_{q+1}^{i+1}[1-F(x_{q+1})]^{R_q+1} - \alpha x_{q-1}^{i+1}[1-F(x_{q-1})]^{R_q+1}}{i+1} + \frac{\theta x_{q+1}^{i+2}[1-F(x_{q+1})]^{R_q+1} - \theta x_{q-1}^{i+2}[1-F(x_{q-1})]^{R_q+1}}{i+2} + \frac{\alpha(R_q+1)}{i+1}\int_{x_{q-1}}^{x_{q+1}} x_q^{i+1} f(x_q)[1-F(x_q)]^{R_q}dx_q + \frac{\theta(R_q+1)}{i+2}\int_{x_{q-1}}^{x_{q+1}} x_q^{i+2} f(x_q)[1-F(x_q)]^{R_q}dx_q \tag{10}$$

Substituting Eq. (10) in Eq. (8) and simplifying, yields Eq. (7).

This completes the proof.

Special case

Theorem (2.1) will be valid for the progressively Type-II right censored order statistics as a special case from the general progressively Type-II right censored order statistics when $r = 0$,

$$\mu_{q:m:n}^{(R_1,\dots,R_m)^{(i+2)}} = \frac{(i+2)}{\theta(R_q+1)}\mu_{q:m:n}^{(R_1,\dots,R_m)^{(i)}} - \frac{\alpha(i+2)}{\theta(i+1)}\mu_{q:m:n}^{(R_1,\dots,R_m)^{(i+1)}}$$
$$+\frac{(n-R_1-\dots-R_{q-1}-q+1)}{(R_q+1)}\left[\mu_{q-1:m-1:n}^{(R_1,\dots,R_{q-2},(R_{q-1}+R_q+1),R_{q+1},\dots,R_m)^{(i+2)}} + \frac{\alpha(i+2)}{\theta(i+1)}\mu_{q-1:m-1:n}^{(R_1,\dots,R_{q-2},(R_{q-1}+R_q+1),R_{q+1},\dots,R_m)^{(i+1)}}\right]$$
$$-\frac{(n-R_1-\dots-R_q-q)}{(R_q+1)}\left[\mu_{q:m-1:n}^{(R_1,\dots,R_{q-1},(R_q+R_{q+1}+1),R_{q+2},\dots,R_m)^{(i+2)}} + \frac{\alpha(i+2)}{\theta(i+1)}\mu_{q:m-1:n}^{(R_1,\dots,R_{q-1},(R_q+R_{q+1}+1),R_{q+2},\dots,R_m)^{(i+1)}}\right]$$

In the next two theorems, we introduce recurrence relations for product moments of general progressively Type-II right censored order statistics from linear failure rate distribution.

Theorem 2.2

For $r+1 \le q < s \le m-1, m \le n$ and $i,j \ge 0$, then

$$\mu_{q,s:m:n}^{(R_{r+1},\dots,R_m)^{(i+2,j)}} = \frac{(i+2)}{\theta(R_q+1)}\mu_{q,s:m:n}^{(R_{r+1},\dots,R_m)^{(i,j)}} - \frac{\alpha(i+2)}{\theta(i+1)}\mu_{q,s:m:n}^{(R_{r+1},\dots,R_m)^{(i+1,j)}}$$
$$+\frac{(n-R_{r+1}-\dots-R_{q-1}-q+1)}{(R_q+1)}\left[\mu_{q-1,s-1:m-1:n}^{(R_{r+1},\dots,R_{q-2},(R_{q-1}+R_q+1),R_{q+1},\dots,R_m)^{(i+2,j)}} + \frac{\alpha(i+2)}{\theta(i+1)}\mu_{q-1,s-1:m-1:n}^{(R_{r+1},\dots,R_{q-2},(R_{q-1}+R_q+1),R_{q+1},\dots,R_m)^{(i+1,j)}}\right]$$
$$-\frac{(n-R_{r+1}-\dots-R_q-q)}{(R_q+1)}\left[\mu_{q,s-1:m-1:n}^{(R_{r+1},\dots,R_{q-1},(R_q+R_{q+1}+1),R_{q+2},\dots,R_m)^{(i+2,j)}} + \frac{\alpha(i+2)}{\theta(i+1)}\mu_{q,s-1:m-1:n}^{(R_{r+1},\dots,R_{q-1},(R_q+R_{q+1}+1),R_{q+2},\dots,R_m)^{(i+1,j)}}\right] \tag{11}$$

Proof

From Eq. (6), we get

$$\mu_{q,s:m:n}^{(R_{r+1},\dots,R_m)^{(i,j)}} = A_{(n,m-1)} \iint \dots \int_{0<x_{r+1}<\dots<x_{q-1}<x_{q+1}<\dots<x_m<\infty} x_s^j\, I_1(x_{q-1},x_{q+1})[F(x_{r+1})]^r$$
$$\times f(x_{r+1})[1-F(x_{r+1})]^{R_{r+1}} \dots f(x_{q-1})[1-F(x_{q-1})]^{R_{q-1}} f(x_{q+1})[1-F(x_{q+1})]^{R_{q+1}} \dots \times$$

$$f(x_m)[1-F(x_m)]^{R_m}dx_{r+1}dx_{r+2}\dots dx_{q-1}dx_{q+1}\dots dx_m \quad (12)$$

Substituting by Eq. (10) in Eq. (12) and simplifying, yields Eq. (11).

This completes the proof.

Special case

Theorem (2.2) will be valid for the progressively Type-II right censored order statistics as a special case from the general progressively Type-II right censored order statistics when $r=0$,

$$\mu_{q,s:m:n}^{(R_1,\dots,R_m)^{(i+2,j)}}=\frac{(i+2)}{\theta(R_q+1)}\mu_{q,s:m:n}^{(R_1,\dots,R_m)^{(i,j)}}-\frac{\alpha(i+2)}{\theta(i+1)}\mu_{q,s:m:n}^{(R_1,\dots,R_m)^{(i+1,j)}}$$

$$+\frac{(n-R_1-\dots-R_{q-1}-q+1)}{(R_q+1)}\left[\mu_{q-1,s-1:m-1:n}^{(R_1,\dots,R_{q-2},(R_{q-1}+R_q+1),R_{q+1},\dots,R_m)^{(i+2,j)}}+\frac{\alpha(i+2)}{\theta(i+1)}\mu_{q-1,s-1:m-1:n}^{(R_1,\dots,R_{q-2},(R_{q-1}+R_q+1),R_{q+1},\dots,R_m)^{(i+1,j)}}\right]$$

$$-\frac{(n-R_1-\dots-R_q-q)}{(R_q+1)}\left[\mu_{q,s-1:m-1:n}^{(R_1,\dots,R_{q-1},(R_q+R_{q+1}+1),R_{q+2},\dots,R_m)^{(i+2,j)}}+\frac{\alpha(i+2)}{\theta(i+1)}\mu_{q,s-1:m-1:n}^{(R_1,\dots,R_{q-1},(R_q+R_{q+1}+1),R_{q+2},\dots,R_m)^{(i+1,j)}}\right]$$

Theorem 2.3

For $r+1\le q<s\le m-1, m\le n$ *and* $i,j\ge 0$, then

$$\mu_{q,s:m:n}^{(R_{r+1},\dots,R_m)^{(i,j+2)}}=\frac{(j+2)}{\theta(R_s+1)}\mu_{q,s:m:n}^{(R_{r+1},\dots,R_m)^{(i,j)}}-\frac{\alpha(j+2)}{\theta(j+1)}\mu_{q,s:m:n}^{(R_{r+1},\dots,R_m)^{(i,j+1)}}$$

$$+\frac{(n-R_{r+1}-\dots-R_{s-1}-s+1)}{(R_s+1)}\left[\mu_{q,s-1:m-1:n}^{(R_{r+1},\dots,R_{s-2},(R_{s-1}+R_s+1),R_{s+1},\dots,R_m)^{(i,j+2)}}+\frac{\alpha(j+2)}{\theta(j+1)}\mu_{q,s-1:m-1:n}^{(R_{r+1},\dots,R_{s-2},(R_{s-1}+R_s+1),R_{s+1},\dots,R_m)^{(i,j+1)}}\right]$$

$$-\frac{(n-R_{r+1}-\dots-R_s-s)}{(R_s+1)}\left[\mu_{q,s:m-1:n}^{(R_{r+1},\dots,R_{s-1},(R_s+R_{s+1}+1),R_{s+2},\dots,R_m)^{(i,j+2)}}+\frac{\alpha(j+2)}{\theta(j+1)}\mu_{q,s:m-1:n}^{(R_{r+1},\dots,R_{s-1},(R_s+R_{s+1}+1),R_{s+2},\dots,R_m)^{(i,j+1)}}\right] \quad (13)$$

Proof

From Eq. (4) and Eq. (6), we get

$$\mu_{q,s:m:n}^{(R_{r+1},\dots,R_m)^{(i,j)}}=A_{(n,m-1)}\iint\dots\int_{0<x_{r+1}<\dots<x_{s-1}<x_{s+1}<\dots<x_m<\infty}x_q^i\ I_2(x_{s-1},x_{s+1})[F(x_{r+1})]^r$$

$$\times f(x_{r+1})[1-F(x_{r+1})]^{R_{r+1}}\dots f(x_{s-1})[1-F(x_{s-1})]^{R_{s-1}}f(x_{s+1})[1-F(x_{s+1})]^{R_{s+1}}\dots\times$$

$$f(x_m)[1-F(x_m)]^{R_m}\ dx_{r+1}dx_{r+2}\dots dx_{s-1}dx_{s+1}\dots d_m \quad (14)$$

where

$$I_2(x_{s-1},x_{s+1})=\int_{x_{s-1}}^{x_{s+1}}x_s^j\ (\alpha+\theta x_s)\ [1-F(x_s)]^{R_s+1}\ dx_s \quad (15)$$

Now, integrating by parts we obtain

$$I_2(x_{s-1},x_{s+1})=\left\{\frac{\alpha x_{s+1}^{j+1}[1-F(x_{s+1})]^{R_s+1}-\alpha x_{s-1}^{j+1}[1-F(x_{s-1})]^{R_s+1}}{j+1}+\frac{\theta x_{s+1}^{j+2}[1-F(x_{s+1})]^{R_s+1}-\theta x_{s-1}^{j+2}[1-F(x_{s-1})]^{R_s+1}}{j+2}+\frac{\alpha(R_s+1)}{j+1}\int_{x_{s-1}}^{x_{s+1}}x_s^{j+1}f(x_s)[1-F(x_s)]^{R_s}dx_s+\frac{\theta(R_s+1)}{j+2}\int_{x_{s-1}}^{x_{s+1}}x_s^{j+2}f(x_s)[1-F(x_s)]^{R_s}dx_s\right\} \quad (16)$$

Substituting by Eq. (16) in Eq. (14) and simplifying, yields Eq. (13).

This completes the proof.

Special case

Theorem (2.2) will be valid for the progressively Type-II right censored order statistics as a special case from the general progressively Type-II right censored order statistics when $r=0$,

$$\mu_{q,s:m:n}^{(R_1,\dots,R_m)^{(i,j+2)}}=\frac{(j+2)}{\theta(R_s+1)}\mu_{q,s:m:n}^{(R_1,\dots,R_m)^{(i,j)}}-\frac{\alpha(j+2)}{\theta(j+1)}\mu_{q,s:m:n}^{(R_1,\dots,R_m)^{(i,j+1)}}$$

$$+\frac{(n-R_1-\dots-R_{s-1}-s+1)}{(R_s+1)}\left[\mu_{q,s-1:m-1:n}^{(R_1,\dots,R_{s-2},(R_{s-1}+R_s+1),R_{s+1},\dots,R_m)^{(i,j+2)}}+\frac{\alpha(j+2)}{\theta(j+1)}\mu_{q,s-1:m-1:n}^{(R_1,\dots,R_{s-2},(R_{s-1}+R_s+1),R_{s+1},\dots,R_m)^{(i,j+1)}}\right]$$

$$-\frac{(n-R_1-\cdots-R_s-s)}{(R_s+1)}\left[\mu_{q,s:m-1:n}^{(R_1,\ldots,R_{s-1},(R_s+R_{s+1}+1),R_{s+2},\ldots,R_m)^{(i,j+2)}}+\frac{\alpha(j+2)}{\theta(j+1)}\mu_{q,s:m-1:n}^{(R_1,\ldots,R_{s-1},(R_s+R_{s+1}+1),R_{s+2},\ldots,R_m)^{(i,j+1)}}\right]$$

3. Characterization for Single and Product Moments

In this section, we introduce the characterization of the linear failure rate distribution using the relation between pdf and cdf and using recurrence relation for single and product moments of general progressively Type-II right censored order statistics from linear failure rate distribution.

In the next theorem, we introduce the characterization of the linear failure rate distribution using relation between pdf and cdf.

Theorem 3.1

Let X be a continuous random variable with pdf $f(\cdot)$, cdf $F(\cdot)$ and survival function $[1-F(\cdot)]$. Then X has linear failure rate distribution iff

$$f(x)=[\alpha+\theta x][1-F(x)].x\geq 0 \tag{17}$$

Proof

Necessity:

From Eq. (2) and Eq. (3) we can easily obtain Eq. (17).

Sufficiency:

Suppose that X is a continuous random variable with pdf $f(\cdot)$ and cdf $F(\cdot)$. Suppose, also, that equation Eq. (17) is true. Then we have:

$$\frac{-d[1-F(x)]}{1-F(x)}=[\alpha+\theta x]dx$$

On integrating, we get

$$-\ln|1-F(x)|=\alpha x+\frac{\theta}{2}x^2+C,$$

where C is an arbitrary constant.

Now, since $[1-F(0)]=1$, then putting $x=0$ in this equation, we get $C=0$.

Therefore,

$$-\ln|1-F(x)|=\alpha x+\frac{\theta}{2}x^2,$$

Hence,

$$F(x)=1-e^{-\alpha x-\frac{\theta}{2}x^2}.$$

That is the distribution function of linear failure rate distribution.

This completes the proof.

In the next theorem, we introduce the characterization of the linear failure rate distribution using recurrence relation for single moment of general progressively Type-II right censored order statistics has introduced in the following theorems.

Theorem 3.2

Let $X_{r+1:n}\leq X_{r+2:n}\leq\ldots\leq X_{m:n}$ be the order stasistics of a random sample of size $(n-r)$. Then X has linear failure rate distribution iff, for $r+2\leq q\leq m-1, m\leq n\ and\ i\geq 0$,

$$\mu_{q:m:n}^{(R_{r+1},\ldots,R_m)^{(i+2)}}=\frac{(i+2)}{\theta(R_q+1)}\mu_{q:m:n}^{(R_{r+1},\ldots,R_m)^{(i)}}-\frac{\alpha(i+2)}{\theta(i+1)}\mu_{q:m:n}^{(R_{r+1},\ldots,R_m)^{(i+1)}}$$

$$+\frac{(n-R_{r+1}-\cdots-R_{q-1}-q+1)}{(R_q+1)}\left[\mu_{q-1:m-1:n}^{(R_{r+1},\ldots,R_{q-2},(R_{q-1}+R_q+1),R_{q+1},\ldots,R_m)^{(i+2)}}+\frac{\alpha(i+2)}{\theta(i+1)}\mu_{q-1:m-1:n}^{(R_{r+1},\ldots,R_{q-2},(R_{q-1}+R_q+1),R_{q+1},\ldots,R_m)^{(i+1)}}\right]$$

$$-\frac{(n-R_{r+1}-\cdots-R_q-q)}{(R_q+1)}\left[\mu_{q:m-1:n}^{(R_{r+1},\ldots,R_{q-1},(R_q+R_{q+1}+1),R_{q+2},\ldots,R_m)^{(i+2)}}+\frac{\alpha(i+2)}{\theta(i+1)}\mu_{q:m-1:n}^{(R_{r+1},\ldots,R_{q-1},(R_q+R_{q+1}+1),R_{q+2},\ldots,R_m)^{(i+1)}}\right] \tag{18}$$

Proof

Necessity:

Theorem 2.1 proved the necessary part of this theorem.

Sufficiency:

Assuming that equation (18) holds, then we have:

$$\mu_{q:m:n}^{(R_{r+1},\ldots,R_m)^{(i)}}=\frac{\theta(R_q+1)}{(i+2)}\mu_{q:m:n}^{(R_{r+1},\ldots,R_m)^{(i+2)}}+\frac{\alpha(R_q+1)}{(i+1)}\mu_{q:m:n}^{(R_{r+1},\ldots,R_m)^{(i+1)}}$$

$$+(n-R_{r+1}-\cdots-R_q-q)\left[\frac{\theta}{(i+2)}\mu_{q:m-1:n}^{(R_{r+1},\ldots,R_{q-1},(R_q+R_{q+1}+1),R_{q+2},\ldots,R_m)^{(i+2)}}+\frac{\alpha}{(i+1)}\mu_{q:m-1:n}^{(R_{r+1},\ldots,R_{q-1},(R_q+R_{q+1}+1),R_{q+2},\ldots,R_m)^{(i+1)}}\right]$$

$$-(n-R_{r+1}-\cdots-R_{q-1}-q+1)\left[\frac{\theta}{(i+2)}\mu_{q-1:m-1:n}^{(R_{r+1},\ldots,R_{q-2},(R_{q-1}+R_q+1),R_{q+1},\ldots,R_m)^{(i+2)}}+\frac{\alpha}{(i+1)}\mu_{q-1:m-1:n}^{(R_{r+1},\ldots,R_{q-2},(R_{q-1}+R_q+1),\ldots,R_m)^{(i+1)}}\right] \tag{19}$$

From Eq. (4) and Eq. (5), we get

$$\mu_{q:m:n}^{(R_{r+1},\dots,R_m)^{(i+1)}} = A_{(n,m-1)} \iint \cdots \int_{0<x_{r+1}<\cdots<x_{q-1}<x_{q+1}<\cdots<x_m<\infty} I_3(x_{q-1}, x_{q+1})[F(x_{r+1})]^r \times$$
$$f(x_{r+1})[1-F(x_{r+1})]^{R_{r+1}} \dots f(x_{q-1})[1-F(x_{q-1})]^{R_{q-1}} f(x_{q+1})[1-F(x_{q+1})]^{R_{q+1}} \dots \times$$
$$f(x_m)[1-F(x_m)]^{R_m} dx_{r+1} \dots dx_{q-1} dx_{q+1} \dots dx_m \tag{20}$$

where

$$I_3(x_{q-1}, x_{q+1}) = \int_{x_{q-1}}^{x_{q+1}} x_q^{i+1} f(x_q)[1-F(x_q)]^{R_q} dx_q \tag{21}$$

Integrating by parts, we obtain

$$I_3(x_{q-1}, x_{q+1}) = \frac{-1}{R_q+1} x_{q+1}^{i+1}[1-F(x_{q+1})]^{R_q+1} + \frac{1}{R_q+1} x_{q-1}^{i+1}[1-F(x_{q-1})]^{R_q+1} + \frac{i+1}{R_q+1}\int_{x_{q-1}}^{x_{q+1}} x_q^{i}[1-F(x_q)]^{R_q+1} dx_q \tag{22}$$

Substituting in Eq. (20), we get

$$\mu_{q:m:n}^{(R_{r+1},\dots,R_m)^{(i+1)}} = \frac{i+1}{R_q+1} A_{(n,m-1)} \iint \cdots \int_{0<x_{r+1}<\cdots<x_{q-1}<x_{q+1}<\cdots<x_m<\infty} [F(x_{r+1})]^r \times$$
$$f(x_{r+1})[1-F(x_{r+1})]^{R_{r+1}} \dots f(x_{q-1})[1-F(x_{q-1})]^{R_{q-1}} \int_{x_{q-1}}^{x_{q+1}} x_q^{i}[1-F(x_q)]^{R_q+1} dx_q$$
$$f(x_{q+1})[1-F(x_{q+1})]^{R_{q+1}} \dots f(x_m)[1-F(x_m)]^{R_m} dx_{r+1} \dots dx_{q-1} dx_{q+1} \dots dx_m$$
$$+\frac{A_{(n,m-1)}}{R_q+1} \iint \cdots \int_{0<x_{r+1}<\cdots<x_{q-1}<x_{q+1}<\cdots<x_m<\infty} x_{q-1}^{i+1}[F(x_{r+1})]^r \times$$
$$f(x_{r+1})[1-F(x_{r+1})]^{R_{r+1}} \dots f(x_{q-1})[1-F(x_{q-1})]^{1+R_{qq}+R_{q-1}} f(x_{q+1})[1-F(x_{q+1})]^{R_{q+1}}$$
$$\times \dots f(x_m)[1-F(x_m)]^{R_m} dx_{r+1} \dots dx_{q-1} dx_{q+1} \dots dx_m$$
$$-\frac{A_{(n,m-1)}}{R_q+1} \iint \cdots \int_{0<x_{r+1}<\cdots<x_{q-1}<x_{q+1}<\cdots<x_m<\infty} x_{q+1}^{i+1}[F(x_{r+1})]^r \times$$
$$f(x_{r+1})[1-F(x_{r+1})]^{R_{r+1}} \dots f(x_{q-1})[1-F(x_{q-1})]^{R_{q-1}} f(x_{q+1})[1-F(x_{q+1})]^{1+R_q+R_{q+1}}$$
$$\times \dots f(x_m)[1-F(x_m)]^{R_m} dx_{r+1} \dots dx_{q-1} dx_{q+1} \dots dx_m$$
$$= A_{(n,m-1)} \frac{i+1}{R_q+1} \iint \cdots \int_{0<x_{r+1}<\cdots<x_{q-1}<x_{q+1}<\cdots<x_m<\infty} [F(x_{r+1})]^r \times$$
$$f(x_{r+1})[1-F(x_{r+1})]^{R_{r+1}} \dots f(x_{q-1})[1-F(x_{q-1})]^{R_{q-1}} \int_{x_{q-1}}^{x_{q+1}} x_q^{i}[1-F(x_q)]^{R_q+1} dx_q \times$$
$$f(x_{q+1})[1-F(x_{q+1})]^{R_{q+1}} \dots f(x_m)[1-F(x_m)]^{R_m} dx_{r+1} \dots dx_{q-1} dx_{q+1} \dots dx_m +$$
$$\frac{(n-R_{r+1}-\cdots-R_q-q)}{R_q+1} \mu_{q:m-1:n}^{(R_{r+1},\dots R_{q-1},(R_q+R_{q+1}+1),R_{q+2},\dots,R_m)^{(i+1)}}$$
$$-\frac{(n-R_{r+1}-\cdots-R_{q-1}-q+1)}{R_q+1} \mu_{q-1:m-1:n}^{(R_{r+1},\dots R_{q-2},(R_{q-1}+R_q+1),R_{q+1},\dots,R_m)^{(i+1)}} \tag{23}$$

and

$$\mu_{q:m:n}^{(R_{r+1},\dots,R_m)^{(i+2)}} = A_{(n,m-1)} \frac{i+2}{R_q+1} \iint \dots \int_{0<x_{r+1}<\dots<x_{q-1}<x_{q+1}<\dots<x_m<\infty} [F(x_{r+1})]^r \times$$

$$f(x_{r+1})[1-F(x_{r+1})]^{R_{r+1}} \dots f(x_{q-1})[1-F(x_{q-1})]^{R_{q-1}} \int_{x_{q-1}}^{x_{q+1}} x_q^{i+1}[1-F(x_q)]^{R_q+1} dx_q \times$$

$$f(x_{q+1})[1-F(x_{q+1})]^{R_{q+1}} \dots f(x_m)[1-F(x_m)]^{R_m} dx_{r+1} \dots dx_{q-1} dx_{q+1} \dots dx_m$$

$$+\frac{(n-R_{r+1}-\dots-R_q-q)}{R_q+1} \mu_{q:m-1:n}^{(R_{r+1},\dots R_{q-1},(R_q+R_{q+1}+1),R_{q+2},\dots,R_m)^{(i+2)}}$$

$$-\frac{(n-R_{r+1}-\dots-R_{q-1}-q+1)}{R_q+1} \mu_{q-1:m-1:n}^{(R_{r+1},\dots R_{q-2},(R_{q-1}+R_q+1),R_{q+1},\dots,R_m)^{(i+2)}} \quad (24)$$

Now by substituting for $\mu_{q:m:n}^{(R_{r+1},\dots,R_m)^{(i+1)}}$ *and* $\mu_{q:m:n}^{(R_{r+1},\dots,R_m)^{(i+2)}}$ from Eq. (23) and Eq. (24) in Eq. (19), we get

$$\mu_{q:m:n}^{(R_{r+1},\dots,R_m)^{(i)}} = A_{(n,m-1)} \iint \dots \int_{0<x_{r+1}<\dots<x_m<\infty} x_q^i [F(x_{r+1})]^r (\alpha+\theta x_q) \times$$

$$f(x_{r+1})[1-F(x_{r+1})]^{R_{r+1}} \dots f(x_{q-1})[1-F(x_{q-1})]^{R_{q-1}} [1-F(x_q)]^{R_q+1} \times$$

$$f(x_{q+1})[1-F(x_{q+1})]^{R_{q+1}} \dots f(x_m)[1-F(x_m)]^{R_m} dx_{r+1} \dots dx_m \quad (25)$$

We get

$$A_{(n,m-1)} \iint \dots \int_{0<x_{r+1}<\dots<x_m<\infty} x_q^i [F(x_{r+1})]^r \left[f(x_q)-(\alpha+\theta x_q)[1-F(x_q)]\right] \times$$

$$[1-F(x_q)]^{R_q} f(x_{r+1})[1-F(x_{r+1})]^{R_{r+1}} \dots f(x_{q-1})[1-F(x_{q-1})]^{R_{q-1}} \times$$

$$f(x_{q+1})[1-F(x_{q+1})]^{R_{q+1}} \dots f(x_m)[1-F(x_m)]^{R_m} dx_{r+1} \dots dx_m = 0$$

Using Muntz-Szasz theorem, (See, Hwang and Lin [10]), we get

$$f(x_q) = (\alpha+\theta x_q)[1-F(x_q)]$$

Using Theorem 3.1, we get

$$F(x) = 1 - e^{-\alpha x - \frac{\theta x^2}{2}}$$

That is the distribution function of linear failure rate distribution.

This completes the proof.

In the next two theorems, we introduce the characterize the linear failure rate distribution using recurrence relation for product moment of general progressively Type-II right censored order statistics.

Theorem 3.3

Let $X_{r+1:n} \le X_{r+2:n} \le \dots \le X_{m:n}$ be the order stasistics of a random sample of size $(n-r)$. Then X has linear failure rate distribution iff, for $r+1 \le q < s \le m-1, m \le n$ *and* $i,j \ge 0$,

$$\mu_{q,s:m:n}^{(R_{r+1},\dots,R_m)^{(i+2,j)}} = \frac{(i+2)}{\theta(R_q+1)} \mu_{q,s:m:n}^{(R_{r+1},\dots,R_m)^{(i,j)}} - \frac{\alpha(i+2)}{\theta(i+1)} \mu_{q,s:m:n}^{(R_{r+1},\dots,R_m)^{(i+1,j)}}$$

$$+\frac{(n-R_{r+1}-\dots-R_{q-1}-q+1)}{(R_q+1)} \left[\mu_{q-1,s-1:m-1:n}^{(R_{r+1},\dots,R_{q-2},(R_{q-1}+R_q+1),R_{q+1},\dots,R_m)^{(i+2,j)}} + \frac{\alpha(i+2)}{\theta(i+1)} \mu_{q-1,s-1:m-1:n}^{(R_{r+1},\dots,R_{q-2},(R_{q-1}+R_q+1),R_{q+1},\dots,R_m)^{(i+1,j)}}\right]$$

$$-\frac{(n-R_{r+1}-\dots-R_q-q)}{(R_q+1)} \left[\mu_{q,s-1:m-1:n}^{(R_{r+1},\dots,R_{q-1},(R_q+R_{q+1}+1),R_{q+2},\dots,R_m)^{(i+2,j)}} + \frac{\alpha(i+2)}{\theta(i+1)} \mu_{q,s-1:m-1:n}^{(R_{r+1},\dots,R_{q-1},(R_q+R_{q+1}+1),R_{q+2},\dots,R_m)^{(i+1,j)}}\right] \quad (26)$$

Proof

Necessity:

Theorem 2.2 proved the necessary part of this theorem

Sufficiency:

Assuming that equation (26) holds, then we have:

$$\mu_{q,s:m:n}^{(R_{r+1},\dots,R_m)^{(i,j)}} = \frac{\theta(R_q+1)}{(i+2)}\mu_{q,s:m:n}^{(R_{r+1},\dots,R_m)^{(i+2,j)}} + \frac{\alpha(R_q+1)}{(i+1)}\mu_{q,s:m:n}^{(R_{r+1},\dots,R_m)^{(i+1,j)}}$$
$$+(n-R_{r+1}-\dots-R_q-q)\left[\frac{\theta}{(i+2)}\mu_{q,s-1:m-1:n}^{(R_{r+1},\dots,R_{q-1},(R_q+R_{q+1}+1),R_{q+2},\dots,R_m)^{(i+2,j)}} + \frac{\alpha}{(i+1)}\mu_{q,s-1:m-1:n}^{(R_{r+1},\dots,R_{q-1},(R_q+R_{q+1}+1),R_{q+2},\dots,R_m)^{(i+1,j)}}\right]$$
$$-(n-R_{r+1}-\dots-R_{q-1}-q+1)\left[\frac{\theta}{(i+2)}\mu_{q-1,s-1:m-1:n}^{(R_{r+1},\dots,R_{q-2},(R_{q-1}+R_q+1),R_{q+1},\dots,R_m)^{(i+2,j)}} + \frac{\alpha}{(i+1)}\mu_{q-1,s-1:m-1:n}^{(R_{r+1},\dots,R_{q-2},(R_{q-1}+R_q+1),R_{q+1},\dots,R_m)^{(i+1,j)}}\right] \quad (27)$$

where

$$\mu_{q,s:m:n}^{(R_{r+1},\dots,R_m)^{(i+1,j)}} = A_{(n,m-1)}\iint\dots\int_{0<x_{r+1}<\dots<x_{q-1}<x_{q+1}<\dots<x_m<\infty}[F(x_{r+1})]^r \times$$
$$I_3(x_{q-1},x_{q+1})x_s^j f(x_{r+1})[1-F(x_{r+1})]^{R_{r+1}}\dots f(x_{q-1})[1-F(x_{q-1})]^{R_{q-1}} \times$$
$$f(x_{q+1})[1-F(x_{q+1})]^{R_{q+1}}\dots f(x_m)[1-F(x_m)]^{R_m}dx_{r+1}\dots dx_{q-1}dx_{q+1}\dots dx_m \quad (28)$$

Substituting by Eq. (22) in Eq. (28), we get

$$\mu_{q,s:m:n}^{(R_{r+1},\dots,R_m)^{(i+1,j)}} = A_{(n,m-1)}\frac{i+1}{R_q+1}\iint\dots\int_{0<x_{r+1}<\dots<x_{q-1}<x_{q+1}<\dots<x_m<\infty}x_s^j[F(x_{r+1})]^r \times$$
$$f(x_{r+1})[1-F(x_{r+1})]^{R_{r+1}}\dots f(x_{q-1})[1-F(x_{q-1})]^{R_{q-1}}\int_{x_{q-1}}^{x_{q+1}}x_q^i[1-F(x_q)]^{R_q+1}dx_q \times$$
$$f(x_{q+1})[1-F(x_{q+1})]^{R_{q+1}}\dots f(x_m)[1-F(x_m)]^{R_m}dx_{r+1}\dots dx_{q-1}dx_{q+1}\dots dx_m$$
$$+\frac{A_{(n,m-1)}}{R_q+1}\iint\dots\int_{0<x_{r+1}<\dots<x_{q-1}<x_{q+1}<\dots<x_m<\infty}x_{q-1}^{i+1}x_s^j[F(x_{r+1})]^r \times$$
$$f(x_{r+1})[1-F(x_{r+1})]^{R_{r+1}}\dots f(x_{q-1})[1-F(x_{q-1})]^{1+R_q+R_{q-1}}f(x_{q+1})[1-F(x_{q+1})]^{R_{q+1}} \times$$
$$\dots f(x_m)[1-F(x_m)]^{R_m}dx_{r+1}\dots dx_{q-1}dx_{q+1}\dots dx_m$$
$$-\frac{A_{(n,m-1)}}{R_q+1}\iint\dots\int_{0<x_{r+1}<\dots<x_{q-1}<x_{q+1}<\dots<x_m<\infty}x_{q+1}^{i+1}x_s^j[F(x_{r+1})]^r \times$$
$$f(x_{r+1})[1-F(x_{r+1})]^{R_{r+1}}\dots f(x_{q-1})[1-F(x_{q-1})]^{R_{q-1}}f(x_{q+1})[1-F(x_{q+1})]^{1+R_q+R_{q+1}} \times$$
$$\dots f(x_m)[1-F(x_m)]^{R_m}dx_{r+1}\dots dx_{q-1}dx_{q+1}\dots dx_m$$
$$= A_{(n,m-1)}\frac{i+1}{R_q+1}\iint\dots\int_{0<x_{r+1}<\dots<x_{q-1}<x_{q+1}<\dots<x_m<\infty}x_s^j[F(x_{r+1})]^r \times$$

$$f(x_{r+1})[1-F(x_{r+1})]^{R_{r+1}} \dots f(x_{q-1})[1-F(x_{q-1})]^{R_{q-1}} \int_{x_{q-1}}^{x_{q+1}} x_q^i [1-F(x_q)]^{R_q+1} dx_q \times$$

$$f(x_{q+1})[1-F(x_{q+1})]^{R_{q+1}} \dots f(x_m)[1-F(x_m)]^{R_m} dx_{r+1} \dots dx_{q-1} dx_{q+1} \dots dx_m +$$

$$\frac{(n-R_{r+1}-\cdots-R_{q-1}-q+1)}{R_q+1} \mu_{q-1,s-1:m-1:n}^{(R_{r+1},\dots,R_{q-2},(R_{q-1}+R_q+1),R_{q+1},\dots,R_m)^{(i+1,j)}}$$

$$-\frac{(n-R_{r+1}-\cdots-R_q-q)}{R_q+1} \mu_{q,s-1:m-1:n}^{(R_{r+1},\dots,R_{q-1},(R_q+R_{q+1}+1),R_{q+2},\dots,R_m)^{(i+1,j)}} \quad (29)$$

and

$$\mu_{q,s:m:n}^{(R_{r+1},\dots,R_m)^{(i+2,j)}} = \frac{i+2}{R_q+1} A_{(n,m-1)} \iint \dots \int_{0<x_{r+1}<\cdots<x_{q-1}<x_{q+1}<\cdots<x_m<\infty} x_s^j [F(x_{r+1})]^r \times$$

$$f(x_{r+1})[1-F(x_{r+1})]^{R_{r+1}} \dots \int_{x_{q-1}}^{x_{q+1}} x_q^{i+1} [1-F(x_q)]^{R_q+1} dx_q \, f(x_{q-1})[1-F(x_{q-1})]^{R_{q-1}}$$

$$f(x_{q+1})[1-F(x_{q+1})]^{R_{q+1}} \dots f(x_m)[1-F(x_m)]^{R_m} dx_{r+1} \dots dx_{q-1} dx_{q+1} \dots dx_m$$

$$+\frac{(n-R_{r+1}-\cdots-R_{q-1}-q+1)}{R_q+1} \mu_{q-1,s-1:m-1:n}^{(R_{r+1},\dots,R_{q-2},(R_{q-1}+R_q+1),R_{q+1},\dots,R_m)^{(i+2,j)}}$$

$$-\frac{(n-R_{r+1}-\cdots-R_q-q)}{R_q+1} \mu_{q,s-1:m-1:n}^{(R_{r+1},\dots,R_{q-1},(R_q+R_{q+1}+1),R_{q+2},\dots,R_m)^{(i+2,j)}} \quad (30)$$

Now by substituting for $\mu_{q,s:m:n}^{(R_{r+1},\dots,R_m)^{(i+1,j)}}$ *and* $\mu_{q,s:m:n}^{(R_{r+1},\dots,R_m)^{(i+2,j)}}$ from Eq. (29) and Eq. (30) in Eq. (27), we get

$$\mu_{q,s:m:n}^{(R_{r+1},\dots,R_m)^{(i,j)}} = A_{(n,m-1)} \iint \dots \int_{0<x_{r+1}<\cdots<x_m<\infty} x_q^i x_s^j [F(x_{r+1})]^r (\alpha+\theta x_q)$$

$$f(x_{r+1})[1-F(x_{r+1})]^{R_{r+1}} \dots f(x_{q-1})[1-F(x_{q-1})]^{R_{q-1}} [1-F(x_q)]^{R_q+1} \times$$

$$f(x_{q+1})[1-F(x_{q+1})]^{R_{q+1}} \dots f(x_m)[1-F(x_m)]^{R_m} dx_{r+1} \dots dx_m \quad (31)$$

Then

$$A_{(n,m-1)} \iint \dots \int_{0<x_{r+1}<\cdots<x_m<\infty} x_q^i x_s^j [F(x_{r+1})]^r \left[f(x_q)-(\alpha+\theta x_q)[1-F(x_q)]\right] \times$$

$$[1-F(x_q)]^{R_q} f(x_{r+1})[1-F(x_{r+1})]^{R_{r+1}} \dots f(x_{q-1})[1-F(x_{q-1})]^{R_{q-1}} \times$$

$$f(x_{q+1})[1-F(x_{q+1})]^{R_{q+1}} \dots f(x_m)[1-F(x_m)]^{R_m} dx_{r+1} \dots dx_m = 0$$

Using Muntz-Szasz theorem, (See, Hwang and Lin [10]), we get

$$f(x_q) = (\alpha+\theta x_q)[1-F(x_q)]$$

Using Theorem 3.1, we get

$$F(x) = 1-e^{-\alpha x-\frac{\theta x^2}{2}}$$

That is the distribution function of linear failure rate distribution.

This completes the proof.

Theorem 3.4

Let $X_{r+1:n} \le X_{r+2:n} \le \dots \le X_{m:n}$ be the order stasistics of a random sample of size $(n-r)$. Then X has linear failure rate distribution iff, for $r+1 \le q < s \le m-1, m \le n \text{ and } i,j \ge 0$,

$$\mu_{r,s:m:n}^{(R_{r+1},\dots,R_m)^{(i,j+2)}} = \frac{(j+2)}{\theta(R_s+1)} \mu_{q,s:m:n}^{(R_{r+1},\dots,R_m)^{(i,j)}} - \frac{\alpha(j+2)}{\theta(j+1)} \mu_{q,s:m:n}^{(R_{r+1},\dots,R_m)^{(i,j+1)}}$$

$$+\frac{(n-R_{r+1}-\cdots-R_{s-1}-s+1)}{(R_s+1)}\left[\mu_{q,s-1:m-1:n}^{(R_{r+1},\dots,R_{s-2},(R_{s-1}+R_s+1),R_{s+1},\dots,R_m)^{(i,j+2)}}+\frac{\alpha(j+2)}{\theta(j+1)}\mu_{q,s-1:m-1:n}^{(R_{r+1},\dots,R_{s-2},(R_{s-1}+R_s+1),R_{s+1},\dots,R_m)^{(i,j+1)}}\right]$$

$$-\frac{(n-R_{r+1}-\cdots-R_s-s)}{(R_s+1)}\left[\mu_{q,s:m-1:n}^{(R_{r+1},\dots,R_{s-1},(R_s+R_{s+1}+1),R_{s+2},\dots,R_m)^{(i,j+2)}}+\frac{\alpha(j+2)}{\theta(j+1)}\mu_{q,s:m-1:n}^{(R_{r+1},\dots,R_{s-1},(R_s+R_{s+1}+1),R_{s+2},\dots,R_m)^{(i,j+1)}}\right] \quad (32)$$

Proof

Necessity:

Theorem 2.3 proved the necessary part of this theorem.

Sufficiency:

Assuming that equation (32) holds, then we have:

$$\mu_{q,s:m:n}^{(R_{r+1},\dots,R_m)^{(i,j)}}=\frac{\theta(R_s+1)}{(j+2)}\mu_{q,s:m:n}^{(R_{r+1},\dots,R_m)^{(i,j+2)}}+\frac{\alpha(R_s+1)}{(j+1)}\mu_{q,s:m:n}^{(R_{r+1},\dots,R_m)^{(i,j+1)}}$$

$$+(n-R_{r+1}-\cdots-R_s-s)\left[\frac{\theta}{(j+2)}\mu_{q,s-1:m-1:n}^{(R_{r+1},\dots,R_{s-1},(R_s+R_{s+1}+1),R_{s+2},\dots,R_m)^{(i,j+2)}}+\frac{\alpha}{(j+1)}\mu_{q,s-1:m-1:n}^{(R_{r+1},\dots,R_{s-1},(R_s+R_{s+1}+1),R_{s+2},\dots,R_m)^{(i,j+1)}}\right]$$

$$-(n-R_{r+1}-\cdots-R_{s-1}-s+1)\left[\begin{array}{l}\frac{\theta}{(j+2)}\mu_{q-1,s-1:m-1:n}^{(R_{r+1},\dots,R_{s-2},(R_{s-1}+R_s+1),R_{s+1},\dots,R_m)^{(i,j+2)}}\\+\frac{\alpha}{(j+1)}\mu_{q,s-1:m-1:n}^{(R_{r+1},\dots,R_{s-2},(R_{s-1}+R_s+1),R_{s+1},\dots,R_m)^{(i,j+1)}}\end{array}\right] \quad (33)$$

From Eq. (4) and Eq. (6), we get

$$\mu_{q,s:m:n}^{(R_{r+1},\dots,R_m)^{(i,j+1)}}=A_{(n,m-1)}\iint\cdots\int_{0<x_{r+1}<\cdots<x_{s-1}<x_{s+1}<\cdots<x_m<\infty}[F(x_{r+1})]^r x_q^i\times$$
$$I_4(x_{s-1},x_{s+1})f(x_{r+1})[1-F(x_{r+1})]^{R_{r+1}}f(x_{s-1})[1-F(x_{s-1})]^{R_{s-1}}\times$$
$$f(x_{s+1})[1-F(x_{s+1})]^{R_{s+1}}\dots f(x_m)[1-F(x_m)]^{R_m}dx_{r+1}\dots dx_{s-1}dx_{s+1}\dots dx_m \quad (34)$$

where

$$I_4(x_{s-1},x_{s+1}) = \int_{x_{s-1}}^{x_{s+1}} x_s^{j+1}f(x_s)[1-F(x_s)]^{R_s}dx_s \quad (35)$$

Integrating by parts, we have obtain

$$I_4(x_{s-1},x_{s+1})=\frac{-1}{R_s+1}x_{s+1}^{j+1}[1-F(x_{s+1})]^{R_s+1}+\frac{x_{s-1}^{j+1}}{1+R_s}[1-F(x_{s-1})]^{R_s+1}$$
$$+\frac{i+1}{R_s+1}\int_{x_{s-1}}^{x_{s+1}}x_s^j[1-F(x_s)]^{R_s+1}dx_s \quad (36)$$

Now by substituting in Eq. (34), we get

$$\mu_{q,s:m:n}^{(R_{r+1},\dots,R_m)^{(i,j+1)}}=A_{(n,m-1)}\frac{j+1}{R_s+1}\iint\cdots\int_{0<x_{r+1}<\cdots<x_{s-1}<x_{s+1}<\cdots<x_m<\infty}x_q^i[F(x_{r+1})]^r\times$$
$$f(x_{r+1})[1-F(x_{r+1})]^{R_{r+1}}\dots f(x_{s-1})[1-F(x_{s-1})]^{R_{s-1}}\int_{x_{s-1}}^{x_{s+1}}x_s^j[1-F(x_s)]^{1+R_s}dx_s\times$$
$$f(x_{s+1})[1-F(x_{s+1})]^{R_{s+1}}\dots f(x_m)[1-F(x_m)]^{R_m}dx_{r+1}\dots dx_{s-1}dx_{s+1}\dots dx_m$$
$$+\frac{A_{(n,m-1)}}{R_s+1}\iint\cdots\int_{0<x_{r+1}<\cdots<x_{s-1}<x_{s+1}<\cdots<x_m<\infty}x_{s-1}^{j+1}x_{qq}^i[F(x_{r+1})]^r\times$$
$$f(x_{r+1})[1-F(x_{r+1})]^{R_{r+1}}\dots f(x_{s-1})[1-F(x_{s-1})]^{1+R_s+R_{s-1}}f(x_{s+1})[1-F(x_{s+1})]^{R_{s+1}}\dots\times$$
$$f(x_m)[1-F(x_m)]^{R_m}dx_{r+1}\dots dx_{s-1}dx_{s+1}\dots dx_m$$

$$-\frac{A_{(n,m-1)}}{R_s+1}\iint\cdots\int_{0<x_{r+1}<\cdots<x_{s-1}<x_{s+1}<\cdots<x_m<\infty} x_{s+1}^{j+1}x_q^i[F(x_{r+1})]^r\times$$

$$f(x_{r+1})[1-F(x_{r+1})]^{R_{r+1}}\dots f(x_{s-1})[1-F(x_{s-1})]^{R_{s-1}}f(x_{s+1})[1-F(x_{s+1})]^{1+R_s+R_{s+1}}\dots\times$$

$$f(x_m)[1-F(x_m)]^{R_m}dx_{r+1}\dots dx_{s-1}dx_{s+1}\dots dx_m$$

$$=A_{(n,m-1)}\frac{j+1}{R_s+1}\iint\cdots\int_{0<x_{r+1}<\cdots<x_{s-1}<x_{s+1}<\cdots<x_m<\infty}[F(x_{r+1})]^r\times$$

$$f(x_{r+1})[1-F(x_{r+1})]^{R_{r+1}}\dots f(x_{s-1})[1-F(x_{s-1})]^{R_{s-1}}\int_{x_{s-1}}^{x_{s+1}} x_s^j x_q^i[1-F(x_s)]^{R_s+1}dx_s$$

$$f(x_{s+1})[1-F(x_{s+1})]^{R_{s+1}}\dots f(x_m)[1-F(x_m)]^{R_m}dx_{r+1}\dots dx_{s-1}dx_{s+1}\dots dx_m+$$

$$\frac{(n-R_{r+1}-\cdots-R_{s-1}-s+1)}{R_s+1}\mu_{q,s-1:m-1:n}^{(R_{r+1},\dots,R_{s-2},(R_{s-1}+R_s+1),R_{s+1},\dots,R_m)^{(i,j+1)}}$$

$$-\frac{(n-R_{r+1}-\cdots-R_s-s)}{R_s+1}\mu_{q,s-1:m-1:n}^{(R_{r+1},\dots,R_{s-1},(R_s+R_{s+1}+1),R_{s+2},\dots,R_m)^{(i,j+1)}} \tag{37}$$

and

$$\mu_{q,s:m:n}^{(R_{r+1},\dots,R_m)^{(i,j+2)}}=A_{(n,m-1)}\frac{j+2}{R_s+1}\iint\cdots\int_{0<x_{r+1}<\cdots<x_{s-1}<x_{s+1}<\cdots<x_m<\infty}x_q^i[F(x_{r+1})]^r\times$$

$$f(x_{r+1})[1-F(x_{r+1})]^{R_{r+1}}\dots f(x_{s-1})[1-F(x_{s-1})]^{R_{s-1}}\int_{x_{s-1}}^{x_{s+1}} x_s^{j+1}[1-F(x_s)]^{R_s+1}dx_s$$

$$f(x_{s+1})[1-F(x_{s+1})]^{R_{s+1}}\dots f(x_m)[1-F(x_m)]^{R_m}dx_{r+1}\dots dx_{s-1}dx_{s+1}\dots dx_m$$

$$+\frac{(n-R_{r+1}-\cdots-R_{s-1}-s+1)}{R_s+1}\mu_{q,s-1:m-1:n}^{(R_{r+1},\dots,R_{s-2},(R_{s-1}+R_s+1),R_{s+1},\dots,R_m)^{(i,j+2)}}$$

$$-\frac{(n-R_{r+1}-\cdots-R_s-s)}{R_s+1}\mu_{q,s-1:m-1:n}^{(R_{r+1},\dots,R_{s-1},(R_s+R_{s+1}+1),R_{s+2},\dots,R_m)^{(i,j+2)}} \tag{38}$$

Now by substituting for $\mu_{q,s:m:n}^{(R_{r+1},\dots,R_m)^{(i,j+1)}}$ *and* $\mu_{q,s:m:n}^{(R_{r+1},\dots,R_m)^{(i,j+2)}}$ from Eq. (37) and Eq. (38) in Eq. (33), we get

$$\mu_{q,s:m:n}^{(R_{r+1},\dots,R_m)^{(i,j)}}=A_{(n,m-1)}\iint\cdots\int_{0<x_{r+1}<\cdots<x_m<\infty}x_q^i x_s^j[F(x_{r+1})]^r(\alpha+\theta x_s)$$

$$f(x_{r+1})[1-F(x_{r+1})]^{R_{r+1}}\dots f(x_{s-1})[1-F(x_{s-1})]^{R_{s-1}}[1-F(x_s)]^{R_s+1}\times$$

$$f(x_{s+1})[1-F(x_{s+1})]^{R_{s+1}}\dots f(x_m)[1-F(x_m)]^{R_m}dx_{r+1}\dots dx_m \tag{39}$$

We get

$$A_{(n,m-1)}\iint\cdots\int_{0<x_{r+1}<\cdots<x_m<\infty}x_q^i x_s^j[F(x_{r+1})]^r\left[f(x_q)-(\alpha+\theta x_q)[1-F(x_q)]\right]\times$$

$$[1-F(x_s)]^{R_s}f(x_{r+1})[1-F(x_{r+1})]^{R_{r+1}}\dots f(x_{s-1})[1-F(x_{s-1})]^{R_{s-1}}f\times$$

$$(x_{s+1})[1-F(x_{s+1})]^{R_{s+1}}\dots f(x_m)[1-F(x_m)]^{R_m}dx_{r+1}\dots dx_m=0$$

Using Muntz-Szasz theorem, (See, Hwang and Lin [10]), we get

$$f(x_s)=(\alpha+\theta x_s)[1-F(x_s)]$$

Using Theorem 3.1 above, we get

$$F(x)=1-e^{-\alpha x-\frac{\theta x^2}{2}}$$

That is the distribution function of linear failure rate distribution.

This completes the proof.

4. Conclusion

We derived recurrence relations among single and product moments of general progressively Type-II right censored order statistics (Theorem 2.1, 2.2 and 2.3). Characterization for the random variable X following the linear failure rate distribution is obtained using the previous recurrence relations (Theorems 3.2, 3.3 and 3.4) and distribution function (Theorem 3.1). For future work, estimation for the scale and location parameters could be obtained by applying the best linear unbiased estimation to the previous results.

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References

[1] Abd El-Aty, Y. and Marwa M. Mohie El-Din. Recurrence relations for single and double moments of GOS from the inverted linear exponential distribution and any continuous function. *J. of probability and statist. science,* 2009; 7:127-137.

[2] Abd El-Hamid, A. H. and AL-Hussaini, E. K. Inference and optimal design based on step-partially accelerated life tests for the generalized Pareto distribution under progressive Type-I censoring. *Communications in Statistics- Simulation and Computation,* 2015; 44:1750-1769.

[3] Aggarwala, R. and Balakrishnan, N. Recurrence relations for single and product moments of progressive Type-II right censored order statistics from exponential and truncated exponential distributions. *Ann. Inst. Statist. Math.*, 1996; 4:757-771.

[4] Athar, H. Akhterand, Z. and Saran, J. Moments of progressive Type-II right censored order statistics from Lindley distribution. *Statist. Research Letters,* 2014; 3:1-6.

[5] Bain, L. J. Analysis for the Linear Failure-Rate Life-Testing Distribution. *Technometrics*, 1974; 16:551-559.

[6] Balakrishnan, N., Cramer, E., Kamps, U., Schenk, N. Progressive Type II censored order statistics from exponential distributions. Statistics, 2001; 35:537-556.

[7] Balakrishnan, N. and Sandhu R. A. Best linear unbiased and maximum likelihood estimation for exponential distributions based on general progressive Type-II censored samples. Sankhya, 58, Series B., 1996; 1:1-9.

[8] Davis, H. T., Feldstein, M. L. The generalized Pareto law as a model for progressively censored survival data. Biometrika, 1979; 66:299-306.

[9] Jagdish Saran and Narinder Pushkarna. Derived moments of progressive Type-II right censored order statistics from a general class of doubly truncated continuous distributions. *J. of Statistical Theory and Applications,* 2014; 13:162-174.

[10] Hwang, J. S. and Lin, G. D. Extensions of Muntz-Szasz theorem and applications. *Analysis*, 1984; 4:143-160.

[11] Mohie El-Din, M. M., Amein, M. M. and Hafez, E. H. Statistical inference and characterizations from independent and identical Exponential-Bernoulli mixture distribution. *Advanced Research in Statistics and Probability, Issue,* 2011; 3:15-31.

[12] Mohie El-Din, M. M. and Kotb, M. S. Recurrence relations for single and product moments of generalized order statistics for modified Burr XII-geometric distribution and characterization. *J. of Advanced Research Statistics and Probability,* 2011; 1:36-46.

[13] Mokhlis, N. A., Abd El-Aty, Y and Marwa M. Mohie El-Din. Recurrence relations for moments of GOS from Marshall-Olkin-Extended Burr XII distribution. *international J. of math. and computation*, 2010; 8:127-136.